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On the Grassmann space representing the lines of an affine space

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ABSTRACT

In 1982, Bichara and Mazzocca characterized the Grassmann space $Gr(1, \mathbb{A})$ of the lines of an affine space $\mathbb{A}$ of dimension at least 3 over a skew-field K by means of the intersection properties of the three disjoint families Σ_1, Σ_2 and $\mathcal{T}$ of maximal singular subspaces of $Gr(1, \mathbb{A})$. In this paper, we deal with the characterization of $Gr(1, \mathbb{A})$ using only the family $\Sigma = \Sigma_1 \cup \Sigma_2$ of maximal singular subspaces.

1. Introduction

A *partial linear space* is an incidence geometry $\Gamma = (\mathcal{P}, \mathcal{L})$, consisting of a non-empty set $\mathcal{P}$, whose elements are called *points*, and of a family $\mathcal{L}$ of subsets of $\mathcal{P}$, called *lines*, satisfying the following properties:

- every point lies on at least one line;
- every two distinct points lie on at most one line;
- every line contains at least two points.

An *irreducible partial linear space* is a partial linear space where every line contains at least three points.

Two distinct points p and q of a partial linear space $\Gamma = (\mathcal{P}, \mathcal{L})$ are *collinear* if there exists a line $L \in \mathcal{L}$ containing them. The symbol $p \sim q$ means that p and q are collinear. For convenience, we also say that p is collinear to itself. More generally, two subsets X and Y are *collinear* ($X \sim Y$) if each point of one of them is collinear with every point of the other.

If two distinct points p and q of a partial linear space $\Gamma = (\mathcal{P}, \mathcal{L})$ are *collinear*, we denote with $p \vee q$ the unique line $L \in \mathcal{L}$ containing them and we say that it is the line joining p e q.

A partial linear space Γ is *proper* if it contains two non-collinear points. When this is not the case, Γ is a *linear space*. For every point $p \in \mathcal{P}$, $p^\perp$ denotes the set of points of $\mathcal{P}$ collinear with p, and, for every subset $X \subseteq \mathcal{P}$, $X^\perp$ denotes the set $\bigcap_{p \in X} p^\perp$.

A *subspace* of Γ is a subset W of $\mathcal{P}$ such that for every two distinct collinear points of W, the line joining them is contained in W. Clearly, any intersection of subspaces is a subspace and $\mathcal{P}$ is a subspace of Γ (the *improper* one); thus it is possible to define the *closure* $[X]$ of a subset X of $\mathcal{P}$ as the intersection of all subspaces of Γ containing X. A *clique* is a subset of pairwise collinear points of $\mathcal{P}$. If a subspace W is a clique, then it is called *singular subspace*.

Bichara and Mazzocca characterized in [1] the *Grassmann space of the lines* of an affine space $\mathbb{A}$ of dimension at least 3. This is the proper partial linear space $Gr(1, \mathbb{A})$ whose points are the lines of $\mathbb{A}$ and whose lines are the proper and improper pencils of lines of $\mathbb{A}$, a proper pencil being the set of all lines passing through a fixed point and contained in a fixed plane, and an improper pencil being the set of all pairwise parallel lines contained in a fixed plane. According to [1], a proper pencil

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of lines of $\mathbb{A}$ will be called a *line of the first kind* of $Gr(1, \mathbb{A})$, and an improper pencil will be called a *line of the second kind*. It follows that the lines of $Gr(1, \mathbb{A})$ are partitioned into two disjoint subsets $\mathcal{L}_1$ and $\mathcal{L}_2$, consisting of lines of the first kind and the second kind, respectively. Two points of $Gr(1, \mathbb{A})$ are collinear if, and only if, they are either two lines intersecting at a point, or two parallel lines of $\mathbb{A}$, and the line through them is a line of the first or the second kind, respectively.

Furthermore, $Gr(1, \mathbb{A})$ contains three pairwise disjoint families of maximal singular subspaces, say Σ_1, Σ_2 and $\mathcal{T}$. Every element of Σ_1 is the family of all lines containing a fixed point (*proper star*), every element of Σ_2 is the family of all pairwise parallel lines (*improper star*), and, finally, every element of $\mathcal{T}$ is the family of all lines contained in a fixed plane of $\mathbb{A}$ (*ruled plane*).

The following result characterizes $Gr(1, \mathbb{A})$ for $\dim \mathbb{A} \geq 3$.

Theorem 1.1 (*Bichara and Mazzocca, [1], 1982*). *Let $\Gamma = (\mathcal{P}, \mathcal{L})$ be a proper partial linear space whose lines are not maximal singular subspaces satisfying the following axioms.*

(A_1) *Any three pairwise collinear points are contained in a singular subspace of Γ.*
(A_2) *There exist three families Σ_1, Σ_2 and $\mathcal{T}$ of maximal singular subspaces of Γ, and every maximal singular subspace of Γ belongs to exactly one of them. Furthermore, if $\Sigma = \Sigma_1 \cup \Sigma_2$, then the following hold:*
 (i) *For every $S \in \Sigma$, $S' \in \Sigma_1$ with $S \neq S'$, we have $|S \cap S'| = 1$.*
 (ii) *For every $S, S' \in \Sigma_2$ with $S \neq S'$, we have $S \cap S' = \emptyset$.*
 (iii) *For every $S \in \Sigma$ and $T \in \mathcal{T}$, $S \cap T$ is either empty or a line of Γ.*
 (iv) *Every line of $\mathcal{L}$ is contained in exactly one subspace of Σ and exactly one subspace of $\mathcal{T}$.*
(A_3) *Every point $p \in \mathcal{P}$ is contained in a maximal singular subspace of Σ_2.*

Then, there exists an affine space $\mathbb{A}$ of dimension at least 3 such that Γ is isomorphic to the Grassmann space $Gr(1, \mathbb{A})$ of the lines of $\mathbb{A}$.

Tallini's paper [3] was the starting point of these types of characterizations of Grassmann spaces. Melone and Olanda, in [2], characterize the well known Grassmann space of the lines of a projective space of dimension at least 3 by using just properties of one family of maximal singular subspaces. More precisely, they prove that the axioms of Tallini [3] on the two families of maximal singular subspaces follow from their axioms. In this paper, similarly to Melone and Olanda, we prove that the axioms of Bichara and Mazzocca [1] for Grassmann spaces of the lines of an affine space follow from suitable axioms on the family Σ of maximal singular subspaces.

It is easy to see that if we denote by $\mathbb{P}(\mathbb{A})$ the projective extension of $\mathbb{A}$ and by $\mathcal{H}_\infty$ the hyperplane at infinity of $\mathbb{A}$ (removed from $\mathbb{P}(\mathbb{A})$ in order to obtain $\mathbb{A}$), then $Gr(1, \mathbb{A})$ is the incidence structure induced by the well known Grassmann space $Gr(1, \mathbb{P}(\mathbb{A}))$ of the lines of $\mathbb{P}(\mathbb{A})$ on the set of all lines of $\mathbb{A}$. For every pair (A, π) consisting of a point A contained in a plane π of $\mathbb{P}(\mathbb{A})$, let $F(A, \pi)$ be the line of $Gr(1, \mathbb{P}(\mathbb{A}))$ coinciding with the pencil of lines of $\mathbb{P}(\mathbb{A})$ passing through A and contained in π. If $A \in \mathcal{H}_\infty$ and $\pi \nsubseteq \mathcal{H}_\infty$, then $F(A, \pi) \setminus \{\pi \cap \mathcal{H}_\infty\}$ is a line of the second kind of $Gr(1, \mathbb{A})$ and, conversely, every line of the second kind is of type $F(A, \pi) \setminus \{\pi \cap \mathcal{H}_\infty\}$, with $A \in \mathcal{H}_\infty$ and $\pi \nsubseteq \mathcal{H}_\infty$. For every pair (A, π) consisting of a point A contained in a plane π of $\mathbb{P}(\mathbb{A})$, $F^*(A, \pi)$ will denote either $F(A, \pi) \in \mathcal{L}_1$ or $F(A, \pi) \setminus \{\pi \cap \mathcal{H}_\infty\} \in \mathcal{L}_2$, according to $A \notin \mathcal{H}_\infty$ and $\pi \nsubseteq \mathcal{H}_\infty$ or $A \in \mathcal{H}_\infty$ and $\pi \nsubseteq \mathcal{H}_\infty$. Furthermore, if A is a point of $\mathbb{P}(\mathbb{A})$, then S_A will denote either the proper or the improper star of $Gr(1, \mathbb{A})$ consisting of all lines passing through A, according to $A \notin \mathcal{H}_\infty$ or $A \in \mathcal{H}_\infty$, respectively. Finally, for every plane π of $\mathbb{P}(\mathbb{A})$ not contained in $\mathcal{H}_\infty$, T_π will denote the subset of $Gr(1, \mathbb{A})$ consisting of all lines contained in π.

It is easy to see that through every line $L = F^*(A, \pi)$ of $Gr(1, \mathbb{A})$ exactly two maximal singular subspaces pass, coinciding with S_A and T_π, and every point p of $Gr(1, \mathbb{A})$ is contained in exactly one improper star, coinciding with $S_{p \cap \mathcal{H}_\infty}$.

Moreover, let p be a point of $Gr(1, \mathbb{A})$, $S_A \in \Sigma_1$ and $p \notin S_A$. Then every star S_X passing through p intersects S_A at the point of $Gr(1, \mathbb{A})$ coinciding with the line $A \vee X$ of $\mathbb{A}$, and these points trace out the line $L(A, A \vee p)$ of $Gr(1, \mathbb{A})$, which is a line of the first kind.

Finally, if $p \in Gr(1, \mathbb{A})$, $S_A \in \Sigma_2$ and $p \notin S_A$, then $S_{p \cap \mathcal{H}_\infty}$ is the unique star of Σ_2 passing through p. The star $S_{p \cap \mathcal{H}_\infty}$ is disjoint from S_A, every star S_X with $X \in p$ and $X \notin \mathcal{H}_\infty$ intersects S_A at the point of $Gr(1, \mathbb{A})$ coinciding with the line of $\mathbb{A}$ passing through X and having the direction of the point at infinity A, and these points trace out the line $L(A, A \vee p)$ of $Gr(1, \mathbb{A})$, which is a line of the second kind.

In what follows, we will prove that these properties characterize $Gr(1, \mathbb{A})$. More precisely, we will prove the following result.

Theorem 1.2. *Let $\Gamma = (\mathcal{P}, \mathcal{L})$ be a proper irreducible partial linear space whose lines are not maximal singular subspaces and let Σ_1 and Σ_2 be two non-empty families of maximal singular subspaces of Γ. Let $\Sigma = \Sigma_1 \cup \Sigma_2$, and suppose that the following axioms hold.*

(P_1) *Every line $L \in \mathcal{L}$ is contained in at most two maximal singular subspaces of Γ.*
(P_2) *For $i = 1, 2$, for every $S \in \Sigma_i$ and for every point $p \notin S$ there exist exactly $i - 1$ elements of Σ passing through p and disjoint from S. Furthermore, every $S' \in \Sigma$ passing through p and not disjoint from S intersects S at a unique point, and these points trace out a line, coinciding with $p^\perp \cap S$.*
(P_3) *There exists a point $\bar{p} \in \mathcal{P}$ which is contained in at most one element of Σ_2.*

Then, there exists an affine space $\mathbb{A}$ of dimension at least 3 such that Γ is isomorphic to the Grassmann space $Gr(1, \mathbb{A})$ of the lines of $\mathbb{A}$.

2. The proof

Let $\Gamma = (\mathcal{P}, \mathcal{L})$ be a proper irreducible partial linear space whose lines are not maximal singular subspaces and let Σ_1 and Σ_2 be two non-empty families of maximal singular subspaces of Γ. If $\Sigma = \Sigma_1 \cup \Sigma_2$, then every maximal singular subspace of Σ will be called a *star* of Γ, and suppose that the stars satisfy axioms (P_2) and (P_3). The following proposition summarizes some properties of the stars of Γ.

Proposition 2.1. *The stars of Γ satisfy the following properties.*

(i) *Two different non-disjoint stars of Γ intersect at a point.*
(ii) *Every point $p \in \mathcal{P}$ is contained in exactly one star of Σ_2.*
(iii) *Every line $L \in \mathcal{L}$ is contained in exactly one star of Γ.*
(iv) *Every point $p \in \mathcal{P}$ is contained in at least two stars.*
(v) *For every star S and for every point $p \notin S$, $p^{\perp} \cap S$ is a line.*

Proof. (i) This easily follows from condition (P_2).

(ii) *Step* 1. *Σ_2 is a covering of $\mathcal{P}$*. Let p be a point of $\mathcal{P}$ and S be a star of Σ_2. If $p \in S$ we are done, otherwise, from property (P_2), there exists a star S' passing through p and disjoint from S. If $S' \in \Sigma_1$, then a fixed point $q \in S$ is contained into a star disjoint from S', contradicting property (P_2). It follows that $S' \in \Sigma_2$.

Step 2. *Every two distinct stars of Σ_2 have empty intersection*. By contradiction, let S and S' be two distinct stars of Σ_2 having non-empty intersection. From (i) above, $S \cap S'$ is a point p. Recall that, from property (P_3), $\bar{p}$ is a fixed point of $\mathcal{P}$ which is contained in at most one star of Σ_2, hence $\bar{p} \neq p$. If either $\bar{p} \in S$ or $\bar{p} \in S'$, from (P_2) there exists a unique star $\bar{S}$ of Σ passing through $\bar{p}$ and disjoint from either S' or S, respectively. If $\bar{S} \in \Sigma_1$, either the star S' or the star S contains p and is disjoint from $\bar{S}$, contradicting property (P_2). It follows that $\bar{S} \in \Sigma_2$, contradicting the assumption that $\bar{p}$ is contained in at most one star of Σ_2. Hence $\bar{p} \notin S \cup S'$. From property (P_2) again, there exist a star S'' and a star S''' passing through $\bar{p}$ and disjoint from S and S', respectively. The arguments of $\bar{S}$ hold for S'' and S''' too, thus $S'', S''' \in \Sigma_2$. If $S'' \neq S'''$, we contradict (P_3), thus $S'' = S'''$. It follows that there exist two stars passing through p and disjoint from S'', contradicting property (P_2).

(iii) From (i), the line L is contained in at most one star. Let p be a point of L and, from (ii) above, let S be a star of Σ_2 passing through p. If $L \subseteq S$ we are done, otherwise, let q be a point of $L \setminus \{p\}$. Since $p \in q^{\perp} \cap S$, from property (P_2) there exists a star S' passing through q and intersecting S at the point p. Thus $L = p \vee q$ is contained in S'.

(iv) Let p be a point of $\mathcal{P}$ and, from (ii), let S be a star of Σ_2 passing through p. Since $\Sigma_1 \neq \emptyset$ there exists a star $S' \in \Sigma_1$ and S' intersects S at a point, from property (P_2). If $p \in S'$ we are done, otherwise, p is collinear with the point $S \cap S'$ of S'; hence, from property (P_2), $p^{\perp} \cap S'$ is a line and there exists a star S'' passing through p and a point of the line $p^{\perp} \cap S'$ different from $S \cap S'$.

(v) It comes directly from property (P_2). □

As a consequence of the previous proposition, we have the following fundamental result.

Proposition 2.2. *For every point $p \in \mathcal{P}$ and for every line $L \in \mathcal{L}$, if $|p^{\perp} \cap L| \geq 2$, then $L \subseteq p^{\perp}$ (i.e. Γ is a gamma-space).*

Proof. Let p be a point and L be a line such that $p^{\perp} \cap L$ contains at least two distinct points a and b. From Proposition 2.1(iii), L is contained in a unique star S. If $p \in S$, then $p \sim L$, otherwise, from (v) of Proposition 2.1, $p^{\perp} \cap S$ is a line M, and M intersects L at least at the two distinct points a and b, thus $M = L$ and $p \sim L$. □

For every line $L \in \mathcal{L}$, from Proposition 2.1(iii), there exists a unique star of Σ, say S_L, passing through L.

For every point p and for every line L such that $p \sim L$ and $p \notin S_L$, we can consider the subspace $[p, L]$. The following proposition characterizes $[p, L]$, but its principal significance is in the assertion that $[p, L]$ is singular.

Proposition 2.3. *Let $L \in \mathcal{L}$, S_L be the unique star passing through L and $p \notin S_L$ be a point of $\mathcal{P}$ collinear with L. Then $[p, L]$ is one of the following subsets.*

(i) *If $S_L \in \Sigma_1$, then $[p, L] = \bigcup_{q \in L}(p \vee q)$.*
(ii) *If $S_L \in \Sigma_2$, then there exists a unique star $\hat{S}$ passing through p and disjoint from S_L, $L^{\perp} \cap \hat{S}$ is a line R and $[p, L] = \bigcup_{q \in L}(p \vee q) \cup R$.*

In both cases, $[p, L]$ is a singular subspace of Γ.

Proof. (i) Let T be $\bigcup_{q \in L}(p \vee q)$. Clearly, $\{p\}, L \subseteq T$, thus we have to prove the following items:

$(i)_1$ T is a subspace of Γ.
$(i)_2$ Every subspace W of Γ containing p and L contains T, too.

(i)$_1$ Let x and y be two distinct collinear points of T. If either $x, y \in L$, or $x = p$, or $y = p$, then $x \vee y \subseteq T$, thus we can suppose that $x, y \neq p$ and $\{x, y\} \not\subseteq L$. Then, there exist $q_1, q_2 \in L$ such that $x \in p \vee q_1$ and $y \in p \vee q_2$. It is not an essential restriction if we suppose that $y \neq q_2$. Since $q_2 \sim q_1, p$, from Proposition 2.2, $q_2 \sim x$. Let $z \in x \vee y \setminus \{x, y\}$. From Proposition 2.2, $p \sim z$ and, from Proposition 2.1(iii), there exists a star S passing through the line $p \vee z$. Since $S_L \in \Sigma_1$, from property (P$_2$), S intersects S_L at a point q of $p^\perp \cap S_L = L$. From Proposition 2.2, x is collinear with L, hence x is collinear with $p, z, q \in S$ and, from Proposition 2.1(v), $z \in p \vee q \subseteq T$.

(i)$_2$ Since $\{p\}, L \subseteq W$, then for every point q of L we have $p \sim q$ and the line $p \vee q$ is contained in W. It follows that $T \subseteq W$.

(ii) Since $S_L \in \Sigma_2$, from property (P$_2$) there exists a unique star $\hat{S}$ passing through p and disjoint from S_L. For every point $x \in L$, $x \sim p$; hence, from property (P$_2$), $x^\perp \cap \hat{S}$ is a line R_x passing through p. Let $y \in L \setminus \{x\}$ and $z \in p \vee y \setminus \{p, y\}$. From Proposition 2.2, $x \sim z$; hence, from Proposition 2.1(iii), the line $x \vee z$ is contained into a star S passing through x. Since x is contained into the star S_L which is disjoint from $\hat{S}$, from property (P$_2$), the star S intersects $\hat{S}$ at a point w of the line R_x. It is $w \in x \vee z$, otherwise p would be collinear with the three non-collinear points x, z, w in S, contradicting Proposition 2.1(v). It follows that y is collinear with $x, z \in S$; hence, from Proposition 2.2, $y \sim w$. Finally, y is collinear with p and w in $\hat{S}$, thus, $R_y := y^\perp \cap \hat{S} = R_x$. Let R be the unique line of $\hat{S}$ collinear with all points of L. Then $L^\perp \cap \hat{S} = R$, otherwise a point of L would be collinear with more than a line in $\hat{S}$. Let $\hat{T}$ be $\bigcup_{q \in L}(p \vee q) \cup R$. Clearly, $\{p\}, L \subseteq \hat{T}$, thus we have to prove the following items:

(ii)$_1$ $\hat{T}$ is a subspace of Γ.
(ii)$_2$ Every subspace W of Γ containing p and L contains $\hat{T}$, too.

(ii)$_1$ Let x and y be two distinct collinear points of $\hat{T}$.

First of all, let us suppose that $x, y \in \bigcup_{q \in L}(p \vee q)$. If either $x, y \in L$, or $x = p$, or $y = p$, then $x \vee y \subseteq \bigcup_{q \in L}(p \vee q) \subseteq \hat{T}$, thus we can suppose that $x, y \neq p$ and $\{x, y\} \not\subseteq L$. Then, there exist $q_1, q_2 \in L$ such that $x \in p \vee q_1$ and $y \in p \vee q_2$. Furthermore, let us suppose that $y \neq q_2$. Since $q_2 \sim q_1, p$, from Proposition 2.2, $q_2 \sim x$. Let $z \in x \vee y \setminus \{x, y\}$. From Proposition 2.2, $p \sim z$ and, from Proposition 2.1(iii), there exists a star S passing through the line $p \vee z$. If $S \neq \hat{S}$, then, from property (P$_2$), S intersects S_L at a point q of $p^\perp \cap S_L = L$. From Proposition 2.2, x is collinear with L; hence x is collinear with $p, z, q \in S$ and, from Proposition 2.1(v), $z \in p \vee q \subseteq \bigcup_{q \in L}(p \vee q) \subseteq \hat{T}$. If $S = \hat{S}$, since $q_2 \sim x, y$, from Proposition 2.2 $q_2 \sim z$; hence $z \in q_2^\perp \cap \hat{S} = R \subseteq \hat{T}$.

If $x, y \in R$, then $x \vee y = R \subseteq \hat{T}$.

Finally, we can suppose that $x \in R \setminus \{p\}$ and $y \in \bigcup_{q \in L}(p \vee q) \setminus \{p\}$. Let $z \in x \vee y \setminus \{x, y\}$. From Proposition 2.2, $p \sim z$ and, from Proposition 2.1(iii), there exists a star S, which is different from $\hat{S}$, passing through the line $p \vee z$. Since $S_L \in \Sigma_2$ and p is contained into the star $\hat{S}$ which is disjoint from S_L, from property (P$_2$), S intersects S_L at a point q of $p^\perp \cap S_L = L$. Since $R = L^\perp \cap \hat{S}$, x is collinear with L; hence x is collinear with $p, z, q \in S$ and, from Proposition 2.1(v), $z \in p \vee q \subseteq \hat{T}$.

(ii)$_2$ Since $\{p\}, L \subseteq W$, then for every point q of L we have $p \sim q$ and the line $p \vee q$ is contained in W. It follows that $\bigcup_{q \in L}(p \vee q) \subseteq W$. For every point $x \in R \setminus \{x\}$, let q be a fixed point of L. Since $L \sim R$, x is collinear with q and, from Proposition 2.2, p is collinear with the line $x \vee q$. Let z be a point of $x \vee q \setminus \{x, q\}$. From Proposition 2.1(iii), there exists a star S passing through the line $p \vee z$. Since $S_L \in \Sigma_2$ and p is contained into the star $\hat{S}$ which is disjoint from S_L, from property (P$_2$), S intersects S_L at a point q' of $p^\perp \cap S_L = L$. Since x is collinear with $p, z, q' \in S$, from Proposition 2.1(v), $z \in p \vee q' \subseteq \bigcup_{q \in L}(p \vee q) \subseteq W$. Finally, since q and z are two collinear points of the subspace W, the line through them is contained in W, hence $x \in W$.

Finally, we have to prove that both T and $\hat{T}$ are cliques.

Let x and y be two distinct points of $\bigcup_{q \in L}(p \vee q)$. If either $x, y \in L$, or $x = p$, or $y = p$, then $x \sim y$, thus we can suppose that $x, y \neq p$ and $\{x, y\} \not\subseteq L$. Then, there exist $q_1, q_2 \in L$ such that $x \in p \vee q_1$ and $y \in p \vee q_2$. It is not an essential restriction if we suppose that $y \neq q_2$. Since $q_2 \sim q_1, p$, from Proposition 2.2, $q_2 \sim x$. It follows that x is collinear with p, q_2 and, from Proposition 2.2 again, $x \sim y$. This proves that T is a clique.

Furthermore, in case (ii), if $x, y \in R$ the assumption easily follows. Finally, let $x \in R$ and $y \in \bigcup_{q \in L}(p \vee q) \setminus \{p\}$. Then, there exists a point $q \in L$ such that $y \in p \vee q$. Since $x \sim p, q$, from Proposition 2.2 it is $x \sim y$. $\square$

For every line $L \in \mathcal{L}$, let T_L be the set $(L^\perp \setminus S_L) \cup L$ and $\mathcal{T}$ be the family $\{T_L : L \in \mathcal{L}\}$. The following proposition holds.

Proposition 2.4. *For every line L of $\mathcal{L}$, the following holds.*

(i) *$L^\perp \setminus S_L$ is non-empty.*
(ii) *T_L is a subspace of Γ.*
(iii) *T_L is a maximal singular subspace of Γ if, and only if, property* (P$_1$) *holds.*

Proof. (i) Let p_0 be a fixed point of the line L. From Proposition 2.1(iv), there exists a star S passing through p_0 and different from S_L. From Proposition 2.1(v), a point $q \in L \setminus \{p_0\}$ is collinear with a line M of S; hence, from Proposition 2.2, every point of $M \setminus \{p_0\}$ is collinear with L, thus $M \setminus \{p_0\} \subseteq L^\perp \setminus S_L$.

(ii) Let x and y be two distinct and collinear points of T_L. If $x, y \in L$, then $x \vee y = L \subseteq T_L$, thus we can suppose that $\{x, y\} \not\subseteq L$. From Proposition 2.2, every point of L is collinear with $x \vee y$, hence every point $z \in x \vee y$ is contained in $L^\perp$. Moreover, if $x \vee y$ intersects S_L at a point w, then $w \in L$, otherwise a point of $x \vee y$ different from w would be collinear with L and $w \notin L$ in S_L, contradicting (v) of Proposition 2.1. It follows that $x \vee y \subseteq T_L$.

(iii) Suppose that T_L is a maximal singular subspace of Γ. Then L is contained in S_L and T_L. If W is a maximal singular subspace of Γ passing through L and different from S_L, from Proposition 2.1(v) $W \cap S_L = L$; hence every point $x \in W \setminus L$ is a point of $L^\perp \setminus L$. It follows that $W \setminus L \subseteq L^\perp \setminus L$ and, from maximality of W, $W = T_L$. Thus, the line L is contained in S_L and T_L, and property (P_1) holds.

Conversely, suppose that (P_1) holds. From (ii), T_L is a subspace, thus we have to prove the following steps.

Step 1. *T_L is a clique*. Let x and y be two distinct points of T_L. If either x or y lies on L, then $x \sim y$, thus we can suppose that $x, y \in T_L \setminus L = L^\perp \setminus S_L$, which is non-empty, from (i). From Proposition 2.3, $[x, L]$ and $[y, L]$ are two singular subspaces of Γ. If $x \not\sim y$, then there exist two different maximal singular subspaces X and Y of Γ containing $[x, L]$ and $[y, L]$, respectively. Furthermore, neither X nor Y coincides with S_L, otherwise, either x or y is contained in S_L, a contradiction, since $x, y \in L^\perp \setminus S_L$. It follows that L is contained into three different maximal singular subspaces of Γ, contradicting property (P_1).

Step 2. *T_L is a maximal singular subspace*. Let X be a singular subspace containing T_L and let us consider a point $x \in X \setminus L$. Then, from $\{x\}, L \subseteq X$ we have $x \sim L$. Furthermore, $x \notin S_L$, otherwise a fixed point $y \in T_L \setminus S_L$ (whose existence is guaranteed by (i)) would be collinear with L and $x \notin L$ in S_L, contradicting (v) of Proposition 2.1. It follows that $x \in L^\perp \setminus S_L \subseteq T_L$, thus $X \subseteq T_L$ and $X = T_L$. □

From the previous proposition, the family $\mathcal{T} = \{T_L : L \in \mathcal{L}\}$ consists of maximal singular subspaces of Γ. In what follows, we will prove that the three families Σ_1, Σ_2 and $\mathcal{T}$ satisfy the hypotheses of Theorem 1.1. We explicitly observe that property (P_1) is a necessary and sufficient condition in order that every T_L is a maximal singular subspace of Γ.

Proposition 2.5. *The partial linear space $\Gamma = (\mathcal{P}, \mathcal{L})$ satisfies the following properties.*

(i) *Any three pairwise collinear points of $\mathcal{P}$ are contained in a maximal singular subspace of $\Sigma \cup \mathcal{T}$.*
(ii) *No subspace of $\mathcal{T}$ is contained in a star of Σ.*
(iii) *If $S \in \Sigma$ and $T \in \mathcal{T}$, then $S \cap T$ either is empty or it is a line.*
(iv) *Every line of $\mathcal{L}$ is contained in exactly one subspace of Σ and exactly one subspace of $\mathcal{T}$.*
(v) *Every maximal singular subspace belongs to $\Sigma \cup \mathcal{T}$.*

Proof. (i) Let a, b and c be three pairwise collinear points of $\mathcal{P}$, and let L be the line of $\mathcal{L}$ passing through a and b. From Proposition 2.2, $c \in L^\perp$; hence either $c \in S_L$, or $c \in L^\perp \setminus S_L \subseteq T_L$.

(ii) If there exists a subspace $T_L \in \mathcal{T}$ which is contained into a star $S \in \Sigma$, then, from Proposition 2.1(iii), S coincides with the unique star S_L containing L, a contradiction, since $L^\perp \setminus S_L \neq \emptyset$ from (i) of Proposition 2.4.

(iii) Clearly, for every $S \in \Sigma$ and $T \in \mathcal{T}$, $S \cap T$ is at most a line, otherwise we can consider a point $x \in T \setminus S$ (from (ii)) and x would be collinear with more than a line into the star S, contradicting Proposition 2.1(v). Since $T \in \mathcal{T}$, there exists a line $L \in \mathcal{L}$ such that $T = T_L$. From Proposition 2.1(iii), let S_L be the unique star passing through L. If $S = S_L$, then $S_L \cap T_L = L$, and we are done. Thus, we can suppose that $S \neq S_L$. From Proposition 2.1(i), $S \cap S_L$ either is empty or is a point y. Moreover, let $S \cap T \neq \emptyset$, and let p be a point of $S \cap T$.

Let $p \in L$. For every $w \in L \setminus \{p\}$, from Proposition 2.1(v), $w^\perp \cap S$ is a line M passing through p. It follows that every point $z \in M \setminus \{p\}$ is collinear with both p and w in L; hence $M \setminus \{p\} \subseteq L^\perp \setminus S_L$ and $M \subseteq T_L$, thus $S \cap T_L = M$.

Now, suppose that $p \notin L$. Moreover, let us suppose that $S \cap S_L = \emptyset$. From property (P_2), $S_L \in \Sigma_2$; hence, from Proposition 2.3(ii), $L^\perp \cap S$ is a line R passing through p. It follows that $R \subseteq L^\perp \setminus S_L \subseteq T_L$, thus $S \cap T_L = R$.

Finally, let us suppose that $p \notin L$ and $S \cap S_L = y$. Clearly, $p \neq y$, otherwise $T_L \cap S_L$ contains $L \cup \{p\}$, contradicting the fact that a star and a subspace of $\mathcal{T}$ intersect in at most a line. If $y \notin L$, p would be collinear with y and L in S_L, contradicting Proposition 2.1(v). It follows that $y \in L$; hence $S \cap T_L = p \vee y$.

(iv) From Proposition 2.1(iii), every line L is contained in exactly one star S_L of Σ. Moreover, the maximal singular subspace $T_L = (L^\perp \setminus S_L) \cup L$ of $\mathcal{T}$ contains L. If T is a maximal singular subspace of $\mathcal{T}$ passing through L and different from T_L, then L is contained in at least three maximal singular subspaces of Γ, contradicting property (P_1).

(v) Let W be a maximal singular subspace of Γ, let us suppose that $W \notin \Sigma$, and let L be a line of W. From (iv) and property (P_1), it is $W \in \mathcal{T}$. □

Let $\Gamma = (\mathcal{P}, \mathcal{L})$ be a proper partial linear space whose lines are not maximal singular subspaces and let Σ_1 and Σ_2 be two non-empty families of maximal singular subspaces of Γ such that the family $\Sigma = \Sigma_1 \cup \Sigma_2$ satisfies property (P_1), (P_2) and (P_3) of Theorem 1.2. From Proposition 2.1(iii), every line $L \in \mathcal{L}$ is contained in exactly one maximal singular subspace $S_L \in \Sigma$ and, from (iii) of Proposition 2.4, the subset $T_L = (L^\perp \setminus S_L) \cup L$ is a maximal singular subspace. Denoted by $\mathcal{T}$, the family of all maximal singular subspaces T_L, $L \in \mathcal{L}$, from Proposition 2.5(v) every maximal singular subspace of Γ belongs to exactly one of the families Σ_1, Σ_2, $\mathcal{T}$. Axiom (A_1) of Theorem 1.1 follows from (i) of Proposition 2.5. Item (i) of Axiom (A_2) easily follows from property (P_2), and items (ii), (iii) and (iv) follow from Proposition 2.1(ii), and from (iii) and (iv) of Proposition 2.5. Finally, Axiom (A_3) follows from (ii) of Proposition 2.1.

Since all the hypotheses of the Theorem 1.1 of Bichara and Mazzocca are satisfied, we can conclude that there exists an affine space $\mathbb{A}$ of dimension at least 3 such that Γ is isomorphic to the Grassmann space $Gr(1, \mathbb{A})$ of the lines of $\mathbb{A}$.

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