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Global asymptotic behavior and boundedness of positive solutions to an odd-order rational difference equation[☆]

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Abstract

In this note we consider the following high-order rational difference equation

$$x_n = 1 + \frac{\prod_{i=1}^{k}(1 - x_{n-i})}{\sum_{i=1}^{k} x_{n-i}}, \quad n = 0, 1, \ldots,$$

where $k \geq 3$ is odd number, $x_{-k}, x_{-k+1}, x_{-k+2}, \ldots, x_{-1}$ is positive numbers. We obtain the boundedness of positive solutions for the above equation, and with the perturbation of initial values, we mainly use the transformation method to prove that the positive equilibrium point of this equation is globally asymptotically stable.

Keywords: Odd-order rational difference equation; Positive equilibrium point; Global asymptotic stability; Boundedness; Transformation method

1. Introduction

Recently, there has been an increasing interest in the study of the global asymptotic properties of rational difference equations. However, there have not been any effective general methods to deal with the global behavior of rational difference equations of order greater than one so far. As we know, it is extremely difficult to understand thoroughly the global behaviors of solutions of rational difference equations although they have simple forms (or expressions). One can refer to [1–9] for examples to illustrate this. Therefore, the study of rational difference equations of order greater than one is worth further consideration.

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The authors [6] proved that the positive equilibrium point of the difference equation

$$x_{n+1} = \frac{x_n x_{n-1} + 1}{x_n + x_{n-1}}, \quad n = 0, 1, 2, \ldots \tag{LZ1}$$

with positive initial values x_{-1}, x_0 is globally asymptotically stable.

In fact, Eq. (LZ1) may be rewritten into

$$x_{n+1} = 1 + \frac{(1 - x_n)(1 - x_{n-1})}{x_n + x_{n-1}}, \quad n = 0, 1, \ldots. \tag{LZ2}$$

Motivated by this kind of form of the above Eq. (LZ2), the first author of this paper studied global asymptotic stability for positive solutions to the equation

$$x_{n+1} = 1 + \frac{(1 - x_n)(1 - x_{n-1})(1 - x_{n-2})}{x_n + x_{n-1} + x_{n-2}}, \quad n = 0, 1, 2, \ldots, \tag{L}$$

where the initial conditions x_{-2}, x_{-1}, x_0 are positive numbers and at least two of them are not larger than one.

In this paper, we consider the following high-order rational difference equation

$$x_n = 1 + \frac{\prod_{i=1}^{k}(1 - x_{n-i})}{\sum_{i=1}^{k} x_{n-i}}, \quad n = 0, 1, \ldots, \tag{1}$$

where $k \geq 3$ is odd number, and the initial values $x_{-k}, x_{-k+1}, x_{-k+2}, \ldots, x_{-1} \in [\alpha, \beta]$, with $0 < \alpha < 1$ and $\beta > 1$.

It is clear that the equilibrium $\overline{x}$ of Eq. (1) satisfies

$$\overline{x} = \frac{(1 - \overline{x})^k + k\overline{x}}{k\overline{x}},$$

from which we can get that Eq. (1) has a unique positive equilibrium $\overline{x} = 1$.

Eq. (1) is interesting in its own right. To the best of our knowledge, however, Eq. (1) has not been investigated so far. Therefore, to study its qualitative properties is theoretically meaningful.

It is worthwhile to note that the global asymptotic stability for Eqs. (LZ2) and (L) is proved in [6] via the analysis of semi-cycle structure (similar methods are also used in [8,9]). Such analysis while computationally feasible for small k, can be very involved for larger values. One can see that it is difficult to depict semi-cycle structure for larger k in Eq. (1). It is fortunate that, in this note, the transformation method used does not require prior determination of detailed semi-cycle structure.

The paper proceeds as follows. In Section 2, we obtain the boundedness of positive solutions for Eq. (1), while by introducing some preliminary lemmas and notation, in Section 3, we get a proof of global asymptotic stability for the solutions of Eq. (1) with the perturbation of initial values.

The results obtained in this paper partly generalize the corresponding ones given in paper [6].

2. Boundedness of positive solutions to Eq. (1)

Theorem 2.1. *Suppose that there exist $\alpha \in (0, 1)$ and $\beta \in (1, +\infty)$, such that*

$$\frac{(1 - \alpha)^p (1 - \beta)^q}{p\alpha + q\beta} \leq \beta - 1, \quad p + q = k, q = 0, 2, 4, \ldots, k - 1 \tag{2}$$

and

$$\frac{(1 - \alpha)^p (1 - \beta)^q}{p\alpha + q\beta} \geq \alpha - 1, \quad p + q = k, q = 1, 3, 5, \ldots, k, \tag{3}$$

then the solution $\{x_n\}$ of Eq. (1) satisfies $x_n \in [\alpha, \beta]$, for $n = 0, 1, 2, \ldots$.

Proof. In Eq. (1), by setting

$$f(\mathcal{Y}_1, \mathcal{Y}_2, \ldots, \mathcal{Y}_k) = \frac{\prod_{i=1}^{k}(1-\mathcal{Y}_i)}{\sum_{i=1}^{k}\mathcal{Y}_i},$$

where $\mathcal{Y}_i \in [\alpha, \beta], i = 1, 2, \ldots, k$, and by deducing, we easily get that the function $f(\mathcal{Y}_1, \mathcal{Y}_2, \ldots, \mathcal{Y}_k)$ is monotone in $\mathcal{Y}_i$ $(i = 1, 2, \ldots, k)$, respectively.

In fact, might as well supposes,

(i) if $\mathcal{Y}_1, \mathcal{Y}_2, \ldots, \mathcal{Y}_p \le 1, \mathcal{Y}_{p+1}, \mathcal{Y}_{p+2}, \ldots, \mathcal{Y}_{p+q} > 1$, q is even number, then we easily prove that f is decreasing in $\mathcal{Y}_1, \mathcal{Y}_2, \ldots, \mathcal{Y}_p$, and increasing in $\mathcal{Y}_{p+1}, \mathcal{Y}_{p+2}, \ldots, \mathcal{Y}_{p+q}$;

(ii) if $\mathcal{Y}_1, \mathcal{Y}_2, \ldots, \mathcal{Y}_p \le 1, \mathcal{Y}_{p+1}, \mathcal{Y}_{p+2}, \ldots, \mathcal{Y}_{p+q} > 1$, q is odd number, then we easily prove that f is increasing in $\mathcal{Y}_1, \mathcal{Y}_2, \ldots, \mathcal{Y}_p$, and decreasing in $\mathcal{Y}_{p+1}, \mathcal{Y}_{p+2}, \ldots, \mathcal{Y}_{p+q}$.

Therefore, we may get the extremum of $f(\mathcal{Y}_1, \mathcal{Y}_2, \ldots, \mathcal{Y}_k)$, that is,

$$0 \le f(\mathcal{Y}_1, \mathcal{Y}_2, \ldots, \mathcal{Y}_k) \le \frac{(1-\alpha)^p(1-\beta)^q}{p\alpha + q\beta}, \quad \mathcal{Y}_i \in [\alpha, \beta], i = 1, 2, \ldots, k,$$

here $p + q = k, q = 0, 2, 4, \ldots, k - 1$; and

$$\frac{(1-\alpha)^p(1-\beta)^q}{p\alpha + q\beta} \le f(\mathcal{Y}_1, \mathcal{Y}_2, \ldots, \mathcal{Y}_k) \le 0, \quad \mathcal{Y}_i \in [\alpha, \beta], i = 1, 2, \ldots, k,$$

here $p + q = k, q = 1, 3, 5, \ldots, k$.

According to (2) and (3), we may get

$$\alpha - 1 \le f(\mathcal{Y}_1, \mathcal{Y}_2, \ldots, \mathcal{Y}_k) \le \beta - 1, \quad \mathcal{Y}_i \in [\alpha, \beta], i = 1, 2, \ldots, k,$$

which implies that the solution $\{x_n\}$ of Eq. (1) satisfies $x_n - 1 \in [\alpha - 1, \beta - 1]$, for $n = 0, 1, 2, \ldots$. The proof of theorem follows. □

3. The global asymptotic behavior of solutions to Eq. (1)

In order to prove the result in this section, we introduce some preliminary lemmas and notation. First, we consider the simple transformed sequence $\{x_n^*\}$ defined by

$$x_n^* = \begin{cases} e^{x_n - 1}, & 1 \le x_n < 2 \\ e^{1 - x_n}, & 0 < x_n < 1. \end{cases} \tag{4}$$

The following elementary lemmas will be useful.

Lemma 3.1. *Suppose that $\{x_n\}$ satisfies (1), that $\{x_n^*\}$ is obtained from $\{x_n\}$ via (4), and that $1 \le x_{n-i} < 2, i = 1, 2, \ldots, p$, $0 < x_{n-p-j} < 1, j = 1, 2, \ldots, q$, $p + q = k$, then,*

$$\ln x_n^* = \frac{\prod_{i=1}^{p}\ln x_{n-i}^* \cdot \prod_{j=1}^{q}\ln x_{n-p-j}^*}{k + \sum_{i=1}^{p}\ln x_{n-i}^* - \sum_{j=1}^{q}\ln x_{n-p-j}^*}. \tag{5}$$

Proof. From (4), we know that, if $1 \le x_n < 2$, then $x_n - 1 = \ln x_n^*$; while if $0 < x_n < 1$, then $x_n - 1 = -\ln x_n^*$.

(i) if q is even number, then p is odd number, and we have

$$-\ln x_n^* = \frac{(-1)^p\prod_{i=1}^{p}\ln x_{n-i}^* \cdot \prod_{j=1}^{q}\ln x_{n-p-j}^*}{k + \sum_{i=1}^{p}\ln x_{n-i}^* - \sum_{j=1}^{q}\ln x_{n-p-j}^*}; \tag{6}$$

(ii) if q is odd number, then p is even number, and we get

$$\ln x_n^* = \frac{(-1)^p \prod_{i=1}^{p} \ln x_{n-i}^* \cdot \prod_{j=1}^{q} \ln x_{n-p-j}^*}{k + \sum_{i=1}^{p} \ln x_{n-i}^* - \sum_{j=1}^{q} \ln x_{n-p-j}^*}. \tag{7}$$

Obviously, from (6) and (7), (5) follows. □

Lemma 3.2. *Suppose that f is defined by*

$$f(y_1, \ldots, y_p; y_{p+1}, \ldots, y_{p+q}) = \frac{y_1 \cdots y_p \cdot y_{p+1} \cdots y_{p+q}}{k + y_1 + \cdots + y_p - y_{p+1} - \cdots - y_{p+q}}, \tag{8}$$

and $y_1, \ldots, y_p, y_{p+1}, \ldots, y_{p+q} \in [0, 1)$, then, f is increasing in $y_1, \ldots, y_p; y_{p+1}, \ldots, y_{p+q}$, respectively. Here, $p + q = k$, k is odd number.

Proof.

$$\frac{\partial f}{\partial y_m} = \frac{E \cdot F}{(k + y_1 + \cdots + y_p - y_{p+1} - \cdots - y_{p+q})^2}, \quad m = 1, 2, \ldots, p,$$

here

$$E = y_1 \cdots y_{m-1} \cdot y_{m+1} \cdots y_p \cdot y_{p+1} \cdots y_{p+q} \geq 0,$$
$$F = k + y_1 + \cdots + y_{m-1} + \cdots + y_{m+1} + \cdots + y_p - y_{p+1} - \cdots - y_{p+q} \geq 0.$$

Similarly,

$$\frac{\partial f}{\partial y_m} = \frac{G \cdot H}{(k + y_1 + \cdots + y_i - y_{i+1} - \cdots - y_{i+j})^2}, \quad m = p+1, p+2, \ldots, p+q,$$

here

$$G = y_1 \cdots y_p \cdot y_{p+1} \cdots y_{m-1} \cdot y_{m+1} \cdots y_{p+q} \geq 0,$$
$$H = k + y_1 + \cdots + y_p - y_{p+1} - \cdots - y_{m-1} - \cdots - y_{m+1} - \cdots - y_{p+q} \geq 0.$$

From the above analysis, the Lemma 3.2 follows. □

By Lemmas 3.1 and 3.2, we have

Lemma 3.3. *Suppose that $p \geq 1$ in Lemma 3.1, then we have*

$$\max_{1 \leq i \leq k} \{\ln x_{n-i}^*\} \geq \ln x_n^* \geq 0, \tag{9}$$

for all $n \geq k$, where k is odd number.

Proof. By the definition of (4), we easily know that in (5), $1 \leq x_n^* < e$ for all n. Setting $M = \max_{1 \leq i \leq k}\{\ln x_{n-i}^*\}$, and applying Lemma 3.2 k times, we obtain

$$\ln x_n^* \leq \frac{M^k}{k + (p - q)M}.$$

Note that $1 \leq x_n^* < e$ for all n, $M = \max_{1 \leq i \leq k}\{\ln x_{n-i}^*\} \in [0, 1)$, then we obtain

$$M^k \leq kM + (1 - k)M^2 \leq kM + (p - q)M^2.$$

Therefore,

$$\ln x_n^* \leq \frac{M^k}{k + (p - q)M} \leq M. \tag{10}$$

Also the proof of the Lemma 3.3 is completed. □

Now, set

$$D_n = \max_{n-k\le i\le n-1}\{\ln x_i^*\}, \quad n = 0, 1, 2, \ldots. \tag{11}$$

By (11) and (9), we have

$$\begin{aligned} D_{n+1} &= \max_{n+1-k\le i\le n}\{\ln x_i^*\} = \max\{\max_{n+1-k\le i\le n-1}\{\ln x_i^*\}, \ln x_n^*\} \\ &\le \max\{\max_{n+1-k\le i\le n-1}\{\ln x_i^*\}, \max_{1\le i\le k}\{\ln x_{n-i}^*\}\} \\ &= \max\{\max_{n+1-k\le i\le n-1}\{\ln x_i^*\}, \max_{n-k\le i\le n-1}\{\ln x_i^*\}\} = \max_{n-k\le i\le n-1}\{\ln x_i^*\} = D_n. \end{aligned}$$

Therefore, we easily get the following lemma.

Lemma 3.4. *The sequence D_n is monotonically non-increasing in n for $n \ge k$.*

Since $D_n \ge 0$ for $n \ge k$, Lemma 3.4 implies that, as n tends to infinity, the sequence D_n converges to some limit, say D, where $D \ge 0$.

Theorem 3.1. *If (2) and (3) hold, and $1 < \beta < 2$, then the positive equilibrium point $\overline{x} = 1$ of Eq. (1) is globally asymptotically stable.*

Proof. The linearized equation of Eq. (1) about the positive equilibrium point $\overline{x}$ is

$$z_n = 0 \cdot z_{n-1} + 0 \cdot z_{n-2} + \cdots + 0 \cdot z_{n-k},$$

and so it is clear from [7, Remarks 1.3.7] that the positive equilibrium point $\overline{x}$ of Eq. (1) is locally asymptotically stable. It remains to verify that every positive solution $\{x_n\}_{n=-k}^{\infty}$ of Eq. (1) converges to $\overline{x}$ as $n \longrightarrow \infty$. Namely, we want to prove

$$\lim_{n\to\infty} x_n = \overline{x} = 1. \tag{12}$$

Might as well supposes that in Eq. (1), $1 \le x_{n-i} < 2, i = 1, 2, \ldots, p$, $0 < x_{n-p-j} < 1, j = 1, 2, \ldots, q$, $p + q = k$, for $n = 0, 1, 2, \ldots$. Especially, $p = 0$ means that $x_{n-j} < 1, j = 1, 2, \ldots, k$.

First, if $p = 0$, the rules for the positive and negative semi-cycles of the solution to Eq. (1) can be periodically expressed as follows: $\ldots, (k+1)^-, 1^+, (k+1)^-, 1^+, (k+1)^-, 1^+, \ldots$. Then we can easily prove (12) by the semi-cycle methods (see References [6,8,9]).

Next, if $p \ge 1$, it is difficult to depict semi-cycle structure of the solution to Eq. (1), so we will show that the transformed sequence $\{x_n^*\}$ converges to 1, which also indicates (12) to be true.

By the definition (11), the values of D_n are taken on by entries in the sequence $\{\ln x_n^*\}$, and as well, by Lemma 3.3, $\ln x_n^* \in [0, D_n]$ for $n \ge k$. It suffices to prove that $D = 0$. Suppose that $D > 0$, then for any $\mathbb{c} \in (0, D)$, we can find an N such that $\ln x_N^* \in [D, D + \mathbb{c}]$, and for $n \ge N - k$,

$$\ln x_n^* \in [0, D + \mathbb{c}].$$

Since $x_n^* \ge 1$ for all n, employing Lemmas 3.1 and 3.2 gives

$$D \le \ln x_N^* \le \frac{(D+\mathbb{c})^k}{k + (p-q)(D+\mathbb{c})},$$

namely,

$$D[k + (p-q)(D+\mathbb{c})] \le (D+\mathbb{c})^k$$

or

$$kD + (p-q)D^2 \le D^k + C_k^1 D^{k-1}\mathbb{c} + C_k^2 D^{k-2}\mathbb{c}^2 + \cdots + C_k^k \mathbb{c}^k + (q-p)D\mathbb{c},$$

which implies that $D = 0$ by $k \ge 3$ and arbitrary $\mathbb{c} > 0$. The proof of this theorem is complete. □

For the further study, we leave the following problem to the interested readers:

Conjecture. *If (2) and (3) hold, and $\beta \geq 2$, then the positive equilibrium point $\overline{x} = 1$ of Eq. (1) is globally asymptotically stable.*

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