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Asymptotic Initial Value Technique for singularly perturbed convection–diffusion delay problems with boundary and weak interior layers

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ABSTRACT

In this paper a numerical method named as Asymptotic Initial Value Technique (AIVT) is suggested to solve the singularly perturbed boundary value problem for the second order ordinary delay differential equation with the discontinuous convection–diffusion coefficient term. In this technique, the original problem of solving the second order differential equation is reduced to solving three first order differential equations, one of which is a delay differential equation and other two are singularly perturbed problems. The singularly perturbed problems are solved by the second order hybrid finite difference scheme, whereas the delay problem is solved by the fourth order Runge–Kutta method with Hermite interpolation. An error estimate is derived by using the supremum norm and it is of order $O(\varepsilon + N^{-2}\ln^2 N)$, where N and ε are the discretization parameter and the perturbation parameter, respectively. Numerical results are provided to illustrate the theoretical results.

1. Introduction

In the recent years, there has been growing interest in developing numerical methods for singularly perturbed delay differential equations. For more details one may refer to [1–9]. The motivation for this study has come from the paper of Lange and Miura [10]. In the present paper, we consider Singularly Perturbed Boundary Value Problems (SPBVPs) for the second order ordinary delay differential equation with the discontinuous convection–diffusion coefficient term and suggest a numerical method namely Asymptotic Initial Value Technique (AIVT). So far, people have considered problems with continuous coefficients. This initial value method was introduced by Gasparo and Macconi [11]. In fact they applied this method to solve singularly perturbed boundary value problems for differential equations without delay. In [12], the authors presented a numerical method namely asymptotic-numerical method for solving singularly perturbed Robin problems.

The present paper is organized as follows. In Section 2, the problem under study is stated. A maximum principle for the Delay Differential Equation (DDE) is derived in Section 3 and using this principle a stability result is obtained. An asymptotic expansion for the current problem is constructed in Section 4. The present numerical method is described in Section 5 and an error estimate is derived in Section 6. Section 7 presents numerical results.

Note. In the following, C denotes a generic positive constant independent of the singular perturbation parameter ε and the discretization parameter N of the discrete problem. It is conventional for the convection–diffusion problem to assume that

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$\varepsilon \le CN^{-1}$ [13]. The supremum norm is used for studying the convergence of the numerical solution to the exact solution of a singular perturbation problem: $\|u\|_{\Omega} = \sup_{x\in\Omega} |u(x)|$.

2. Statement of the problem

Motivated by the work of [10,13], we consider the following boundary value problem for SPDDE.
Find $u \in Y = C^0(\overline{\Omega}) \cap C^1(\Omega) \cap C^2(\Omega^- \cup \Omega^+)$ such that

$$\begin{cases} -\varepsilon u''(x) + a(x)u'(x) + b(x)u(x-1) = f(x), \quad x \in \Omega^- \cup \Omega^+, \\ u(x) = \phi(x), \quad x \in [-1, 0], \qquad u(2) = l, \end{cases} \tag{2.1}$$

$a(x) = a_1(x),\ x \in [0, 1],\ a(x) = a_2(x),\ x \in (1, 2],\ a_1(1-) \ne a_2(1+)$, where $a_i(x) \ge \alpha_i > 0,\ i = 1, 2, \alpha = \min\{\alpha_1, \alpha_2\}$, $\beta_0 \le b(x) \le \beta_1 \le 0, 2\alpha + 5\beta_0 \ge \eta > 0, \alpha(\alpha_2 - \alpha) > -2\beta_0, |a_i'(x)| \le C, b,\ f$ are sufficiently smooth functions on $\overline{\Omega},\ \Omega = (0, 2),\ \overline{\Omega} = [0, 2],\ \Omega^- = (0, 1),\ \Omega^+ = (1, 2)$ and ϕ is smooth on $[-1, 0]$. Using the stepwise integration procedure [10] and the method of proof adopted in the Theorem 1 of [14], one can show that the above boundary value problem (2.1) has a solution.

The above problem is equivalent to

$$Pu(x) := \begin{cases} -\varepsilon u''(x) + a_1(x)u'(x) = f_1(x) - b(x)\phi(x-1), \quad x \in \Omega^-, \\ -\varepsilon u''(x) + a_2(x)u'(x) + b(x)u(x-1) = f_2(x), \quad x \in \Omega^+, \end{cases} \tag{2.2}$$
$$u(0) = \phi(0), \qquad u(1-) = u(1+), \qquad u'(1-) = u'(1+), \qquad u(2) = l,$$

where $u(1-)$ and $u(1+)$ denote the left and right limits of u at $x = 1$, respectively.

3. Stability result

The differential–difference operator P defined in (2.2) satisfies the following maximum principle.

Theorem 3.1 (*Maximum Principle*). *Let $w \in C^0(\overline{\Omega}) \cap C^2(\Omega^- \cup \Omega^+)$ be any function satisfying $w(0) \ge 0, w(2) \ge 0$, $Pw(x) \ge 0,\ \forall x \in \Omega^- \cup \Omega^+$ and $w'(1+) - w'(1-) = [w'](1) \le 0$. Then $w(x) \ge 0,\ \forall x \in \overline{\Omega}$.*

Proof. Define $s(x) = \frac{1}{8} + \frac{x}{2},\ x \in [0, 1],\ s(x) = \frac{3}{8} + \frac{x}{4},\ x \in [1, 2]$. Note that $s(x) > 0, \forall x \in \overline{\Omega},\ Ps(x) > 0, \forall x \in \Omega^- \cup \Omega^+$ and $[s'](1) < 0$. Let $\mu = \max\{\frac{-w(x)}{s(x)} : x \in \overline{\Omega}\}$. Then there exists $x_0 \in \Omega$ such that $w(x_0) + \mu s(x_0) = 0$ and $w(x) + \mu s(x) \ge 0,\ \forall x \in \overline{\Omega}$. Therefore the function $(w + \mu s)$ attains its minimum at $x = x_0$. Suppose the theorem does not hold true. Then $\mu > 0$.

Case (i). $(x_0 \in \Omega^-)$

$$0 < P(w + \mu s)(x_0) = -\varepsilon(w + \mu s)''(x_0) + a_1(x_0)(w + \mu s)'(x_0) \le 0.$$

It is a contradiction.

Case (ii). $(x_0 \in \Omega^+)$

$$0 < P(w + \mu s)(x_0) = -\varepsilon(w + \mu s)''(x_0) + a_2(x_0)(w + \mu s)'(x_0) + b(x_0)(w + \mu s)(x_0 - 1) \le 0.$$

It is a contradiction.

Case (iii). $(x_0 = 1)$

$$0 \le [(w + \mu s)'](1) = [w'](1) + \mu[s'](1) < 0.$$

It is a contradiction. Hence the proof of the theorem. □

Theorem 3.2 (*Stability Result*). *For any $u \in Y$ we have*

$$|u(x)| \le C \max\left\{ |u(0)|, |u(2)|, \sup_{\xi \in \Omega^- \cup \Omega^+} |Pu(\xi)| \right\}, \quad \forall x \in \overline{\Omega}. \tag{3.1}$$

Proof. Define $\psi^{\pm}(x) = CMs(x) \pm u(x),\ x \in \overline{\Omega}$, where $M = \max\{|u(0)|, |u(2)|, \sup_{\xi\in\Omega^-\cup\Omega^+} |Pu(\xi)|\}$. Then, $\psi^{\pm}(0) = CMs(0) \pm u(0) > 0$ and $\psi^{\pm}(2) = CMs(2) \pm u(2) > 0$, by a proper choice of C.

$$P\psi^{\pm}(x) = CMPs(x) \pm Pu(x) \ \ge \frac{CM}{8}(4\alpha) \pm Pu(x) \ge 0, \quad x \in \Omega^-$$

by a proper choice of C. Similarly one can obtain $P\psi^{\pm}(x) \geq 0,\ x \in \Omega^+$ and $[\psi^{\pm'}](1) = CM[s'](1) \pm [u'](1) < 0$ by a proper choice of C. Then by Theorem 3.1 we have $\psi^{\pm}(x) \geq 0,\ x \in \overline{\Omega}$. Therefore

$$|u(x)| \leq C\ \max\left\{|u(0)|, |u(2)|, \sup_{\xi\in\Omega^-\cup\Omega^+} |Pu(\xi)|\right\}, \quad \forall x \in \overline{\Omega}. \quad \square$$

An immediate consequence of the above Theorem 3.2 is that, the solution of the BVP (2.1) is unique.

4. An asymptotic expansion

In this section, an asymptotic expansion approximation for the solution of problem (2.1) is constructed by using the fundamental idea of the WKB method [15,12].

Let $u_0 \in C^0(\overline{\Omega}) \cap C^1(\Omega^- \cup \Omega^+ \cup \{2\})$ be the solution of the reduced problem of (2.1) given by

$$\begin{cases} a(x)u_0'(x) + b(x)u_0(x-1) = f(x), \quad x \in \Omega^- \cup \Omega^+ \cup \{2\}, \\ u_0(x) = \phi(x), \quad x \in [-1, 0]. \end{cases} \tag{4.1}$$

Further, let $v_1(x) = \exp(-\int_x^1 \frac{a_1(s)}{\varepsilon}ds),\ \forall x \in [0, 1]$ and $v_2(x) = \exp(-\int_x^2 \frac{a_2(s)}{\varepsilon}ds),\ \forall x \in [1, 2]$ be the solutions of the following Terminal Value Problems (TVPs), respectively:

$$L_1v_1(x) = \varepsilon v_1'(x) - a_1(x)v_1(x) = 0, \quad x \in [0, 1), \qquad v_1(1) = 1 \tag{4.2}$$

$$\text{and} \quad L_2v_2(x) = \varepsilon v_2'(x) - a_2(x)v_2(x) = 0, \quad x \in [1, 2), \qquad v_2(2) = 1. \tag{4.3}$$

An asymptotic expansion approximation to the solution of the original problem (2.1) is given by

$$u_{as}(x) = \begin{cases} u_0(x) + k_1[v_1(x) - v_1(0)], \quad x \in [0, 1], \\ u_0(x) + k_2v_2(x) + k_3, \quad x \in [1, 2]. \end{cases} \tag{4.4}$$

Here the constants $k_1,\ k_2,$ and k_3 are to be determined such that $u_{as} \in Y$. In fact the constants $k_1,\ k_2,$ and k_3 are given by

$$\begin{cases} k_1 = \dfrac{[l - u_0(2)]}{1 - v_1(0)}v_2(1) + k_3\dfrac{1 - v_2(1)}{1 - v_1(0)}, \qquad k_2 = l - u_0(2) - k_3, \\ k_3 = \dfrac{\varepsilon[u_0'(1+) - u_0'(1-)][1 - v_1(0)] + a_2(1+)[l - u_0(2)][1 - v_1(0)]v_2(1)}{a_1(1-) + a_2(1+) - [a_2(1+)v_1(0) + a_1(1-)v_2(1)]}. \end{cases} \tag{4.5}$$

It is easy to see that k_2 is bounded, $|k_1| \leq C\varepsilon$ and $|k_3| \leq C\varepsilon$.

Theorem 4.1. *Let u be the solution of (2.1) and u_{as} be its asymptotic expansion approximation given by (4.4). Then $\|u - u_{as}\|_{\overline{\Omega}} \leq C\varepsilon$.*

Proof. Consider the barrier function

$$\varphi^{\pm}(x) = \begin{cases} M\varepsilon[s(x) + e^{\frac{-\alpha(1-x)}{\varepsilon}} + e^{\frac{-\alpha(2-x)}{\varepsilon}}] \pm (u(x) - u_{as}(x)), \quad x \in [0, 1], \\ M\varepsilon[s(x) + 1 + e^{\frac{-\alpha(2-x)}{\varepsilon}}] \pm (u(x) - u_{as}(x)), \quad x \in [1, 2], \end{cases}$$

where M is a positive constant independent of perturbation parameter ε.

It is obvious that $\varphi^{\pm} \in C^0(\overline{\Omega}) \cap C^2(\Omega^- \cup \Omega^+)$, $\varphi^{\pm}(0) > 0$ and $\varphi^{\pm}(2) > 0$.

Case (i). ($x \in \Omega^-$). First we have

$$P(u(x) - u_{as}(x)) = \varepsilon u_0'' + k_1[a_1'(x)]v_1(x) \geq -C\varepsilon[1 + e^{\frac{-\alpha(1-x)}{\varepsilon}}],$$

since $|k_1| \leq C\varepsilon,\ \|u_0''\|_{\Omega^-\cup\Omega^+} \leq C$ and $|a_1'(x)| \leq C$. Then,

$$\begin{aligned} P\varphi^{\pm}(x) &= M\left[\alpha[a_1(x) - \alpha][e^{(\frac{-\alpha(1-x)}{\varepsilon})} + e^{(\frac{-\alpha(2-x)}{\varepsilon})}] + \varepsilon\frac{a_1(x)}{2}\right] \pm P(u(x) - u_{as}(x)) \\ &\geq M\left[\alpha[\alpha_1 - \alpha][e^{(\frac{-\alpha(1-x)}{\varepsilon})} + e^{(\frac{-\alpha(2-x)}{\varepsilon})}] + \varepsilon\frac{\alpha_1}{2}\right] \mp C\varepsilon[1 + e^{(\frac{-\alpha(1-x)}{\varepsilon})}]. \end{aligned}$$

Then $P\varphi^{\pm}(x) \geq 0$ for a suitable choice of M.

Case (ii). ($x \in \Omega^+$). First we have

$$P(u(x) - u_{as}(x)) = \varepsilon u_0'' + k_2[a_2'(x)v_2(x) - b(x)v_2(x-1)] - k_3[b(x)] \geq -C(\varepsilon + e^{\frac{-\alpha(2-x)}{\varepsilon}}),$$

since $|k_3| \leq C\varepsilon$, $\|u_0''\|_{\Omega^- \cup \Omega^+} \leq C$, $v_2(x) \leq e^{-\frac{\alpha(2-x)}{\varepsilon}}$, $v_2(x-1) \leq C\varepsilon$, $|a_2'(x)| \leq C$ and b is a smooth function on [0, 2]. Then,

$$P\varphi^{\pm}(x) = M\left[\alpha[a_2(x)-\alpha]e^{(\frac{-\alpha(2-x)}{\varepsilon})} + \varepsilon\left[\frac{a_2(x)}{4} + b(x)[s(x-1)+1+e^{\frac{-\alpha(3-x)}{\varepsilon}}]\right]\right] \pm P(u(x)-u_{as}(x))$$
$$\geq M\left[\alpha[\alpha_2-\alpha]e^{(\frac{-\alpha(2-x)}{\varepsilon})} + \varepsilon\left[\frac{\alpha_2}{4} + \frac{5\beta_0}{8} + \beta_0 + \beta_0 e^{\frac{-\alpha(3-x)}{\varepsilon}}\right]\right] \mp C(\varepsilon + e^{(\frac{-\alpha(2-x)}{\varepsilon})}).$$

Then $P\varphi^{\pm}(x) \geq 0$ for a suitable choice of M. Further $[\varphi^{\pm'}](1) < 0$. Then by Theorem 3.1 we have $\varphi^{\pm}(x) \geq 0,\ x \in \overline{\Omega}$, that is, $|u(x) - u_{as}(x)| \leq C\varepsilon,\ x \in \overline{\Omega}$. Hence the proof of the theorem. □

Note. From the expansion (4.4) and Theorem 4.1 we see that problem (2.1) exhibits a weak interior layer at $x = 1$ and a strong boundary layer at $x = 2$.

5. Numerical methods

In this section, a hybrid finite difference scheme for the singularly perturbed TVPs (4.2)–(4.3) and a classical numerical method for the IVP (4.1) are described.

5.1. Mesh selection strategy

Since the BVP (2.1) exhibits a strong boundary layer at $x = 2$ and a weak interior layer at $x = 1$, we choose a piece-wise uniform Shishkin mesh on [0, 2]. For this we divide the interval [0, 2] in to four subintervals, namely $\Omega_1 = [0, 1-\tau]$, $\Omega_2 = [1-\tau, 1]$, $\Omega_3 = [1, 2-\tau]$, $\Omega_4 = [2-\tau, 2]$, where $\tau = \min\{0.5, \frac{2\varepsilon \ln N}{\alpha^*}\}$, where $\alpha^* > \max\{\alpha_1, \alpha_2\}$. Let $h = 2N^{-1}\tau$ and $H = 2N^{-1}(1-\tau)$. The mesh $\overline{\Omega}^{2N} = \{x_0, x_1, \ldots, x_{2N}\}$ is defined by

$$\begin{cases} x_0 = 0.0, \quad x_i = x_0 + iH, \quad i = 1(1)\frac{N}{2}, \quad x_{i+\frac{N}{2}} = x_{\frac{N}{2}} + ih, \quad i = 1(1)\frac{N}{2}, \\ x_{i+N} = x_N + iH, \quad i = 1(1)\frac{N}{2}, \quad x_{i+\frac{3N}{2}} = x_{\frac{3N}{2}} + ih, \quad i = 1(1)\frac{N}{2}. \end{cases}$$

5.2. A hybrid finite difference scheme for the TVPs (4.2)–(4.3)

Applying the hybrid finite difference scheme given in [5,16] to the TVPs (4.2) and (4.3) on $[0, 1] \cap \overline{\Omega}^{2N}$ and $[1, 2] \cap \overline{\Omega}^{2N}$ separately, we get

$$L_1^N V_{1_i} = 0, \quad i = 0(1)N-1, \qquad V_{1_N} = 1, \tag{5.1}$$
$$L_2^N V_{2_i} = 0, \quad i = N(1)2N-1, \qquad V_{2_{2N}} = 1, \tag{5.2}$$

where

$$L_1^N V_{1_i} = \begin{cases} \varepsilon \dfrac{V_{1_{i+1}} - V_{1_i}}{H} - a_1(x_i)V_{1_i}, \quad i = 0(1)\dfrac{N}{2} - 1, \\ \varepsilon \dfrac{V_{1_{i+1}} - V_{1_i}}{h} - a_1\left(\dfrac{x_i + x_{i+1}}{2}\right)\dfrac{V_{1_i} + V_{1_{i+1}}}{2}, \quad i = \dfrac{N}{2}(1)N-1, \end{cases}$$

$$L_2^N V_{2_i} = \begin{cases} \varepsilon \dfrac{V_{2_{i+1}} - V_{2_i}}{H} - a_2(x_i)V_{2_i}, \quad i = N(1)\dfrac{3N}{2} - 1, \\ \varepsilon \dfrac{V_{2_{i+1}} - V_{2_i}}{h} - a_2\left(\dfrac{x_i + x_{i+1}}{2}\right)\dfrac{V_{2_i} + V_{2_{i+1}}}{2}, \quad i = \dfrac{3N}{2}(1)2N-1. \end{cases}$$

The following theorem gives an error estimate for this scheme.

Theorem 5.1. *Let $v_1(x)$ and $v_2(x)$ be the solutions of problems* (4.2) *and* (4.3)*, respectively. Further let V_{1_i} and V_{2_i} be their numerical solutions defined by* (5.1) *and* (5.2)*, respectively. Then*

$$\|v_1 - V_1\|_{\overline{\Omega}^{2N} \cap [0,1]} \leq CN^{-2}\ln^2 N \quad \text{and} \quad \|v_2 - V_2\|_{\overline{\Omega}^{2N} \cap [1,2]} \leq CN^{-2}\ln^2 N.$$

Proof. Application of the method of proof described in [16,5] on the intervals [0, 1] and [1, 2] yields the desired result. □

5.3. A numerical method for problem (4.1)

In order to obtain a numerical solution for problem (4.1), we apply the fourth order Runge–Kutta method with piecewise cubic Hermite interpolation on $\overline{\Omega}^{2N}$ [17]. In fact we have

$$\begin{cases} U_0(x_0) = \phi(x_0), \\ U_{0_{i+1}} = U_{0_i} + \dfrac{1}{6}(K_1 + 2K_2 + 2K_3 + K_4), \quad i = 0(1)2N-1, \end{cases} \tag{5.3}$$

where

$$\begin{cases} K_1 = h^*\left[\dfrac{f(x_i)}{a(x_i)} - \dfrac{b(x_i)}{a(x_i)} U_0^{h^*}(x_i)\right] \\ K_2 = h^*\left[\dfrac{f\left(x_i + \frac{h^*}{2}\right)}{a\left(x_i + \frac{h^*}{2}\right)} - \dfrac{b\left(x_i + \frac{h^*}{2}\right)}{a\left(x_i + \frac{h^*}{2}\right)} U_0^{h^*}\left(x_i + \dfrac{h^*}{2}\right)\right] \\ K_3 = h^*\left[\dfrac{f\left(x_i + \frac{h^*}{2}\right)}{a\left(x_i + \frac{h^*}{2}\right)} - \dfrac{b\left(x_i + \frac{h^*}{2}\right)}{a\left(x_i + \frac{h^*}{2}\right)} U_0^{h^*}\left(x_i + \dfrac{h^*}{2}\right)\right] \\ K_4 = h^*\left[\dfrac{f(x_i + h^*)}{a(x_i + h^*)} - \dfrac{b(x_i + h^*)}{a(x_i + h^*)} U_0^{h^*}(x_i + h^*)\right], \end{cases} \quad h^* = \begin{cases} H, & i = 0(1)\dfrac{N}{2} - 1, \quad i = N(1)\dfrac{3N}{2} - 1 \\ h, & i = \dfrac{N}{2}(1)N - 1, \quad i = \dfrac{3N}{2}(1)2N - 1, \end{cases}$$

$$U_0^{h^*}(x) = \begin{cases} \phi(x-1), \quad x \in [x_i, x_{i+1}], \quad i = 0(1)N-1, \\ U_{0_{i-N}} A_i(x) + U_{0_{i-N+1}} A_{i+1}(x) + B_i(x) f^*(x_{i-N}) + B_{i+1}(x) f^*(x_{i-N+1}), \\ \quad x \in [x_i, x_{i+1}], \ i = N(1)2N-1, \end{cases}$$

$$A_i(x) = \left[1 - \frac{2(x - x_i)}{x_i - x_{i+1}}\right] \frac{(x - x_{i+1})^2}{(x_i - x_{i+1})^2}, \qquad A_{i+1}(x) = \left[1 - \frac{2(x - x_{i+1})}{x_{i+1} - x_i}\right] \frac{(x - x_i)^2}{(x_{i+1} - x_i)^2},$$

$$B_i(x) = \frac{(x - x_i)(x - x_{i+1})^2}{(x_i - x_{i+1})^2}, \qquad B_{i+1}(x) = \frac{(x - x_{i+1})(x - x_i)^2}{(x_{i+1} - x_i)^2}$$

$$\text{and} \quad f^*(x_{i-N}) = \frac{f(x_{i-N})}{a(x_{i-N})} - \frac{b(x_{i-N})}{a(x_{i-N})}\phi(x_{i-N} - 1), \qquad f^*(x_{i-N+1}) = \frac{f(x_{i-N+1})}{a(x_{i-N+1})} - \frac{b(x_{i-N+1})}{a(x_{i-N+1})}\phi(x_{i-N+1} - 1).$$

Here $a(x_i) = a_1(x_i),\ i = 0(1)N,$ and $a(x_i) = a_2(x_i),\ i = N+1(1)2N$. The following theorem gives an error estimate for the above method.

Theorem 5.2. *Let $u_0(x)$ be the solution of problem (4.1). Further let U_{0_i} be its numerical solution defined by (5.3). Then $\|u_0 - U_0\|_{\overline{\Omega}^{2N}} \le C\overline{h}^4$, where $\overline{h} = \max\{H, h\}$.*

Proof. Applying the method described in [17] on the intervals [0, 1] and [1, 2] separately we get the desired proof. □

5.4. A numerical solution to the BVP (2.1)

A numerical solution to the original problem (2.1) is given by

$$U_i = \begin{cases} U_{0_i} + k_1[V_{1_i} - v_1(0)], \quad i = 0(1)N, \\ U_{0_i} + k_2 V_{2_i} + k_3, \quad i = N+1(1)2N, \end{cases} \tag{5.4}$$

where U_{0_i}, V_{1_i} and V_{2_i} are numerical solutions of problems (4.1)–(4.3) respectively and $k_1,\ k_2$ and k_3 are defined by (4.5). An error estimate for the above numerical solution is derived in the following section.

6. Error estimate

Theorem 6.1. *Let $u(x)$ be the solution of problem (2.1). Further let U_i be its numerical solution defined by (5.4). Then $\|u - U\|_{\overline{\Omega}^{2N}} \le C(\varepsilon + N^{-2}\ln^2 N)$.*

Proof. From Theorems 4.1, 5.1 and 5.2, we have

$$\|u - u_{as}\|_{\overline{\Omega}} \le C\varepsilon, \qquad \|u_0 - U_0\|_{\overline{\Omega}^{2N}} \le C\overline{h}^4,$$

$$\|v_1 - V_1\|_{\overline{\Omega}^{2N} \cap [0,1]} \le CN^{-2}\ln^2 N \quad \text{and} \quad \|v_2 - V_2\|_{\overline{\Omega}^{2N} \cap [1,2]} \le CN^{-2}\ln^2 N.$$

Then, $|u(x_i) - U_i| \le |u(x_i) - u_{as}(x_i)| + |u_{as}(x_i) - U_i|,\ i = 0(1)2N$

$$\le \begin{cases} |u(x_i) - u_{as}(x_i)| + |u_0(x_i) - U_{0_i}| + |k_1|\,|v_1(x_i) - V_{1_i}|, \quad i = 0(1)N, \\ |u(x_i) - u_{as}(x_i)| + |u_0(x_i) - U_{0_i}| + |k_2|\,|v_2(x_i) - V_{2_i}|, \quad i = N+1(1)2N, \end{cases}$$

$$\le \begin{cases} C\varepsilon + C\varepsilon N^{-2}\ln^2 N, \quad i = 0(1)N, \\ C\varepsilon + CN^{-2}\ln^2 N, \quad i = N+1(1)2N. \end{cases}$$

That is, $|u(x_i) - U_i| \le C(\varepsilon + N^{-2}\ln^2 N), \quad i = 0(1)2N.$ □ (6.1)

Table 1
Numerical results.

	M (Number of mesh points)						
	64	128	256	512	1024	2048	4096
	Example 7.1, $\varepsilon \in \{2^{-17}, 2^{-16}, 2^{-15}, 2^{-14}, 2^{-13}, 2^{-12}, 2^{-11}\}$						
Max. error	5.1438e−4	1.7672e−4	5.9056e−5	1.9387e−5	6.3586e−6	2.1222e−6	7.3739e−7
Order	1.5413	1.5813	1.6070	1.6083	1.5832	1.5250	–
	Example 7.2, $\varepsilon \in \{2^{-12}, 2^{-11}, 2^{-10}, 2^{-9}, 2^{-8}, 2^{-7}, 2^{-6}\}$						
D^M	4.3408e−3	1.5033e−3	5.0751e−4	1.5480e−4	4.8023e−5	1.4576e−5	4.2949e−6
p^M	1.5298	1.5666	1.7131	1.6886	1.7202	1.7629	–

7. Numerical results

In this section, two examples (one constant coefficient problem and one variable coefficient problem) are given to illustrate the numerical method discussed in this paper. The exact solution of the variable coefficient problem is not known. Therefore, in this case we use the double mesh principle to estimate the error and compute the experiment rate of convergence in our computed solution. For this we put $D_\varepsilon^M = \max_{0\le i\le M} |U_i^M - U_{2i}^{2M}|$, where U_i^M and U_{2i}^{2M} are the ith components of the numerical solutions on meshes of M and $2M$ points, respectively. We compute the uniform error and rate of convergence as $D^M = \max_\varepsilon D_\varepsilon^M$ and $P^M = \log_2(\frac{D^M}{D^{2M}})$. For the following examples the numerical results are presented for the values of perturbation parameter $\varepsilon \in \{2^{-17}, 2^{-16}, \dots, 2^{-6}\}$.

Example 7.1 (*Constant Coefficient Homogeneous Problem*).

$$\begin{cases} -\varepsilon u''(x) + 3u'(x) - u(x-1) = 0, & x \in \Omega^-, \\ -\varepsilon u''(x) + 4u'(x) - u(x-1) = 0, & x \in \Omega^+, \\ u(x) = 1, \quad x \in [-1, 0], & u(2) = 2. \end{cases} \tag{7.1}$$

The exact solution of this problem is given by

$$u(x) = \begin{cases} 1 + B[e^{\frac{3x}{\varepsilon}} - 1] + \dfrac{x}{3}, & x \in [0, 1], \\ 2 - D[e^{\frac{8}{\varepsilon}} - e^{\frac{4x}{\varepsilon}}] - \dfrac{(13+\varepsilon)}{24} + \dfrac{B}{2} - \dfrac{B\varepsilon}{3} e^{\frac{3}{\varepsilon}} + \dfrac{x(1-B)}{4} + \dfrac{(x-1)^2}{24} + \dfrac{\varepsilon x}{48} + \dfrac{B\varepsilon}{3} e^{\frac{3(x-1)}{\varepsilon}}, & x \in [1, 2], \end{cases}$$

$$\text{where } D = e^{-\frac{8}{\varepsilon}} D1, \qquad B = e^{-(3/\varepsilon)} B_1, \qquad B_1 = \frac{4D_1 e^{-(4/\varepsilon)} + \varepsilon^2/48 - \varepsilon/12}{3 - (3\varepsilon/4)e^{-(3/\varepsilon)}},$$

$$D_1 = \frac{\frac{18-\varepsilon}{16} + \frac{\varepsilon}{12} + \varepsilon^2\left(\frac{1}{36} + \frac{1}{48}\right) - \frac{\varepsilon^3}{144} + \varepsilon e^{\frac{-3}{\varepsilon}} \left[\frac{5\varepsilon}{192} + \frac{\varepsilon^2}{144} - \frac{18-\varepsilon}{64} - \frac{\varepsilon}{36} - \frac{5}{48}\right]}{3 - e^{(-3/\varepsilon)} \frac{3\varepsilon}{4} + e^{-(4/\varepsilon)}\left(1 - \frac{4\varepsilon}{3}\right) - e^{-(7/\varepsilon)}\left(5 + \frac{\varepsilon}{2}\right)}.$$

Example 7.2 (*Variable Coefficient Non-Homogeneous Problem*).

$$\begin{cases} -\varepsilon u''(x) + (3 + x^2)u'(x) - u(x-1) = \sin(x), & x \in \Omega^-, \\ -\varepsilon u''(x) + (4 + e^x)u'(x) - u(x-1) = \sin(x), & x \in \Omega^+, \\ u(x) = 1, \quad x \in [-1, 0], & u(2) = 2 \end{cases} \tag{7.2}$$

Table 1 presents the numerical results for the problems given in Examples 7.1–7.2.

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