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Existence and Uniqueness of Solutions for a General Nonlinear Variational Inequality

X. P. DING
Department of Mathematics, Sichuan Normal University
Chengdu, Sichuan, P.R. China 61006

E. TARAFDAR
Department of Mathematics, The University of Queensland
St. Lucia, Brisbane, Australia 4072

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Abstract—In this paper, two existence and uniqueness theorems of solutions for a general nonlinear variational inequality under reflexive Banach space settings are proved.

Keywords—Nonlinear variational inequality, 0-diagonally relation, Antimonotone mapping, Contact problems, Lipschitz continuous mapping.

1. INTRODUCTION

The theory of variational inequalities provides very effective and powerful techniques for studying a wide class of problems arising in mechanics, optimization and control problems, transportation and economics equilibrium, contact problems in elasticity, and other branches of mathematics and engineering sciences. In recent years, variational inequalities have been generalized and applied in various directions.

Recently Noor [1–4], Siddiqi-Ansari [5], Bose [6], and Ding [7] studied a class of general nonlinear variational inequalities with bilinear functionals in Hilbert spaces. By applying the fixed point technique of Glowinski-Lions-Tremolieres [8] and Lions-Stampacchia [9], they have proved the existence and uniqueness of solutions for these variational inequalities and give the iterative algorithms for finding the approximate solutions.

In this paper, we consider and study a new class of general nonlinear variational inequalities in reflexive Banach spaces. Some existence and uniqueness theorems of solutions for the general nonlinear variational inequalities are proved by applying minimax inequality and fixed point techniques. Several special cases are discussed, which can be obtained from the main results.

2. PRELIMINARIES

Let B be a Banach space with norm $\|\cdot\|$, B^* be the topological dual space of B and $\langle u, v\rangle$ be the pairing between $u \in B^*$ and $v \in B$. Let M be a nonempty closed convex subset of B, and $a(\cdot, \cdot) : B \times B \to \mathbb{R}$ be a continuous function which is linear in both arguments such that there

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exist constants $\alpha > 0$, $\beta > 0$ satisfying

$$a(v,\, v) \geq \alpha \left\| v \right\|^2, \qquad \text{for all } v \in B, \tag{2.1}$$

$$a(u,\, v) \leq \beta \left\| u \right\| \left\| v \right\|, \qquad \text{for all } u, v \in B. \tag{2.2}$$

Let the functional $b(\cdot,\, \cdot) : B \times B \to \mathbb{R}$ satisfy the following properties:

(1) $b(u,\, v)$ is linear in the first argument;

(2) for each $u \in B$, $b(u,\, \cdot)$ is a convex functional;

(3) $b(u,\, v)$ is bounded, that is, there exists a constant $\nu > 0$ such that

$$b(u,\, v) \leq \nu \left\| u \right\| \left\| v \right\|, \qquad \text{for all } u, v \in B; \tag{2.3}$$

(4) for all $u, v, w \in B$

$$b(u,\, v) - b(u,\, w) \leq b(u,\, v - w). \tag{2.4}$$

Let $A : B \to B^*$, $g : B \to B$ be nonlinear operators and M be a nonempty convex subset of B. A and g are said to have 0-diagonally concave relation on M if the function $\varphi(w,\, v) : M \times M \to \mathbb{R}$ defined by

$$\varphi(w,\, v) = \langle A(w),\, g(v) - g(w) \rangle$$

is 0-diagonally concave in v (cf., [10]), i.e., for any finite set $\{v_1,\, \dots,\, v_m\} \subset M$ and any $w_\lambda = \sum_{j=1}^{m} \lambda_j v_j \left(\lambda_j \geq 0,\, \sum_{j=1}^{m} \lambda_j = 1 \right)$,

$$\sum_{j=1}^{m} \lambda_j \varphi(w_\lambda,\, v_j) \leq 0.$$

A is called antimonotone on M with respect to g if for all $u, v \in M$,

$$\langle Au - Av,\, g(u) - g(v) \rangle \leq 0.$$

A is said to be ξ-Lipschitz continuous on M if there exists a constant $\xi > 0$ such that

$$\left\| Au - Av \right\| \leq \xi \left\| u - v \right\|, \qquad \text{for all } u, v \in M.$$

We consider the following general nonlinear variational inequality problem: find $u \in M$ such that

$$a(u, v - u) + b(u,\, v) - b(u,\, u) \geq \langle A(u),\, g(v) - g(u) \rangle, \qquad \text{for all } v \in M. \tag{2.5}$$

Special Cases

(I) If $g(M) = M$, then the problem (2.5) is equivalent to find $u \in H$ such that $g(u) \in M$ and

$$a(u,\, v - u) + b(u,\, v) - b(u,\, u) \geq \langle A(u),\, g(v) - g(u) \rangle, \qquad \text{for all } g(v) \in M. \tag{2.6}$$

(II) If $g = I$ is the identity operator, the problem (2.5) is equivalent to finding $u \in M$ such that

$$a(u,\, v - u) + b(u,\, v) - b(u,\, u) \geq \langle A(u),\, v - u \rangle, \qquad \text{for all } v \in K. \tag{2.7}$$

Variational inequality (2.7), which models contact problems with friction in elastostatics, has been studied by Duvant-Lions [11], Noor [2,3], and Bose [6] in Hilbert space setting and has been generalized by Ding [7] to quasi-variational variational inequalities in Hilbert space setting.

(III) If $a \equiv 0$ and $A = -I$, then the problem (2.6) is equivalent to finding $u \in B$ such that $g(u) \in M$ and

$$\langle Tu,\, g(v) - g(u) \rangle + b(u,\, v) - b(u,\, u) \geq 0, \qquad \text{for all } g(v) \in K. \tag{2.8}$$

Variational inequality (2.8) and its special cases have been studied by Noor [4] in Hilbert space and Yao [12] in Banach space with $b \equiv 0$.

The following result is Theorem 1 of Ding-Tan [13].

LEMMA 2.1. *Let X be a nonempty convex subset of a topological vector space and let $f : X \times X \to \mathbb{R} \cup \{-\infty, +\infty\}$ be such that*

(i) *for each $x \in X$, $y \mapsto f(x, y)$ is lower semi-continuous on each nonempty compact subset of X,*
(ii) *for each nonempty finite set $A \subset X$ and each $y \in \mathrm{co}(A)$, $\min_{x \in A} f(x, y) \le 0$,*
(iii) *there exist a nonempty compact convex subset X_0 of X and a nonempty compact subset K of X such that for each $y \in X \setminus K$, there is an $x \in \mathrm{co}(X_0 \cup \{y\})$ with $f(x, y) > 0$.*

Then there exists $\hat{y} \in K$ such that $f(x, \hat{y}) \le 0$ for all $x \in X$.

3. MAIN RESULTS

In this section, we use minimax inequality and fixed point technique to prove the existence and uniqueness of solutions for the general nonlinear variational inequality (2.5).

THEOREM 3.1. *B be a reflexive Banach space, B^* be the topological dual of B, and M be a nonempty closed convex subset of B. Let $a : B \times B \to \mathbb{R}$ be a continuous function which is linear in both arguments and satisfies (2.1) and (2.2) and $b : B \times B \to \mathbb{R}$ satisfy the conditions (1)–(4) where $\nu \in (0, \alpha)$. Let $A : B \to B^*$ and $g : B \to B$ be two nonlinear operators such that A is continuous and antimonotone with respect to g, A and g have 0-diagonally concave relation on M and g is θ-Lipschitz continuous. Then, the variational inequality (2.5) has an unique solution $\hat{u}$ in M.*

PROOF. First of all, we prove that for each given $\overline{u} \in M$, there exists an unique $\overline{w} \in M$ such that for all $v \in M$.

$$a(\overline{w}, v - \overline{w}) + b(\overline{u}, v) - b(\overline{u}, \overline{w}) \ge \langle A(\overline{w}), g(v) - g(\overline{w}) \rangle. \tag{3.1}$$

For any fixed $\overline{u} \in M$, define a mapping $f : M \times M \to \mathbb{R}$ by

$$f(v, w) = \langle A(w), g(v) - g(w) \rangle + b(\overline{u}, w) - b(\overline{u}, v) - a(w, v - w), \qquad \text{for all } v, w \in M.$$

Since $b(\cdot, \cdot)$ satisfies the conditions (3) and (4), it is easy to see that $b(\cdot, \cdot)$ also satisfies

$$|b(u, v) - b(u, w)| \le \nu \|u\| \|v - w\|,$$

and hence, $b(u, v)$ is continuous in the second argument. Since $a(\cdot, \cdot)$ is a continuous function which is linear in both arguments and the operators A and g are continuous, we must have that for each fixed $v \in M$, $f(v, \cdot)$ is weak lower semicontinuous. We claim that $f(v, w)$ satisfies the condition (ii) of Lemma 2.1. If it were not true, then there exist $A = \{v_1, \dots, v_n\} \subset M$ and $w = \sum_{u=1}^{n} \lambda_i v_i$, $\lambda_i \ge 0$ and $\sum_{i=1}^{n} \lambda_i = 1$ such that $f(v_i, w) > 0$ for all $i = 1, \dots, n$, that is,

$$\langle A(w), g(v_i) - g(w) \rangle + b(\overline{u}, w) - b(\overline{u}, v_i) - a(w, v_i - w) > 0, \qquad \text{for all } i = 1, \dots, n.$$

It follows that

$$\left\langle A(w), \sum_{i=1}^{n} \lambda_i g(v_i) - g(w) \right\rangle + b(\overline{u}, w) - \sum_{i=1}^{n} \lambda_i b(\overline{u}, v_i) - a(w, w - w) > 0.$$

Since $b(u, v)$ is convex in the second argument, we have

$$b(\overline{u}, w) = b\left(\overline{u}, \sum_{i=1}^{n} \lambda_i v_i\right) \le \sum_{i=1}^{n} b(\overline{u}, v_i),$$

and hence, we have

$$\left\langle A(w),\, \sum_{i=1}^{n} \lambda_i g(v_i) \right\rangle > \langle A(w),\, g(w) \rangle = \left\langle A(w),\, g\left(\sum_{i=1}^{n} \lambda_i v_i\right) \right\rangle,$$

which contradicts the fact that A and g have 0-diagonally concave relation on M. Thus, the condition (ii) of Lemma 2.1 is also satisfied. Let

$$\rho = \frac{1}{\alpha}[(\beta + \nu)\,\|\overline{u}\| + \theta\,\|A(\overline{u})\|],$$
$$K = \{w \in M :\ \|w - \overline{u}\| \le \rho\}.$$

Then K is a bounded closed convex subset of M and hence, it is weakly compact convex. Let $X_0 = \{\overline{u}\}$. Clearly, X_0 is also weakly compact convex subset of M. For each $w \in M \setminus K$, there exists $\overline{u} \in \mathrm{co}(X_0 \cup \{w\})$ such that

$$\begin{aligned} f(\overline{u},\, w) &= \langle A(w),\, g(\overline{u}) - g(w)\rangle + b(\overline{u},\, w) - b(\overline{u},\, \overline{u}) - a(w,\, \overline{u} - w) \\ &\ge a(w - \overline{u},\, w - \overline{u}) - a(\overline{u},\, \overline{u} - w) - b(\overline{u},\, \overline{u} - w) + \langle A(w),\, g(\overline{u}) - g(w)\rangle. \end{aligned}$$

Since A is antimonotone with respect to g, we have

$$\langle A(w), g(\overline{u}) - g(w)\rangle \ge \langle A(\overline{u}),\, g(\overline{u}) - g(w)\rangle.$$

By the assumptions on $a(\cdot,\, \cdot)$ and g, we have

$$\begin{aligned} f(\overline{u},\, w) &\ge \alpha\,\|w - \overline{u}\|^2 - \beta\,\|\overline{u}\|\,\|w - \overline{u}\| - \nu\,\|\overline{u}\|\,\|w - \overline{u}\| - \theta\,\|A(\overline{u})\|\,\|w - \overline{u}\| \\ &= \|w - \overline{u}\|\,[\alpha\,\|w - \overline{u}\| - (\beta + \nu)\,\|\overline{u}\| - \theta\,\|A(\overline{u})\|] > 0. \end{aligned}$$

Hence, the condition (iii) of Lemma 2.1 is also satisfied. By Lemma 2.1, there exists $\overline{w} \in M$ such that $f(v,\, \overline{w}) \le 0$ for all $v \in M$, that is,

$$a(\overline{w},\, v - \overline{w}) + b(\overline{u},\, v) - b(\overline{u},\, \overline{w}) \ge \langle A(\overline{w}),\, g(v) - g(\overline{w})\rangle, \qquad \text{for all } v \in M. \tag{3.1}$$

Now we prove that the $\overline{w}$ is unique. Suppose that for given $\overline{u} \in M$, there exist $w_1, w_2 \in M$ such that (3.1) holds for all $v \in M$. Then we have

$$a(w_1,\, v - w_1) + b(\overline{u},\, v) - b(\overline{u},\, w_1) \ge \langle A(w_1),\, g(v) - g(w_1)\rangle, \text{ and} \tag{3.2}$$
$$a(w_2, v - w_2) + b(\overline{u},\, v) - b(\overline{u},\, w_2) \ge \langle A(w_2),\, g(v) - g(w_2)\rangle, \tag{3.3}$$

for all $v \in M$. Taking $v = w_2$ in (3.2) and $v = w_1$ in (3.3) and adding these inequalities, we obtain

$$a(w_2 - w_1,\, w_2 - w_1) \le \langle A(w_2) - A(w_1),\, g(w_2) - g(w_1)\rangle.$$

Since A is antimonotone with respect to g and $a(\cdot,\, \cdot)$ satisfies (2.1), we must have $w_1 = w_2$, and hence, $\overline{w} \in M$ is unique for a given $\overline{u} \in M$. Thus, we have proved that given $u \in M$, there is a unique solution $w(u)$ satisfying (3.1). Define a mapping $F : M \to M$ by $u \to w(u)$. We prove that the mapping F is a Banach contraction mapping. Indeed, for any $u_1, u_2 \in M$, there exist unique $w_1 = F(u_1)$ and $w_2 = F(u_2)$ such that for each $v \in M$

$$a(F(u_1),\, v - F(u_1)) + b(u_1,\, v) - b(u_1,\, F(u_1)) \ge \langle A(F(u_1)),\, g(v) - g(F(u_1))\rangle, \tag{3.4}$$
$$a(F(u_2),\, v - F(u_2)) + b(u_2,\, v) - b(u_2,\, F(u_2)) \ge \langle A(F(u_2)),\, g(v) - g(F(u_2))\rangle. \tag{3.5}$$

Taking $v = F(u_2)$ in (3.4), and $v = F(u_1)$ in (3.5) and adding these inequalities and by the assumptions of $a(\cdot,\cdot)$, $b(\cdot,\cdot)$, and A, we have

$$a(F(u_1) - F(u_2),\, F(u_1) - F(u_2)) + b(u_1 - u_2,\, F(u_1)) - b(u_1 - u_2,\, F(u_2)) \le \langle A(F(u_1)) - A(F(u_2)),\, g(F(u_1)) - g(F(u_2))\rangle \le 0,$$

and hence,

$$\begin{aligned} a(F(u_1) - F(u_2),\, F(u_1) - F(u_2)) &\le b(u_2 - u_1,\, F(u_2)) - b(u_2 - u_1,\, F(u_1)) \\ &\le b(u_2 - u_1,\, F(u_2) - F(u_1)) \\ &\le \nu \left\| u_2 - u_1 \right\| \left\| F(u_2) - F(u_1) \right\|. \end{aligned}$$

It follows that

$$\alpha \left\| F(u_1) - F(u_2) \right\|^2 \le \nu \left\| u_2 - u_1 \right\| \left\| F(u_2) - F(u_1) \right\|,$$

and

$$\left\| F(u_1) - F(u_2) \right\| \le \frac{\nu}{\alpha} \left\| u_2 - u_1 \right\|.$$

Since $\nu \in (0, \alpha)$, $F : M \to M$ is a Banach contraction mapping. Hence, there exists an unique point $\hat{u} \in M$ such that $\hat{u} = F(\hat{u})$, that is,

$$a(\hat{u},\, v - \hat{u}) + b(\hat{u},\, v) - b(\hat{u},\, \hat{u}) \ge \langle A(\hat{u}),\, g(v) - g(\hat{u})\rangle, \qquad \text{for all } v \in M. \qquad \blacksquare$$

REMARK 3.1. If $g : H \to H$ be the identity mapping, then Theorem 3.1 improves and generalizes the corresponding results of Duvant-Lions [11], Glowinski-Lions-Tremolieres [8], Noor [2] and Necas-Jerusek-Haslinger [14] to reflexive Banach spaces.

REMARK 3.2. Instead of assuming B to be a reflexive Banach, we can easily see that the Theorem 3.1 will hold if we assume $B = C^*$ when C is any Banach space.

In Theorem 3.1, if A is Lipschitz continuous, then the antimonotonity of A is not necessary. This has been proved in the following theorem.

THEOREM 3.2. *Let M be a nonempty closed convex subset of a reflexive Banach space B and B^* be the topological dual space of B. Let $a : B \times B \to \mathbb{R}$ be a continuous function which is linear in both arguments and satisfies* (2.1), (2.2) *and $b : B \times B \to \mathbb{R}$ satisfy conditions* (1)–(4). *Suppose that $A : B \to B^*$ is ξ-Lipschitz continuous, $g : B \to B$ is θ-Lipschitz continuous and A and g have 0-diagonally concave relation such that $\nu + \xi\theta < \alpha$. Then, the variational inequalities* (2.5) *have a unique solution $\hat{u} \in M$.*

PROOF. For any fixed $\overline{u} \in M$, define a mapping $f : M \times M \to \mathbb{R}$ by

$$f(v,\, w) = \langle A(w),\, g(v) - g(w)\rangle + b(\overline{u},\, w) - b(\overline{u},\, v) - a(w,\, v - w), \qquad \text{for all } v, w \in M.$$

By the same argument as in the proof of Theorem 3.1, f satisfies the conditions (i) and (ii) of Lemma 2.1. Let $X_0 = \{\overline{u}\}$ and $K = \{w \in M : \|w - \overline{u}\| \le \rho\}$ where $\rho = 1/(\alpha - \xi\theta)[(\beta + \nu)\|\overline{u}\| + \theta \|A(\overline{u})\|]$. Then, for each $w \in M \setminus K$, there exists $\overline{u} \in \mathrm{co}(X_0 \cup \{w\})$ such that

$$\begin{aligned} f(\overline{u},\, w) &= \langle A(w),\, g(\overline{u}) - g(w)\rangle + b(\overline{u},\, w) - b(\overline{u},\, \overline{u}) - a(w,\, \overline{u} - w) \\ &\ge a(w - \overline{u},\, w - \overline{u}) - a(\overline{u},\, \overline{u} - w) - \langle A(\overline{u}) - A(w),\, g(\overline{u}) - g(w)\rangle \\ &\qquad + \langle A(\overline{u}),\, g(\overline{u}) - g(w)\rangle - b(\overline{u},\, \overline{u} - w) \\ &\ge \alpha \|w - \overline{u}\|^2 - \beta \|\overline{u}\| \|w - \overline{u}\| - \theta\xi \|w - \overline{u}\|^2 - \theta \|A(\overline{u})\| \|w - \overline{u}\| - \nu \|\overline{u}\| \|w - \overline{u}\| \\ &= \|w - \overline{u}\| \left[(\alpha - \theta\xi)\|w - \overline{u}\| - (\beta + \nu)\|\overline{u}\| - \theta \|A(\overline{u})\|\right] > 0, \end{aligned}$$

and hence, by Lemma 2.1, there exists $\overline{w} \in M$ such that (3.1) holds for all $v \in M$. Noting that A and g both are Lipschitz continuous and the condition $\nu + \theta\xi < \alpha$, we can prove by using similar argument as in the proof of Theorem 3.1, that given $u \in M$, there is a unique solution $w(u) \in M$ satisfying (3.1) and that the mapping $F : M \to M$ defined by $F(u) = w(u)$ is a contraction mapping.

REMARK 3.3. If $g : H \to H$ is the identity mapping, it is clear that A and g have the 0-diagonally concave relation. Hence Theorem 3.2 improves and generalizes Theorem 3.1 of Noor in [2,3] and Bose [6, Theorem 2–4] to reflexive Banach spaces.

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