Equivalence of inverse Sturm–Liouville problems with boundary conditions rationally dependent on the eigenparameter

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Abstract
Three inverse problems for a Sturm–Liouville boundary value problem \(-y'' + qy = \lambda y, y(0) \cos \alpha = y'(0) \sin \alpha \) and \(y'(1) = f(\lambda)y(1)\) are considered for rational \(f\). It is shown that the Weyl \(m\)-function uniquely determines \(\alpha, f,\) and \(q\), and is in turn uniquely determined by either two spectra from different values of \(\alpha\) or by the Prüfer angle. For this it is necessary to produce direct results, of independent interest, on asymptotics and oscillation.

Keywords: Sturm–Liouville; Eigenparameter dependent boundary conditions

1. Introduction

Broadly, inverse spectral theory for Sturm–Liouville problems seeks information about the original problems in terms of spectral constructions that they generate. Particular constructions of interest here will be Weyl’s \(m\)-function, Prüfer’s angle, and sequences of
eigenvalues. The information that we seek will be “uniqueness,” i.e., whether the mapping from the original problem to the spectral construction is 1-1. Our setting involves a general type of eigenvalue dependent boundary condition, and to our knowledge all our inverse results are new. In fact, we shall do somewhat better, by exhibiting explicit relationships between the various constructions. We shall also give various results about the eigenvalues and eigenvectors of the direct problem.

In this section we shall review some of the concepts introduced above. The Sturm–Liouville problem that we consider involves the regular equation

$$ly := -y'' + qy = \lambda y \text{ on } [0, 1],$$

with $q \in AC[0, 1]$, subject to the boundary conditions

$$y(0) \cos \alpha = y'(0) \sin \alpha, \quad \alpha \in [0, \pi),$$

and

$$\frac{y'}{y}(1) = f(\lambda),$$

which we denote by $(\alpha, f, q)$. When $f(\lambda) = \infty$, (1.3) is interpreted as a Dirichlet condition $y(1) = 0$. In general, we consider

$$f(\lambda) = \frac{h(\lambda)}{g(\lambda)},$$

where $g$ and $h$ are polynomials with real coefficients and no common zeros. In addition, if $\deg(g) \geq \deg(h)$ then we set $M = \deg(g)$ and assume that $g$ is monic, and if $\deg(g) < \deg(h)$ then we set $M = \deg(h)$ and assume that $h$ is monic. “Standard” boundary conditions will refer to the case when $f$ is constant, i.e., independent of $\lambda$.

Weyl introduced the $m$-function in 1910 [25] in order to study singular problems on an interval $(a, \infty)$, but the construction involves regular problems on $(a, b)$ for increasing $b$. Setting $a = 0, b = 1$ for notational simplicity, we find that the $m$-function involves two solutions of (1.1) with initial and terminal values corresponding to standard boundary conditions (1.2) and (1.3), respectively. Thus, the construction encodes information about the differential equation and both boundary conditions. The $m$-function has become established as a standard tool in Sturm–Liouville theory, first for singular problems (cf. [24]), but later for regular problems as well (cf. [14]). It has also been used for inverse Sturm–Liouville theory (cf. [4,11,22]), again for standard boundary conditions.

Prüfer introduced his angle in 1926 [19] as an alternative to Riccati equations for the study of Sturm–Liouville oscillation theory. It is now the standard tool for this purpose, and many variants have been proposed for diverse topics including eigenvalue asymptotics (cf. [2]) and interlacing (cf. [1]). Most applications have been to standard boundary conditions, but see [5,10] for periodic and $\lambda$ dependent boundary conditions respectively, and [3,12] for singular problems. As far as we know, Prüfer angles have not been used as inverse Sturm–Liouville data before.

The idea of using two sequences of eigenvalues, with the same boundary condition at one end and different conditions at the other end, seems to have originated in 1946 with Borg’s classic paper [8] on inverse spectral theory. Borg’s work led to much activity in
this area for standard boundary conditions, and it is known that these spectral data give
existence and uniqueness of a corresponding Sturm–Liouville problem [15,16]. Recon-
struction of this problem is a nontrivial task (cf. [20]), but here we shall show how to
reconstruct the m-function, building on some ideas used in [11] for standard boundary
conditions.

Eigenvalue dependent boundary conditions were examined even before the time of
Sturm and Liouville [18]. Rational conditions like (1.3) were investigated in [21] and by
several subsequent authors. Most of this work has been on Hilbert and Pontryagin space
formulations, leading to completeness and expansion theory (cf. [9]). There seems to be
little on inverse theory, but we cite [6,7] where the spectral data consisted of two spectra
and one spectrum and norming constants respectively. In these references \( f \) was a special
type of Nevanlinna function, corresponding to a Hilbert space formulation as above, and
one of our motivations in the present work was to admit general rational dependence of \( f \),
corresponding to an indefinite space situation which allows nonreal and nonsemisimple
eigenvalues.

We conclude this introduction with a brief summary. In Section 2 we discuss eigenvalue
existence, location, and asymptotics and also oscillation of the eigenfunctions. This ma-
terial is needed for the subsequent inverse theory, but we believe is interesting in its own
right, and is new at least in this generality. Section 3 concerns the relation between the Weyl
and Prüfer functions, and prepares the way for Section 4 on the inverse problem when the
m-function is given. In Section 5 we discuss the relation between two given spectra and
the m-function, and the main inverse uniqueness result is deduced as a corollary.

2. Asymptotics

If \( \deg(h) \leq \deg(g) = M \), let \( h(\lambda) = A_M \lambda^M + \cdots + A_0 \) where \( A_M \in \mathbb{R} \) (it may be zero);
while if \( M = \deg(h) > \deg(g) \), let \( g(\lambda) = A_{M-1} \lambda^{M-1} + \cdots + A_0 \) where \( A_{M-1} \in \mathbb{R} \) (it
may be zero).

Let \( v \) be the solution of (1.1) satisfying the terminal conditions

\[
\begin{align*}
v(1, \lambda) & = g(\lambda), \\
v'(1, \lambda) & = h(\lambda),
\end{align*}
\]

and write

\[
D(\lambda, \alpha, f, q) = v'(0, \lambda) \sin \alpha - v(0, \lambda) \cos \alpha.
\]

**Lemma 2.1.** \( D(\lambda, \alpha, f, q) \) is analytic in \( \lambda \); its zeros are precisely the eigenvalues of
(1.1)–(1.3), and the multiplicity of each zero of \( D \) is the same as the algebraic multiplicity
of the corresponding eigenvalue.

For the proof, see [17, Section 2.3].
From the asymptotics in Appendix A, we have:
Let \( \deg(h) \leq \deg(g) \). Then the eigenvalues \( \lambda_n \) are given asymptotically for \( n \to \infty \) by

\[
D(\lambda, \alpha, f, g) = \frac{\sin \alpha \left[ \sqrt{\lambda} \sin \sqrt{\lambda} + \cos \sqrt{\lambda} \left( A_M - \frac{Q}{2} - \cot \alpha \right) \right] + O\left( \frac{e^{\sqrt{\lambda}M}}{\sqrt{\lambda}} \right)}{\lambda^M},
\]

if \( \deg(h) > \deg(g) \),

\[
D(\lambda, \alpha, f, g) = \frac{\sin \alpha \left[ \cos \sqrt{\lambda} \frac{\sin \sqrt{\lambda}}{\lambda} \left( A_{M-1} + \frac{Q}{2} + \cot \alpha \right) \right] + O\left( \frac{e^{\sqrt{\lambda}M}}{\lambda^{3/2}} \right)}{\lambda^M},
\]

if \( \deg(h) = \deg(g) \).

Theorem 2.2. Let \( f(\lambda) = h(\lambda)/g(\lambda) \) where \( h \) and \( g \) are real polynomials with no common zeros. The eigenvalues \( \lambda_n, n = 0, 1, \ldots \) of (1.1)–(1.3), repeated according to algebraic multiplicity and listed in increasing absolute value, are given asymptotically for \( n \to \infty \) by

\[
\lambda_n = \begin{cases} 
\frac{\pi}{2} + 2 \cot \alpha - 2 A_M + Q + o\left( \frac{1}{n} \right), & \alpha \neq 0, \ \deg(h) \leq \deg(g) = M, \\
\frac{\pi}{2} + 2 \cot \alpha - 2 A_M + Q + o\left( \frac{1}{n} \right), & \alpha = 0, \ \deg(h) \leq \deg(g) = M, \\
\frac{\pi}{2} + 2 \cot \alpha + 2 A_{M-1} + Q + o\left( \frac{1}{n} \right), & \alpha \neq 0, \ \deg(h) > \deg(g), \\
\frac{\pi}{2} + Q + 2 A_{M-1} + o\left( \frac{1}{n} \right), & \alpha = 0, \ \deg(h) > \deg(g).
\end{cases}
\]

For large \( n \) all eigenvalues are algebraically simple and real.

Proof. It follows from [21] that only finitely many eigenvalues are non-simple or non-real. We proceed via Rouche’s theorem [23, p. 116] and give the case \( \alpha \neq 0, \ \deg(h) \leq \deg(g) = M \) in complete detail; the other three cases are similar.
We write
\[ c = A_M - \cot \alpha - \frac{Q}{2} \]
and we abbreviate \( D(\lambda, \alpha, f, q) \) to \( D(\lambda) \). Note that
\[ D(\lambda) = \lambda M \sin \alpha \left[ \sqrt{\lambda} \sin \sqrt{\lambda} + c \cos \sqrt{\lambda} \right] + O(\lambda^{M-1/2}) \]
as \( \lambda \to +\infty \). Thus
\[ \frac{(-1)^j D(\lambda)}{c \lambda^{M+1/2} \sin \alpha} = \frac{1 - \eta}{j\pi} + O\left(\frac{1}{j^2}\right) \quad \text{if } \sqrt{\lambda} = j\pi - \frac{\eta c}{j\pi}, \quad 0 < \eta < 2, \]
as \( j \to \infty \). Since \( D(\lambda) \) is continuous and changes sign for \( \eta \) in the above interval, it has a zero at \( \lambda = \tau_j^2 \), which is easily seen to correspond to \( \eta = 1 + O(1/j) \), i.e., to \( \tau_j = j\pi - \frac{\xi_j}{j\pi} + O(1/j^2) \) for large \( j \).

Note that
\[ R(\lambda) = \lambda^{M+1} \sin \alpha \frac{\sin \sqrt{\lambda}}{\sqrt{\lambda}} \]
is entire if we define \( R(0) = 0 \). Let \( \Gamma_n = \{ \lambda = \xi^2 \mid \xi \in \gamma_n \} \) where \( \gamma_n \) is as indicated in Fig. 1 and
\[ \zeta_n = \left( n - \frac{1}{2} \right) \pi, \quad n = 1, 2, 3, \ldots. \]

Let \( Z_n(F) \) denote the numbers of zeros of an entire function \( F \) in the region enclosed by \( \Gamma_n \).

On the curve \( \Gamma_n \) for large \( n \) we have
\[ |R(\lambda)| \geq \kappa |\lambda|^{M+1/2} e^{i|3\sqrt{\lambda}|}, \]
where \( \kappa \) is a positive constant not depending on \( n \) or \( \lambda \), and as
\[ D(\lambda) - R(\lambda) = O\left(\lambda^M e^{i|3\sqrt{\lambda}|}\right). \]

![Fig. 1. \( \gamma_n \) in the \( \xi \)-plane.](image)
we have that on \( \Gamma_n \), for \( n \) large,
\[
|D(\lambda) - R(\lambda)| < |R(\lambda)|.
\]
From Rouché’s theorem we may thus conclude that
\[
Z_n(D) = Z_n(R) = n - 1 + M + 1.
\]
Consequently, in the annulus between \( \Gamma_n \) and \( \Gamma_{n+1} \), \( D \) has precisely one zero, namely \( \tau_n^2 \).
We have thus also proved that \( \tau_n^2 \) is the \((n + M + 1)\)th zero of \( D \), i.e.,
\[
\lambda_{n+M} = \tau_n^2.
\]

**Theorem 2.3.** For large \( n \) eigenfunctions corresponding to the eigenvalue \( \lambda_n \) have oscillation count \( n - M \) in the case of \( \deg(h) \leq \deg(g) = M \) and, for the case of \( M = \deg(h) > \deg(g) \), the oscillation count is \( n - M + 1 \) if \( \lim_{\lambda \to -\infty} f(\lambda) = +\infty \) and \( n - M \) if \( \lim_{\lambda \to -\infty} f(\lambda) = -\infty \).

**Proof.** We shall depend on well-known asymptotic and oscillation results for standard Sturm–Liouville problems (i.e., with constant boundary conditions), cf. [5]. We begin with
\[
\Gamma_n
\]
we have that on
\[
\text{Theorem 2.3. For large } n \text{ eigenfunctions corresponding to the eigenvalue } \lambda_n \text{ have oscillation count } n - M \text{ in the case of } \deg(h) \leq \deg(g) = M \text{ and, for the case of } M = \deg(h) > \deg(g), \text{ the oscillation count is } n - M + 1 \text{ if } \lim_{\lambda \to -\infty} f(\lambda) = +\infty \text{ and } n - M \text{ if } \lim_{\lambda \to -\infty} f(\lambda) = -\infty.

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**Proof.** We shall depend on well-known asymptotic and oscillation results for standard Sturm–Liouville problems (i.e., with constant boundary conditions), cf. [5]. We begin with
the simplest case, \( \deg(h) \leq \deg(g) = M \).

For \( \alpha \neq 0 \), let \( \lambda_n^D \) denote the eigenvalues of \((\alpha, \infty, q)\). Then
\[
\lambda_n^D = \left(n + \frac{1}{2}\right)^2 \pi^2 + O(1)
\]
and all solutions of (1.1)–(1.2) for \( \lambda \in [\lambda_n^D, \lambda_n^{D+1}] \) have \( n \) zeros in \((0, 1)\) for \( n = 1, 2, \ldots \), and none for \( \lambda < \lambda_n^D \). As \( \lambda_{n+M} \in [\lambda_n^{D+1}, \lambda_n^D] \), the eigenfunction of (1.1)–(1.3) with eigenvalue \( \lambda_n \) has oscillation count \( n - M \) in \((0, 1)\).

For \( \alpha = 0 \), let \( \lambda_n^{DD} \) denote the eigenvalues of \((0, \infty, q)\), then
\[
\lambda_n^{DD} = (n + 1)^2 \pi^2 + O(1)
\]
and all solutions of (1.1)–(1.2) for \( \lambda \in [\lambda_n^{DD}, \lambda_n^{DD+1}] \) have \( n \) zeros in \((0, 1)\) for \( n = 1, 2, \ldots \), and none for \( \lambda < \lambda_n^{DD} \). As \( \lambda_{n+M} \in [\lambda_n^{DD+1}, \lambda_n^{DD}] \), the eigenfunction of (1.1)–(1.3) with eigenvalue \( \lambda_n \) has oscillation count \( n - M \) in \((0, 1)\).

We now proceed to \( M = \deg(h) > \deg(g) \), where more care is needed.

For \( \alpha \neq 0 \), let \( \lambda_n^M \) denote the eigenvalues of \((\alpha, 0, q)\) and recall that
\[
\lambda_n^0 < \lambda_n^D < \lambda_n^{N} < \lambda_n^1 < \lambda_n^D < \cdots, \quad \lambda_n^N = n^2 \pi^2 + O(1)
\]
If \( y \) is a solution of (1.1)–(1.2) for \( \lambda \in (\lambda_n^N, \lambda_n^D) \) we have \( |y'/y|(1) < 0 \), while for \( \lambda \in (\lambda_n^D, \lambda_n^N+1) \) we have \( |y'/y|(1) > 0 \).

From the previous theorem we observe that
\[
\lambda_{n+M} = \lambda_n^D + O(1).
\]
Thus either \( \lambda_{n+M} \in [\lambda_n^D+1, \lambda_n^D] \) or \( \lambda_{n+M} \in [\lambda_n^D, \lambda_n^{D+1}] \). But as \( \lambda_{n+M} \in (\lambda_n^N, \lambda_n^{N+1}) \) it is enough to determine the sign of \( f(\lambda_{n+M}) = |y'/y|(1) \). If \( \lim_{\lambda \to +\infty} f(\lambda) = +\infty \), then for large \( n \), \( \lambda_{n+M} \in [\lambda_n^D, \lambda_n^{N}] \), giving an oscillation count of \( n + 1 \), while if \( \lim_{\lambda \to +\infty} f(\lambda) = -\infty \), then for large \( n \), \( \lambda_{n+M} \in (\lambda_n^N, \lambda_n^D) \), giving an oscillation count of \( n \).
The case $\alpha = 0$ is similar, except that one replaces the role of $\lambda_n^N$ by that of $\lambda_n^{DN}$, the eigenvalues of $(0, 0, q)$. □

3. Prüfer angle and $m$-function

We begin by recalling the definitions of these two classical constructions. The Prüfer angle $\phi = \phi(x, \lambda)$ satisfies the first order differential equation

$$\phi' = \cos^2 \phi + (\lambda - q) \sin^2 \phi$$

on $(0, 1)$. The definition is completed by specifying the value of $\phi$ at some point. This is traditionally the initial point, but for our purposes it is more convenient to use the final point, and we set

$$\cot \phi(1, \lambda) = f(\lambda)$$

with $\phi(1, \lambda) \in (0, \pi]$. The $n$th eigenvalue $\lambda_n$ then satisfies

$$\phi(0, \lambda_n) = \alpha - n\pi.$$

To define the Weyl $m$-function, we first let

$$\psi(x, \lambda) = v(x, \lambda) \frac{v(x, \lambda)}{v'(0, \lambda) \sin \alpha - v(0, \lambda) \cos \alpha} \tag{3.1}$$

where $v$ is defined via (2.1), (2.2). Then for each $x$, $\psi(x, \lambda)$ is analytic in $\lambda$ except at the eigenvalues. We set

$$R(\alpha) = \begin{bmatrix} -\cos \alpha & \sin \alpha \\ \sin \alpha & \cos \alpha \end{bmatrix}$$

and

$$W(x, \lambda) = \begin{bmatrix} w_1(x, \lambda) & w_2(x, \lambda) \\ w_1'(x, \lambda) & w_2'(x, \lambda) \end{bmatrix},$$

where $w_1$ and $w_2$ are solutions of (1.1) so that the initial conditions

$$W(0, \lambda) = R(\alpha)$$

are satisfied. It should be noted that $W(x, \lambda)$ has entries which are entire functions of $\lambda$ and its determinant is the Wronskian of $w_1$ and $w_2$ and thus equals $-1$ for all $x$ and $\lambda$.

The Weyl $m$-function of (1.1)–(1.3) is defined by

$$\psi = w_1 + mw_2. \tag{3.2}$$

If we also define

$$\Psi(x, \lambda) = \begin{bmatrix} \psi(x, \lambda) & w_2(x, \lambda) \\ \psi'(x, \lambda) & w_2'(x, \lambda) \end{bmatrix}, \tag{3.3}$$

then it easily follows that

$$\Psi(x, \lambda) = W(x, \lambda) \begin{bmatrix} 1 & 0 \\ m(\lambda) & 1 \end{bmatrix} \text{ for all } x. \tag{3.4}$$

Now we are ready to relate the Weyl $m$-function to the Prüfer angle $\phi$. 
Theorem 3.1. The \( m \)-function satisfies \( m(\lambda) = \cot(\alpha - \phi(0, \lambda)) \). Its poles are precisely the eigenvalues of (1.1)–(1.3) and the order of each pole coincides with the algebraic multiplicity of the corresponding eigenvalue.

Proof. Note by (3.2) that \( \psi(0, \lambda) = w_1(0, \lambda) + mw_2(0, \lambda) \) so by (3.1),

\[
\frac{v(0, \lambda)}{v'(0, \lambda) \sin \alpha - v(0, \lambda) \cos \alpha} = -\cos \alpha + m(\lambda) \sin \alpha,
\]

whence

\[
m(\lambda) = \frac{v(0, \lambda) \sin \alpha + v'(0, \lambda) \cos \alpha}{v'(0, \lambda) \sin \alpha - v(0, \lambda) \cos \alpha}.
\]

(3.5)

Since

\[
\frac{v'}{v}(x, \lambda) = \cot \phi(x, \lambda),
\]

(3.6)

we obtain

\[
m(\lambda) = \frac{\sin \phi(0, \lambda) \sin \alpha + \cos \phi(0, \lambda) \cos \alpha}{\cos \phi(0, \lambda) \sin \alpha - \sin \phi(0, \lambda) \cos \alpha} = \cot(\alpha - \phi(0, \lambda)).
\]

Now \( \lambda \) is a pole of \( m \) precisely when \( \alpha \) and \( \phi(0, \lambda) \) differ by a multiple of \( \pi \), or equivalently, when \( v(0, \lambda) \) is an eigenfunction of (1.1)–(1.3) with eigenvalue \( \lambda \).

Finally, the order of a pole of \( m \) equals the order of a zero of the denominator of (3.5) which is \( D \), and by Lemma 2.1 this is the algebraic multiplicity of the corresponding eigenvalue. \( \Box \)

Corollary 3.2. \( \phi(0, \lambda) \) and \( \alpha \) together determine \( m(\lambda) \).

4. \( m \)-function inverse problem

We start with some asymptotics for \( m(\lambda) \) as \( \lambda \to -\infty \).

Lemma 4.1. For \( \lambda \to -\infty \) we have

\[
m(\lambda) = \begin{cases} 
\cot \alpha + O\left(\frac{1}{\sqrt{|\lambda|}}\right), & \alpha \neq 0, \\
\sqrt{|\lambda|} + O(1), & \alpha = 0.
\end{cases}
\]

Proof. From (3.5), (3.6), and the asymptotics for \( v(0) \) and \( v'(0) \) in Appendix A, we conclude for \( \alpha \neq 0, \)

\[
m(\lambda) = \frac{\cos \alpha + O(\tan \phi(0, \lambda))}{\sin \alpha + O(\tan \phi(0, \lambda))} = \cot \alpha + O\left(\frac{1}{\sqrt{|\lambda|}}\right),
\]

and for \( \alpha = 0, \)

\[
m(\lambda) = -\cot \phi(0, \lambda) = \sqrt{|\lambda|} + O(1). \quad \Box
\]
The main result of this section shows that the \( m \)-function uniquely determines \( \alpha, f \), and \( q \).

**Theorem 4.2.** If the problems \((\alpha, f, q)\) and \((\tilde{\alpha}, \tilde{f}, \tilde{q})\) have the same \( m \)-function, \( m \), then \( \alpha = \tilde{\alpha}, f = \tilde{f}, \) and \( q = \tilde{q} \).

**Proof.** From Lemma 4.1 it follows that \( \alpha = \tilde{\alpha} \). From Theorems 2.2 and 3.1 it follows that \( M = \tilde{M} \) and \( \deg(g) \geq \deg(h) \) if and only if \( \deg(\tilde{g}) \geq \deg(\tilde{h}) \).

Recall the definition of \( \Psi \) (3.3) and define \( \Psi \) similarly for \((\tilde{\alpha}, \tilde{f}, \tilde{g})\).

Let
\[
P(x, \lambda) = \Psi(x, \lambda)\tilde{\Psi}^{-1}(x, \lambda).
\]
As \( m(\lambda) = \tilde{m}(\lambda) \), from (3.4) we may thus conclude that
\[
P(x, \lambda) = W(x, \lambda)\tilde{W}^{-1}(x, \lambda).
\]
Since \( \det W = \det \tilde{W} = -1 \), the entries of \( P(x, \lambda) \) are entire functions of \( \lambda \) for fixed \( x \), and \( \det P(x, \lambda) = 1 \) for all \( x \) and \( \lambda \).

As \( m(\lambda) = \tilde{m}(\lambda) \) and \( \alpha = \tilde{\alpha} \), from the definition of \( m(\lambda) \) we have
\[
\cot \phi(0, \lambda) \cos \alpha + \sin \alpha = \cot \phi(0, \lambda) \cos \alpha + \sin \alpha
\]
giving
\[
\cot \phi(0, \lambda) = \cot \tilde{\phi}(0, \lambda) \quad \text{for all } \lambda.
\]
As \( \det \Psi = -1 = \det \tilde{\Psi} \), by (3.4) we have
\[
P_{11} = \frac{\tilde{v}'(x)w_2(x)}{\tilde{v}(x)\sin \alpha - \tilde{v}'(x)\cos \alpha} - \frac{v(x)w_2'(x)}{v'(x)\sin \alpha - v(0)\cos \alpha},
\]
\[
P_{12} = \frac{v(x)w_2'(x)}{v'(x)\sin \alpha - v(0)\cos \alpha} - \frac{\tilde{v}(x)w_2(x)}{\tilde{v}'(x)\sin \alpha - \tilde{v}(0)\cos \alpha}.
\]

Let \( \delta \) be fixed in \((0, \pi/4)\), and define
\[
\Omega_\delta = \left\{ \lambda \in \mathbb{C}: \left| \sqrt{\lambda} - \frac{n\pi}{2} \right| > \delta \quad \text{for all } n \in \mathbb{N} \right\}.
\]

By Appendix A and (3.6), there exist constants \( K_1, K_2 > 0 \) such that
\[
\left| \cot \phi(0, \lambda) - \cot \alpha \right| > K_1\sqrt{|\lambda|}
\]
for all \( \lambda \in \Omega_\delta \) with \( |\lambda| > K_2 \) provided \( \alpha \neq 0 \).

We consider \( P_{12} \) for large \( \lambda \) in \( \Omega_\delta \). Now (3.6), (4.1), (4.2), and the asymptotic estimates in Appendix A give
\[
\frac{\tilde{v}(x)w_2(x)}{\tilde{v}'(x)\sin \alpha - \tilde{v}(0)\cos \alpha} = \frac{\tilde{v}(x)w_2(x)}{	ilde{v}(0)\sin \alpha - \tilde{v}(0)\cos \alpha} = O(\lambda^{-1/2})
\]
provided \( \alpha \neq 0 \). Similar arguments hold if \( \alpha = 0 \) and if the second term of \( P_{12} \) is considered. Thus \( P_{12} = O(r^{-1/2}) \) on any large circle \( \Gamma(r) \) with centre 0 and radius \( r \) lying...
inside \( \Omega_{\delta} \). On the other hand, \( P_{12} \) is analytic on the disc enclosed by \( \Gamma(r) \), so by the Maximum Modulus Principle \( P_{12} = O(r^{1/2}) \) on this disc. Hence, for each large \( r \) with \( \Gamma(r) \subset \Omega_{\delta} \) we have \( P_{12} = O(r^{-1/2}) \) and so \( P_{12} \equiv 0 \).

We now consider \( P_{11} \). For \( \lambda \) large and in \( \Omega_{\delta} \),

\[
P_{11} = \frac{\tilde{v}'(x)w_2(x)}{\tilde{v}'(0) \sin \alpha - \tilde{v}(0) \cos \alpha} - \frac{v(x)\tilde{w}'_2(x)}{v'(0) \sin \alpha - v(0) \cos \alpha} = 1 + \frac{\tilde{v}(x)(w_2(x) - \tilde{w}_2(x))}{\cot \phi(0) \sin \alpha - \cos \alpha} \]

by (4.1)

\[
= 1 + O\left(\frac{1}{\sqrt{\lambda}}\right).
\]

Thus reasoning for \( P_{11} - 1 \) as we did for \( P_{12} \), we obtain

\[ P_{11} - 1 \equiv 0. \]

But \( \tilde{P} \tilde{\Psi} = \Psi \) so \( w_2 = \tilde{w}_2 \) and therefore \( q = \tilde{q} \). Since \( q = \tilde{q} \), (4.1) gives

\[
f(\lambda) = \cot \phi(1, \lambda) = \cot \tilde{\phi}(1, \lambda) = \tilde{f}(\lambda). \quad \square
\]

5. Two spectrum inverse problem

The first result of this section gives an explicit expression for \( D(\lambda, \alpha, f, q) \) as an infinite product.

**Lemma 5.1.** Under the conditions on \( f(\lambda) \) stated earlier we have

\[
D(\lambda, \alpha, f, q) = \begin{cases} 
\sin \alpha \prod_{n=0}^{M} (\lambda - \lambda_0) \prod_{n=1}^{\infty} \frac{\lambda_n + M - \lambda}{n^2 \pi^2}, & \alpha \neq 0, \\
\sin \alpha \prod_{n=0}^{M-1} (\lambda - \lambda_0) \prod_{n=0}^{\infty} \frac{\lambda_n + M - \lambda}{(n + 1/2)^2 \pi^2}, & \alpha \neq 0, \\
- \prod_{n=0}^{M-1} (\lambda - \lambda_0) \prod_{n=0}^{\infty} \frac{\lambda_n + M - \lambda}{(n + 1/2)^2 \pi^2}, & \alpha = 0, \\
\prod_{n=0}^{M} (\lambda - \lambda_0) \prod_{n=1}^{\infty} \frac{\lambda_n + M - 1 - \lambda}{n^2 \pi^2}, & \alpha = 0, \quad \deg(g) < \deg(h) = M, \\
\prod_{n=0}^{M} (\lambda - \lambda_0) \prod_{n=1}^{\infty} \frac{\lambda_n + M - 1 - \lambda}{n^2 \pi^2}, & \deg(g) < \deg(h) = M.
\end{cases}
\]

**Proof.** From (2.4) it follows that \( D(\lambda, \alpha, f, q) \) is an entire function of order 1/2, with zeros at precisely the eigenvalues \( \lambda_0 \). Let each eigenvalue be repeated according to algebraic multiplicity. Then from the Hadamard product theorem [23, p. 250] we have

\[
D(\lambda, \alpha, f, q) = C \prod_{n=0}^{\infty} \left(1 - \frac{\lambda}{\lambda_n}\right).
\]

(If \( \lambda = 0 \) is an eigenvalue of multiplicity \( k \) then we consider \( \lambda^{-k}D(\lambda, \alpha, f, q) \) instead).
Case 1. \( \alpha \neq 0, M = \deg(g) \geq \deg(h) \).

From (2.4) we have

\[
\lim_{\lambda \to -\infty} \lambda^{M+1/2} \sin \alpha \sin \sqrt{\lambda} D(\lambda, \alpha, f, q) = 1
\]

and from the infinite product representation

\[
\sin \sqrt{\lambda} = \prod_{n=1}^{\infty} \left(1 - \frac{\lambda}{n^2 \pi^2}\right)
\]

[23, p. 114] we obtain

\[
C = \lim_{\lambda \to -\infty} \frac{\lambda^{M+1} \sin \alpha \prod_{n=0}^{\infty} \left(1 - \frac{\lambda}{n^2 \pi^2}\right)}{\prod_{n=0}^{\infty} (1 - \frac{\lambda}{n^2 \pi^2})}
= \lim_{\lambda \to -\infty} \sin \alpha \prod_{n=0}^{M} \frac{\lambda \lambda_n}{\lambda_n - \lambda_n^2} \prod_{n=1}^{\infty} \frac{n^2 \pi^2 - \lambda}{\lambda_n + M - \lambda},
\]

where use has been made of the asymptotic distribution of the eigenvalues to ensure that all infinite products involved converge. From the asymptotic form of the eigenvalues we see that the limit can be taken through the product to give

\[
\lim_{\lambda \to -\infty} \prod_{n=1}^{\infty} \frac{n^2 \pi^2 - \lambda}{\lambda_n + M - \lambda} = 1.
\]

Combining these results, we have

\[
C = \sin \alpha \prod_{n=0}^{M} (-\lambda_n) \prod_{n=1}^{\infty} \frac{\lambda_n + M}{n^2 \pi^2}
\]

and thus

\[
D(\lambda, \alpha, f, q) = \sin \alpha \prod_{n=0}^{M} (\lambda - \lambda_n) \prod_{n=1}^{\infty} \frac{\lambda_n + M - \lambda}{n^2 \pi^2},
\]

where again the asymptotics for \( \lambda_n + M \) have been used to ensure the infinite product, which completes the proof for this case.

Case 2. \( \alpha \neq 0, \deg(g) < \deg(h) = M \).

From (2.4) we have

\[
\lim_{\lambda \to -\infty} \lambda^M \sin \alpha \cos \sqrt{\lambda} D(\lambda, \alpha, f, q) = 1
\]

and from the infinite product representation

\[
\cos \sqrt{\lambda} = \prod_{n=0}^{\infty} \left(1 - \frac{\lambda}{(n + 1/2) \pi^2}\right)
\]
[23, p. 114], we obtain

\[ C = \lim_{\lambda \to -\infty} \lambda^M \sin \alpha \prod_{n=0}^{\infty} \left(1 - \frac{\lambda}{(n+1/2)^2 \pi^2}\right) \]

\[ = \lim_{\lambda \to -\infty} \sin \alpha \prod_{n=0}^{M-1} \frac{\lambda \lambda_n}{\lambda_n - \lambda} \prod_{n=0}^{\infty} \frac{\lambda_n + M - \lambda}{\lambda_n + M - \lambda}, \]

where the asymptotic distribution of the eigenvalues ensures that all infinite products involved converge. Again from the asymptotic form of the eigenvalues we see that the limit can be taken through the product to give

\[ \lim_{\lambda \to -\infty} \prod_{n=0}^{\infty} \frac{\lambda_n + M - \lambda}{\lambda_n + M - \lambda} = 1. \]

Combination of these results yields

\[ C = \sin \alpha \prod_{n=0}^{M-1} (-\lambda_n) \prod_{n=0}^{\infty} \frac{\lambda_n + M - \lambda}{(n+1/2)^2 \pi^2} \]

and thus

\[ D(\lambda, \alpha, f, q) = \sin \alpha \prod_{n=0}^{M-1} (\lambda - \lambda_n) \prod_{n=0}^{\infty} \frac{\lambda_n + M - \lambda}{(n+1/2)^2 \pi^2}, \]

which completes the proof for this case.

**Case 3.** \( \alpha = 0, M = \deg(g) \geq \deg(h). \)

Proof as for Case 2 but with \( \sin \alpha \) replaced by \(-1\).

**Case 4.** \( \alpha = 0, \deg(g) < \deg(h) = M. \)

From (2.4) we have

\[ \lim_{\lambda \to -\infty} \lambda^{-M/2} \sin \sqrt{\lambda} D(\lambda, \alpha, f, q) = 1 \]

and thus from the infinite product representation of sine we have

\[ C = \lim_{\lambda \to -\infty} \prod_{n=0}^{M-1} \frac{\lambda \lambda_n}{\lambda_n - \lambda} \prod_{n=1}^{\infty} \frac{\lambda_n + M - 1 - \lambda}{n^2 \pi^2} \prod_{n=1}^{\infty} \frac{n^2 \pi^2 - \lambda}{\lambda_n + M - 1 - \lambda}, \]

where the asymptotic distribution of the eigenvalues ensures that all infinite products involved converge. Again from the asymptotic form of the eigenvalues the limit can be taken through the product to give

\[ \lim_{\lambda \to -\infty} \prod_{n=1}^{\infty} \frac{n^2 \pi^2 - \lambda}{\lambda_n + M - 1 - \lambda} = 1. \]
We combine these results to yield
\[ C = \prod_{n=0}^{M-1} (-\lambda_n) \prod_{n=1}^{\infty} \frac{\lambda_{n+M-1}}{n^2\pi^2} \]
and thus
\[ D(\lambda, \alpha, f, q) = \prod_{n=0}^{M-1} (\lambda - \lambda_n) \prod_{n=1}^{\infty} \frac{\lambda_{n+M-1} - \lambda}{n^2\pi^2} ,\]
which completes the proof. □

We are now ready to relate the data from two spectra to the construction of \( m \).

**Theorem 5.2.** Let \( \lambda_n \) and \( \mu_n \), \( n = 0, 1, 2, \ldots \) be spectra of \((\alpha, f, q)\) and \((\beta, f, q)\) where \( \alpha \neq \beta \). Then \( m \) is uniquely determined by \( \lambda_n, \mu_n, n = 0, 1, 2, \ldots \).

**Proof.** We start with some notation. The \( \lambda_n \) obey one of four asymptotics (say the \( i \)th in the order presented in Theorem 2.2), and similarly \( \mu_n \) obey the \( j \)th. We shall call this “case \((i, j)\).” Note that the relative magnitudes of the degrees of \( g \) and \( h \) force \( 1 \leq i, j \leq 2 \) or \( 3 \leq i, j \leq 4 \). We write \( (n - l/2)^2/\pi^2 \) for the leading term in \( \lambda_n \), and
\[ \hat{\lambda}_n = \lambda_n - \left(n - \frac{l}{2}\right)^2\pi^2. \]
Evidently \( l = 2M, 2M - 2 \) if \( i = 1, 4 \), respectively, and \( l = 2M - 1 \) otherwise. Similarly we write
\[ \hat{\mu}_n = \mu_n - \left(n - \frac{k}{2}\right)^2\pi^2 \]
for the remainder after removing the leading term in \( \mu_n \). Finally, we write \( d \) for the limit of \( \hat{\lambda}_n - \hat{\mu}_n \) as \( n \to \infty \).

It is clear that \( d, l, k \) can be assumed known, and we claim that they suffice to determine \( \alpha, \beta, M \) and the relative magnitudes of the degrees of \( g \) and \( h \).

(a) Suppose first that \( l, k \) have opposite parity and \( l < k \). If \( l \) is odd and \( k \) is even then we must be in case \((2, 1)\) so \( \alpha = 0, d = -2\cot\beta, M = k/2 \), and \( \deg(h) \leq \deg(g) \). If \( l \) is even and \( k \) is odd, then we are in case \((4, 3)\) and again \( \alpha = 0, d = -2\cot\beta \), but now \( M = l/2 + 1 \) and \( \deg(h) > \deg(g) \).

(b) As for (a) but with \( l > k \). Then we are in case \((1, 2)\) or \((3, 4)\), \( d = 2\cot\alpha, \beta = 0 \) and \( M \) and the relative magnitudes of the degrees of \( g \) and \( h \) can be determined. Note that \( l = k \) is impossible.

(c) Now suppose that \( l, k \) have the same parity. If both are even then we are in case \((1, 1)\), since \( \alpha \neq \beta \) precludes case \((4, 4)\). Similarly, if both are odd then we are in case \((3, 3)\) and in both situations \( M \) and the relative magnitudes of the degrees of \( g \) and \( h \) can be determined, and \( d = 2(\cot\alpha - \cot\beta) \). Now (2.3) shows that
\[ \frac{D(\lambda, \alpha, f, q)}{\sin\alpha} = v'(0, \lambda) - v(0, \lambda) \cot\alpha \]
and the left side is known from Lemma 5.1. A similar equation holds with \( \beta \) replacing \( \alpha \), and so

\[
v(0, \lambda) = \frac{1}{d} \left( \frac{D(\lambda, \alpha, f, q)}{\sin \beta} - \frac{D(\lambda, \alpha, f, q)}{\sin \alpha} \right)
\]
can be assumed known. In particular, its zeros, which form a third eigenvalue sequence \( v_n \) say, are known. But the \( v_n \) correspond to initial angle \( \alpha' = 0 \) and so we can repeat the analysis of (a) to obtain \( \beta, M \) and the relative magnitudes of the degrees of \( g \) and \( h \). Finally, we use \( d \) to obtain \( \alpha \).

With our claim established, it now follows from Lemma 5.1 that we can in all cases obtain explicit expressions for \( D(\lambda, \alpha, f, q) \) and \( D(\lambda, \beta, f, q) \). Since

\[
\begin{bmatrix}
v(0, \lambda) \\
v'(0, \lambda)
\end{bmatrix} = \frac{1}{\sin(\alpha - \beta)} \begin{bmatrix}
\sin \beta & -\sin \alpha \\
\cos \beta & -\cos \alpha
\end{bmatrix} \begin{bmatrix}
D(\lambda, \alpha, f, q) \\
D(\lambda, \beta, f, q)
\end{bmatrix},
\]
the Weyl \( m \)-function for \((\alpha, f, q)\) is

\[
m(\lambda) = \cot(\alpha - \beta) - \frac{D(\lambda, \beta, f, q)}{D(\lambda, \alpha, f, q)} \cosec(\alpha - \beta).
\]

In conclusion we can make precise the inverse spectral claims of the introduction.

**Corollary 5.3.** The triple \((\alpha, f, q)\) is uniquely determined by any of the following spectral data:

(i) \( m(\lambda) \);
(ii) \( \alpha \) and \( \phi(0, \lambda) \);
(iii) two eigenvalue sequences as in Theorem 5.2.

**Proof.** (i) follows from Theorem 4.2. Then (ii) follows from Corollary 3.2 and (iii) from Theorem 5.2. □

**Appendix A**

We collect here some asymptotic estimates which can be derived by bootstrapping one step further than was done in [13].

Proceeding as in [13], we can easily verify that the solutions \( s_1, s_2 \) of (1.1) with terminal conditions

\[
s_1(1) = 1 = s'_2(1), \quad s_2(1) = 0 = s'_1(1)
\]
are given asymptotically for large \( \lambda \) by

\[
s_1(0) = \cos \sqrt{\lambda} + \frac{Q}{2} \sin \frac{\sqrt{\lambda}}{\sqrt{\lambda}} + O\left( \frac{e^{1/3 \sqrt{\lambda}}}{\sqrt{\lambda}} \right),
\]

\[
s'_1(0) = \sqrt{\lambda} \sin \sqrt{\lambda} - \frac{Q}{2} \cos \sqrt{\lambda} + O\left( \frac{e^{1/3 \sqrt{\lambda}}}{\sqrt{\lambda}} \right).
\]
where \( Q = \int_0^1 q \, dt \). Hence, as \( v = g s_1 + h s_2 \), \( v \) and \( v' \) have the following asymptotic forms at 0.

**Case: \( M = \deg(g) \geq \deg(h) \).**

\[
\begin{align*}
v(0, \lambda) &= \lambda^M \cos \sqrt{\lambda} + \lambda^{M-1/2} \left[ \frac{Q}{2} - A_M \right] \sin \sqrt{\lambda} + O\left(\lambda^{M-1} e^{\frac{1}{\sqrt{\lambda}}}\right), \\
v'(0, \lambda) &= \lambda^{M+1/2} \sin \sqrt{\lambda} + \lambda^M \left[ A_M - \frac{Q}{2} \right] \cos \sqrt{\lambda} + O\left(\lambda^{M-1/2} e^{\frac{1}{\sqrt{\lambda}}}\right).
\end{align*}
\]

Asymptotics for the functions \( v \) and \( v' \) at general points are given by

\[
\begin{align*}
v(x, \lambda) &= \begin{cases} 
\lambda^M \cos \sqrt{\lambda}(1 - x) + O\left(\lambda^{M-1/2} e^{[\frac{3}{\lambda}]^{[1-x]}}\right), & M = \deg(g) \geq \deg(h) \\
-\lambda^{M-1/2} \sin \sqrt{\lambda}(1 - x) + O\left(\lambda^{M-1} e^{[\frac{3}{\lambda}]^{[1-x]}}\right), & M = \deg(g) > \deg(h) \end{cases}, \\
v'(x, \lambda) &= \begin{cases} 
\lambda^{M+1/2} \sin \sqrt{\lambda}(1 - x) + O\left(\lambda^M e^{[\frac{3}{\lambda}]^{[1-x]}}\right), & M = \deg(g) \geq \deg(h) \\
\lambda^M \cos \sqrt{\lambda}(1 - x) + O\left(\lambda^{M-1/2} e^{[\frac{3}{\lambda}]^{[1-x]}}\right), & M = \deg(g) > \deg(h) \end{cases}.
\end{align*}
\]

The solutions \( w_2(x, \lambda) \) have the following asymptotic forms:

\[
w_2(x, \lambda) = \begin{cases}
\sin \alpha \cos \sqrt{\lambda}x + \frac{\sin \alpha}{\sqrt{\lambda}} \int_0^x q(t) \cos \sqrt{\lambda}t \sin \sqrt{\lambda}(x - t) \, dt \\
+ \frac{\cos \alpha \sin \sqrt{\lambda}x}{\sqrt{\lambda}} + O\left(\frac{e^{[\frac{3}{\lambda}]^{[1-x]}}}{\lambda}\right), & \alpha \neq 0,
\end{cases}
\]

\[
\begin{align*}
\sin \sqrt{\lambda}x + \frac{1}{\sqrt{\lambda}} \int_0^x q(t) \sin \sqrt{\lambda}t \sin \sqrt{\lambda}(x - t) \, dt \\
+ O\left(\frac{e^{[\frac{3}{\lambda}]^{[1-x]}}}{\lambda^{3/2}}\right), & \alpha = 0.
\end{align*}
\]

\[
\begin{align*}
v_2(0) &= \frac{-\sin \sqrt{\lambda} + Q}{2 \lambda} \cos \sqrt{\lambda} + O\left(\frac{e^{[\frac{3}{\lambda}]^{[1-x]}}}{\lambda^{3/2}}\right), \\
v_2'(0) &= \frac{Q}{2 \lambda} \sin \sqrt{\lambda} + O\left(\frac{e^{[\frac{3}{\lambda}]^{[1-x]}}}{\lambda^{3/2}}\right).
\end{align*}
\]
\[
\begin{align*}
\frac{w'_2(x, \lambda)}{w_2(x, \lambda)} &=
\begin{cases}
-\sqrt{\lambda} \sin \alpha \sin \sqrt{\lambda} x + O\left(e^{\frac{|\alpha \sqrt{\lambda} x|}{|\lambda|}}\right), & \alpha \neq 0,
\cos \frac{\sqrt{\lambda} x}{\lambda} + O\left(e^{\frac{\left|\alpha \sqrt{\lambda} x\right|}{|\alpha|}}\right), & \alpha = 0.
\end{cases}
\end{align*}
\]

References