NORTH-HOLLAND

On the Exponents of Primitive, Ministrong Digraphs With Shortest Elementary Circuit Length s

Xiao-jun Wu and Jia-yu Shao

Department of Applied Mathematics
Tongji University
Shanghai 200092, China

Zhi-ming Jiang

Department of Mathematics
Shanghai College of Petrochemical Technology
Shanghai 200540, China

and

Xi-zhao Zhou

Department of Road and Traffic Engineering
Tongji University
Shanghai 200092, China

Submitted by Richard A. Brualdi

ABSTRACT

Let $\mathrm{MD}_s(n) = \{D \mid D$ is a primitive ministrong digraph with n vertices, and the shortest cycle length of D is $s\}$, and $b_s(n) = \max\{\gamma(D) \mid D \in \mathrm{MD}_s(n)\}$, where $\gamma(D)$ is the primitive exponent of D. Our main results are: (1) we give explicit expressions for $b_s(n)$; (2) for $s \neq 2, 6$, we give a necessary and sufficient condition for a digraph $D \in \mathrm{MD}_s(n)$ with $\gamma(D) = b_s(n)$.

1. INTRODUCTION

The terminology and notation used in this paper will basically follow those in [3, 4, 6]. The definitions of strong digraphs, ministrong (minimally strong) digraphs, primitive digraphs, and their exponents $\gamma(D)$ can be found in [4] and [6]. Let $\mathrm{MD}(n) = \{D \mid D$ is a primitive ministrong digraph with n vertices$\}$, $\mathrm{MD}_s(n) = \{D \in \mathrm{MD}(n) \mid$ the shortest cycle length of D

LINEAR ALGEBRA AND ITS APPLICATIONS 222:41–61 (1995)

0024-3795/95/$9.50
SSDI 0024-3795(93)00274-4

is $s\}$, and $b_s(n) = \max\{\gamma(D) \mid D \in \mathrm{MD}_s(n)\}$, where $\gamma(D)$ is the primitive exponent of D. In the following, we denote by $E(D)$ the set of arcs of D, and call a cycle of length t a *t-cycle*. Also, we use $d_t(x)$ to denote the shortest distance from a vertex x to the set of vertices of all t-cycles. When $x, y \in V(D)$, let $d(x, y)$ denote the distance from x to y as usual. If W is a walk of D, then we denote by $|W|$ the length of W.

Let

$$W_0(s,n) = n + s(n-3) - \tfrac{1}{2}s^2 \tag{1.1}$$

and

$$W_1(s,n) = n + s(r_0 - 2), \tag{1.2}$$

where

$$r_0 = \max\{m \in Z \mid 2 \le m \le n-1,\ \gcd(m,s) = 1\}. \tag{1.3}$$

Let $L(D)$ be the set of distinct lengths of cycles in digraph D, and let $\psi_s(n) = \{D \in \mathrm{MD}_s(n)$ and D satisfies the following condition $(**)\}$:

$$L(D) = \{s, r_0, r_1, \dots, r_\lambda\}; \qquad r_1, r_2, \dots, r_\lambda \text{ are all multiples of } s. \qquad (**)$$

In [6], R. A. Brualdi, and J. A. Ross showed that

$$\gamma(D) \le n^2 - 4n + 6 \qquad (D \in \mathrm{MD}(n)).$$

They also characterized the case of equality. In [4], J. A. Ross further proved that $\gamma(D) \le n + s(n-3)$ for any $D \in \mathrm{MD}_s(n)$, with equality if and only if $\gcd(n-1, s) = 1$ and D is isomorphic to the digraph in Figure 1. Therefore, in the case $\gcd(n-1, s) = 1$, we have $b_s(n) = n + s(r_0 - 2) = n + s(n-3) = W_1(s,n)$, and $\gamma(D) = b_s(n)$ for a digraph $D \in \mathrm{MD}_s(n)$ if and only if D is isomorphic to the digraph in Figure 1. But what is the expression for $b_s(n)$ in the case $\gcd(n-1, s) \ne 1$? We will consider this problem in this paper.

In [1], Wei-Quan Dong, Jia-Yu Shao, and Chun-Fei Dong proved the following theorem.

THEOREM 1.1. ([1]). *Let $\widetilde{b}_s(n) = \max\{\gamma(D) \mid D$ is a primitive digraph with n vertices and shortest cycle length $s\}$, $\widetilde{r}_0 = \max\{m \mid 1 \le m \le n,\ \gcd(m,s) = 1\}$. Then:*

(a) $\widetilde{b}_s(n) = n + s(n-2)$ *when* $\gcd(n,s) = 1$.
(b) $\widetilde{b}_s(n) = n + s(\widetilde{r}_0 - 2)$ *when* $\gcd(n,s) > 1$ *and* $s \nmid n$ *or* $n = ks$, *but* $(k-1) \mid (s-2)$.
(c) $\widetilde{b}_s(n) = n + s(\widetilde{r}_0 - 2) + 1$ *when* $\gcd(n,s) > 1$ *and* $n = ks$, $(k-1) \nmid (s-2)$.

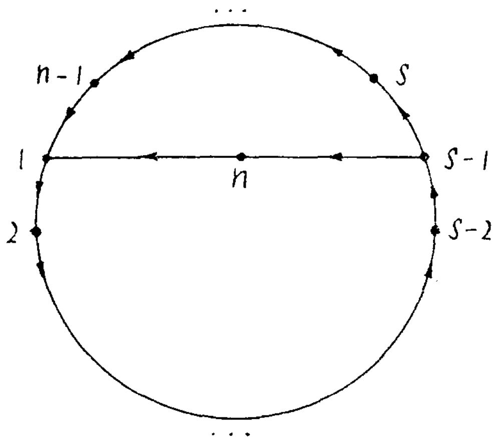

FIG. 1.

They also gave a necessary and sufficient condition for a primitive digraph D to have $\gamma(D) = \widetilde{b}_s(n)$ when $s \neq 2, 6([1])$. In this paper, we obtain the following three results for $\gcd(n-1, s) \neq 1$:

(1) If $\gcd(n-1, s) > 1$ and $s \geq 3$, then $b_s(n) = W_1(s, n)$.
(2) If $\gcd(n-1, s) > 1$ and $s = 2$, then $b_2(n) = W_1(s, n) + 1$.
(3) In Theorems 5.1 and 5.2, we characterize the digraph $D \in \mathrm{MD}_s(n)$ with $\gamma(D) = b_s(n)$ when $\gcd(n-1, s) > 1$ and $s \neq 2, 6$.

To prove our main results, we need the following basic properties of primitive ministrong digraphs.

THEOREM 1.2. [4, 6]. *Let $D \in \mathrm{MD}_s(n)$. Then the following properties hold:*

(1) *D has no loops and no cycles of length n, so $t \leq n-1$ for any $t \in L(D)$, and $2 \leq s \leq n-2$.*
(2) *If $(x, y) \in E(D)$, then (x, y) is an arc of every path from x to y.*
(3) *No cycle of D has chords.*

2. SOME SPECIAL UPPER BOUNDS FOR $\gamma(D)$

Let D be a primitive digraph with n vertices, and $\gamma(D)$ be the exponent of D. Also, let $L(D) = \{r_1, r_2, \ldots, r_\lambda\}$ be the set of distinct lengths of the elementary cycles of D, where $r_\lambda = s$ is the shortest cycle length of D. Then it is well known that D is strong and $\gcd(r_1, r_2, \ldots, r_\lambda) = 1$, where gcd means the greatest common divisor.

DEFINITION 2.1. The (local) exponent from vertex x to y, denoted by $\gamma(x, y)$, is the least integer γ such that there exists a walk of length m from x to y for all integers $m \geq \gamma$.

From Definition 2.1 it is easy to see that

$$\gamma(D) = \max\{\gamma(x, y) \mid x, y \in V(D)\}.$$

Now suppose $B = \{a_1, \ldots, a_k\}$ is a set of distinct positive integers with $\gcd(a_1, \ldots, a_k) = 1$. The Frobenius number $\Phi(B) = \Phi(a_1, \ldots, a_k)$ is defined to be the least integer Φ such that every integer $m \geq \Phi$ can be expressed in the form $m = c_1 a_1 + \cdots + c_k a_k$, where $c_1, \ldots, c_k$ are nonnegative integers. A result due to Schur shows that $\Phi(a_1, \ldots, a_k)$ is finite if $\gcd(a_1, \ldots, a_k) = 1$. In the case $k = 2$, we have $\Phi(a_1, a_2) = (a_1 - 1)(a_2 - 1)$.

DEFINITION 2.2. Let $x, y \in V(D)$ and $B = \{r_{i_1}, \ldots, r_{i_k}\} \subseteq L(D)$. The relative distance $d_B(x, y)$ from x to y is defined to be the length of the shortest walk from x to y that meets (has a common vertex with) at least one cycle of each length r_{i_j} for $j = 1, 2, \ldots, k$.

Let

$$d_B = d(r_{i_1}, \ldots, r_{i_k}) = \max\{d_B(x, y) \mid x, y \in V(D)\},$$

where $B = \{r_{i_1}, \ldots, r_{i_k}\} \subseteq L(D)$.

The following basic upper bound for $\gamma(x, y)$ will be used in the proof of our main results.

THEOREM 2.1. ([5]). *Let D be a digraph, and $B = \{r_{i_1}, \ldots, r_{i_k}\} \subseteq L(D)$ with* $\gcd(r_{i_1}, \ldots, r_{i_k}) = 1$. *Then $\gamma(x, y) \leq d_B(x, y) + \Phi(r_{i_1}, \ldots, r_{i_k})$ and $\gamma(D) \leq d_B + \Phi(B)$.*

3. IMPROVED ESTIMATIONS OF THE RELATIVE DISTANCES

It is not difficult to verify the following estimate for the relative distances: for a primitive digraph D with n vertices, we have ([2])

$$d_B \leq \sum_{a \in B} (n - a) + (n - 1), \tag{3.1}$$

where $B \subseteq L(D)$. However, we want to improve the inequality (3.1) in the case $D \in \mathrm{MD}_s(n)$. For this purpose we first give two lemmas.

LEMMA 3.1. *Let $D \in \mathrm{MD}_s(n)$ with $V(D) = \{1, 2, \ldots, n\}$ and $t \in L(D)$. Then $d_{\{t\}}(u, v) \leq n - t + n - 2$ for all $u, v \in V(D)$.*

PROOF. We consider the following two cases.

Case 1. u belongs to some t-cycle. Then $d_{\{t\}}(u, v) = d(u, v) \leq n-1 \leq n - t + n - 2$.

Case 2. u does not belong to any t-cycle. Let $d_t(u) = d(u, z)$, where z belongs to some t-cycle C. Then $d(u, z) \leq n - t$ and $u \notin V(C)$. We want to show $d(z, v) \leq n - 2$.

Suppose $d(z, v) = n - 1$. Let $W(z, v) = 123 \cdots n$ be a path from z to v of length $n - 1$, where (we suppose) $z = 1, v = n$. Let $W'(u, z)$ be a path from u to z of length $d(u, z)$; then $W'(u, z)$ is the shortest path from u to all vertices of all t-cycles. For $D \in \mathrm{MD}_s(n)$, D has no arcs (i, j) such that $j - i > 1$. Hence $C = 123 \cdots t1$. Since $u \notin V(C)$, there exists an arc (x, z) on the path $W'(u, z)$ such that $x \notin V(C)$ and $x > t$. Thus a cycle $C' = 123 \cdots t\, t+1 \cdots x1$ is formed, and C' has a chord $(t, 1)$. This contradicts $D \in \mathrm{MD}_s(n)$. So $d(z, v) \leq n - 2$. Then

$$d_{\{t\}}(u, v) \leq d(u, z) + d(z, v) \leq n - t + n - 2.$$

The lemma is proved. ■

Let $W_1 = x_1 x_2 \cdots x_m$ and $W_2 = y_1 y_2 \cdots y_t$ be two walks. If $x_m = y_1$, then we denote by $W_1 \cup W_2$ the walk $x_1 x_2 \cdots x_m y_2 y_3 \cdots y_t$.

LEMMA 3.2. *Let $D \in \mathrm{MD}_s(n)$, $B = \{a_1, a_2, \ldots, a_m\} \subseteq L(D)$, $a_1 < a_2 < \cdots < a_m$. Then for any $x, y \in V(D)$, there exists a positive integer i, $1 \leq i \leq m$, such that*

$$d_B(x, y) \leq (n-2-a_1)+(n-2-a_2)+\cdots+(n-2-a_{i-1})+(n-a_i)+(n-2).$$

PROOF. Let $W(x_{k-1}, x_k)$ be a path from x_{k-1} to x_k of length $d_{a_k}(x_{k-1})$, where x_k is a vertex of some a_k-cycle for $k = 1, 2, \ldots, m$, and $x_0 = x$. In the following, we consider three different cases.

Case 1. There exists some integer $i, 2 \leq i \leq m-1$, such that $d_{a_i}(x_{i-1}) \geq n - a_i - 1$ but $d_{a_k}(x_{k-1}) \leq n - a_k - 2$ for $k = 1, 2, \ldots, i-1$. Let $W(x_{i-1}, y)$

be a walk from x_{i-1} to y of length $d_{\{a_i\}}(x_{i-1}, y)$ which meets some a_i-cycle. Because $d_{a_i}(x_{i-1}) \geq n - a_i - 1 = n - (a_i + 1) \geq n - a_t$ for any integer $t > i$, $W(x_{i-1}, y)$ meets some a_{i+1}-cycle, some a_{i+2}-cycle,..., and some a_m-cycle. Thus $W(x_0, x_1) \cup W(x_1, x_2) \cup \cdots \cup W(x_{i-2}, x_{i-1}) \cup W(x_{i-1}, y)$ meets at least one cycle of each length a_k for $k = 1, 2, \ldots, m$. From Lemma 3.1, we have

$$|W(x_{i-1}, y)| = d_{\{a_i\}}(x_{i-1}, y) \leq n - a_i + n - 2.$$

So $d_B(x, y) \leq |W(x_0, x_1)| + |W(x_1, x_2)| + \cdots + |W(x_{i-2}, x_{i-1})| + |W(x_{i-1}, y)| \leq (n - 2 - a_1) + (n - 2 - a_2) + \cdots + (n - 2 - a_{i-1}) + (n - a_i) + (n - 2)$.

Case 2. $d_{a_j}(x_{j-1}) \leq (n - a_j - 2)$ *for* $j = 1, 2, \ldots, m - 1$. For the same reason as in case 1, we have

$$\begin{aligned} d_B(x, y) \leq\ & (n - 2 - a_1) + (n - 2 - a_2) + \cdots + (n - 2 - a_{m-1}) \\ & + (n - a_m) + (n - 2). \end{aligned}$$

Here, taking $i = m$, we obtain the result.

Case 3. $d_{a_1}(x_0) \geq n - a_1 - 1$, *where* $x_0 = x$. Also, for the same reason as in case 1, we obtain that $d_B(x, y) \leq (n - a_1) + (n - 2)$. Here, taking $i = 1$, we obtain the result.

Combining cases 1, 2, and 3, the proof of the lemma is completed. ■

From Lemma 3.2 we easily get the main theorem of this section, which improves the inequality (3.1).

THEOREM 3.1. *Let* $D \in \mathrm{MD}_s(n)$, $B = \{a_1, \ldots, a_m\} \subseteq L(D)$. *Then*

$$\begin{aligned} d_B &= d(a_1, \ldots, a_m) \leq (n - 1 - a_1) + (n - 1 - a_2) \\ &\quad + \cdots + (n - 1 - a_m) + (n - 1) \\ &= \sum_{j=1}^{m} (n - 1 - a_j) + (n - 1). \end{aligned}$$

4. THE EXPRESSIONS FOR $b_s(n)$ IN THE CASE $\gcd(n - 1, s) > 1$

Throughout this section, we will assume that $\gcd(n - 1, s) > 1$. We will prove that $\gamma(D) \leq W_1(s, n)$ for any $D \in \mathrm{MD}_s(n) \backslash \psi_s(n)$ from Lemmas 4.1

to 4.6 when $s \geq 3$ [furthermore, $\gamma(D) < W_1(s,n)$ when $s \neq 2, 6$]. We will also prove from Lemmas 4.7 to 4.10 that $\gamma(D) \leq W_1(s,n)$ for any $D \in \psi_s(n)$ when $s \geq 3$. Finally we obtain $b_s(n) = W_1(s,n)$ with $s \geq 3$ in Theorem 4.1 and $b_2(n) = W_1(2,n) + 1$ in Theorem 4.2. Recall that we have defined r_0, $W_0(s,n)$, and $W_1(s,n)$ in (1.1), (1.2), and (1.3).

LEMMA 4.1. *If* $\gcd(n-1, s) > 1$, *then* $(n-1) - r_0 \leq \frac{1}{2}s$; *furthermore,* $(n-1) - r_0 < \frac{1}{2}s$ *when* $s \neq 2, 6$.

PROOF. See [1, Lemma 4.3]. ■

LEMMA 4.2. *If* $\gcd(n-1, s) > 1$, *then* $W_0(s,n) \leq W_1(s,n)$; *furthermore* $W_0(s,n) < W_1(s,n)$ *when* $s \neq 2, 6$.

PROOF. See [1, Lemma 4.4]. ■

LEMMA 4.3. *Let* $D \in \mathrm{MD}_s(n)$, $L(D) = \{r_1, r_2, \ldots, r_\lambda\}$, *where* $r_\lambda = s$. *If* $\gcd(n-1, s) > 1$, *and* $\gcd(r_i, s) > 1$ *for all* $r_i \in L(D)$, *then* $\gamma(D) \leq W_0(s,n)$.

PROOF. By the hypothesis and the primitivity of D we see that s is not a prime power, so $s \geq 6$. Let $p_1, \ldots, p_t$ be distinct prime divisors of s with $t \geq 2$. Take $r_{i_1}, \ldots, r_{i_t} \in L(D)$ such that $\gcd(p_k, r_{i_k}) = 1$ for $k = 1, \ldots, t$. [Such an r_{i_k} exists because $\gcd(r_1, \ldots, r_\lambda) = 1$, where $r_\lambda = s$.]

Now, take $R = \{s, r_{i_1}, \ldots, r_{i_t}\} \subseteq L(D)$; then $\gcd(s, r_{i_1}, \ldots, r_{i_t}) = 1$. We may use Vitek's upper bound for the Frobenius number $\Phi(R)$, since $\gcd(s, r_{i_k}) > 1$ for $k = 1, \ldots, t$ means that $R = \{s, r_{i_1}, \ldots, r_{i_t}\}$ satisfies the hypothesis of [2, Theorem 4], to obtain that $\Phi(R) \leq s(r-2)/2 \leq s(n-3)/2$, where $r = \max_{1 \leq k \leq t}\{r_{i_k}\} \leq n-1$. Also, by using induction on t ($t \geq 2$), we have $p_1 p_2 \cdots p_t \geq 2(t+1)$, so $s \geq p_1 p_2 \cdots p_t \geq 2(t+1)$, i.e., $t \leq \frac{1}{2}s - 1$. Now $\gcd(n-1, s) > 1$, so $s \leq n-3$, and from Theorem 3.1 we have

$$\begin{aligned} d_R &\leq \sum_{a \in R}(n-1-a) + (n-1) \\ &\leq (n-1-s)(t+1) - 2t - 1 + n - 1 \qquad (*) \\ &= (n-1-s) + t(n-1-s-2) + (n-2) \\ &\leq (n-1-s) + \left(\tfrac{1}{2}s - 1\right)(n-1-s-2) + n - 2 \\ &= n + \tfrac{1}{2}s(n-3) - \tfrac{1}{2}s^2 \end{aligned}$$

[(*) holds because $|R| = t+1 \geq 3$ and $a \geq s+2$ for all $a \in R \backslash \{s\}$ by the hypothesis $\gcd(r_i, s) > 1$ for all $r_i \in L(D)$]. So

$$\begin{aligned}\gamma(D) &\leq d_R + \Phi(R) \leq n + \tfrac{1}{2}s(n-3) - \tfrac{1}{2}s^2 + \tfrac{1}{2}s(n-3) \\ &= n + s(n-3) - \tfrac{1}{2}s^2 = W_0(s,n).\end{aligned}$$

■

The following lemma further improves the estimation of the relative distance. It will be used in the proof of Lemma 4.5.

LEMMA 4.4. *Let* $D \in \mathrm{MD}_s(n)$, $B = \{r_1, r_2\} \subseteq L(D)$, $r_1 > r_2$, $x, y \in V(D)$. *If* $d_{r_2}(x) \geq n - r_1 - (r_2 - 1)$, *then* $d_B(x,y) \leq n - r_2 + n - 2$.

PROOF. Let xP_1zP_2y be a walk from x to y of length $d_{\{r_2\}}(x,y)$ which meets some r_2-cycle, where z is the first vertex on this walk which belongs to some r_2-cycle C, xP_1z is the shortest path from x to z, and zP_2y is the shortest path from z to y. Then $n - r_2 \geq |xP_1z| = d(x,z) \geq d_{r_2}(x) \geq n - r_1 - (r_2 - 1)$.

Case 1. $|zP_2y| = d(z,y) \geq n - r_1$. Then $d_B(x,y) \leq |xP_1zP_2y| = d_{\{r_2\}}(x,y) \leq n - r_2 + n - 2$ by Lemma 3.1.

Case 2. $|zP_2y| = d(z,y) \leq n - r_1 - 1$. Since $|xP_1z \cup zCz| > n - r_1 - (r_2 - 1) + r_2 - 1 = n - r_1$, $xP_1z \cup zCz$ meets some r_1-cycle. So $xP_1z \cup zCz \cup zP_2y$ meets some r_1-cycle and some r_2-cycle. Then we have

$$\begin{aligned}d_B(x,y) &\leq |xP_1z \cup zCz \cup zP_2y| \leq n - r_2 + r_2 + n - r_1 - 1 \\ &= n - r_1 + n - 1 \leq n - r_2 + n - 2 \qquad (\text{by } r_1 \geq r_2 + 1).\end{aligned}$$

Combining cases 1 and 2, we complete the proof of Lemma 4.4. ■

LEMMA 4.5. *Let* $D \in \mathrm{MD}_s(n)$, $\gcd(n-1, s) > 1$. *If there exists* $r \in L(D)$ *with* $r < r_0$ *(*r_0 *as defined above) such that* $\gcd(r,s) = 1$, *then* $\gamma(D) \leq W_1(s,n)$; *furthermore, when* $s \neq 2, 6$, *we have* $\gamma(D) < W_1(s,n)$.

PROOF. Take $B = \{r, s\} \subseteq L(D)$, and let x, y be arbitrary vertices of D. We estimate the upper bounds of $d_B(x,y)$ by considering the following two cases.

Case 1. $n - s - 2 \leq d_s(x) \leq n - s$. Since $r \geq s+1$ and $s \geq 2$, we have $d_s(x) \geq n - s - 2 \geq n - r - 1 \geq n - r - (s-1)$. From Lemma 4.4 we have that

$$d_B(x,y) \le n-s+n-2 \le 3n-s-r-5 \qquad (\text{by} \quad r<r_0\le n-2).$$

Case 2. $d_s(x) \le n-s-3$. Then by Lemma 3.1 we have

$$\begin{aligned} d_B(x,y) &\le (n-s-3)+n-r+n-2 \\ &= 3n-s-r-5. \end{aligned}$$

Combining cases 1 and 2, and using $r<r_0$ and Lemma 4.1, we have

$$\begin{aligned} \gamma(x,y) &\le d_B(x,y)+\Phi(B) \\ &\le 3n-s-r-5+(s-1)(r-1) \\ &= n+2n-s-6+(r-1)(s-2) \\ &\le n+2n-s-6+(r_0-2)(s-2) \\ &= n+s(r_0-2)+2(n-1-r_0)-s \\ &\le n+s(r_0-2) = W_1(s,n). \end{aligned}$$

Furthermore $2(n-1-r_0) < s$ for $s \ne 2,6$, so $\gamma(x,y) < W_1(s,n)$ when $s \ne 2,6$. So $\gamma(D) \le W_1(s,n)$, and $\gamma(D) < W_1(s,n)$ when $s \ne 2,6$. The lemma is proved. ■

LEMMA 4.6. *Let $D \in \mathrm{MD}_s(n)$, $\gcd(n-1,s)>1$, $s\ge 3$. If $r_0 \in L(D)$ and $\gcd(r,s)>1$ for all $r \in L(D)\backslash\{r_0\}$, but there exists $r_{i_0} \in L(D)\backslash\{r_0\}$ such that $s \nmid r_{i_0}$, then $\gamma(D) \le W_1(s,n)$; furthermore, $\gamma(D) < W_1(s,n)$ when $s \ne 6$.*

PROOF. Take $B=\{s,r_{i_0},r_0\}$. Because $\gcd(r_{i_0},s)>1$, r_0 is not a nonnegative integral combination of r_{i_0} and s. Since $r_{i_0} \le n-1 \le r_0+\frac{1}{2}s < r_0+s \le 2r_0$ and $s\nmid r_{i_0}$, r_{i_0} is not a nonnegative integral combination of r_0 and s. Hence $B=\{s,r_{i_0},r_0\}$ satisfies the hypothesis of [2, Theorem 4]. So we have $\Phi(B) \le \frac{1}{2}s(n-3)$. Let x,y be arbitrary vertices of D. We estimate upper bounds of $\gamma(x,y)$ by considering the following three cases.

Case 1. $d(x,y) \ge n-s$. Then $d_B(x,y)=d(x,y)\le n-1<n+r_0-s-1$. So using $r_0\le n-2$ and $s \le n-3$, we have

$$\begin{aligned} \gamma(x,y) &\le d_B(x,y)+\Phi(B) \\ &< n+r_0-s-1+\tfrac{1}{2}s(n-3) \\ &\le n+s(n-3)-\tfrac{1}{2}s(n-3)+(n-2)-s-1 \\ &= n+s(n-3)-\tfrac{1}{2}(n-3)(s-2)-s \\ &\le n+s(n-3)-\tfrac{1}{2}s(s-2)-s \\ &= W_0(s,n) \le W_1(s,n). \end{aligned}$$

Case 2. $n-r_0 \le d(x,y) \le n-s-1$. Because $r_0+s \ge n-1-\frac{1}{2}s+s = n-1+\frac{1}{2}s > n$ by $s \ge 3$, we have $d_B(x,y) \le d(x,y)+r_0 \le n-s+r_0-1$. So using similar arguments to case 1, we have

$$\gamma(x,y) \le W_0(s,n) \le W_1(s,n).$$

Case 3. $d(x,y) \le n-r_0-1$. Suppose x belongs to a cycle of length b.

Subcase 3.1. $b+s>n$. Then

$$d_B(x,y) \le d(x,y)+b \le n-r_0-1+b \le n-r_0-1+n-1 = 2n-r_0-2.$$

So by $n-3 \ge s$ we have

$$\begin{aligned}
\gamma(x,y) &\le 2n-r_0-2+\tfrac{1}{2}s(n-3)\\
&= n+s(n-3)-\tfrac{1}{2}s(n-3)+n-r_0-2\\
&= n+s(n-3)-\tfrac{1}{2}(n-3)(s-2)-r_0+1\\
&\le n+s(n-3)-\tfrac{1}{2}s(s-2)-r_0+1\\
&= W_0(s,n)+s+1-r_0\\
&\le W_0(s,n) \le W_1(s,n).
\end{aligned}$$

Subcase 3.2. $b+s \le n$. Because $\gcd(r_{i_0},s)>1$ and $s \nmid r_{i_0}$, s is not a prime number, so $s \ge 4$, and $r_{i_0} \ge s+2 \ge 6$.

(1) $b = s$ *and* $r_{i_0} > n-s$. Then

$$d_B(x,y) \le d(x,y)+b \le n-r_0-1+s \le 2n-r_0-2.$$

So using similar arguments to subcase 3.1, we have

$$\gamma(x,y) \le W_0(s,n) \le W_1(s,n).$$

(2) $b = s$ *and* $r_{i_0} \le n-s$. Then

$$d_B(x,y) \le d(x,y)+b+r_0 \le n-r_0-1+s+r_0 = n+s-1.$$

Since $r_{i_0} \le n - s$ and $r_{i_0} \ge 6$, we have $n - s \ge 6$. So

$$\begin{aligned}\gamma(x,y) &\le n + s - 1 + \tfrac{1}{2}s(n-3)\\ &= n + s(n-3) - \tfrac{1}{2}s^2 - \tfrac{1}{2}s(n-3-s) + s - 1\\ &= n + s(n-3) - \tfrac{1}{2}s^2 - \tfrac{1}{2}s(n-s-5) - 1\\ &< n + s(n-3) - \tfrac{1}{2}s^2 = W_0(s,n) \le W_1(s,n).\end{aligned}$$

(3) $b > s$. Then $n \ge b + s \ge 2s + 1$, i.e., $n - 1 \ge 2s$, and

$$b + r_0 \ge b + n - 1 - \tfrac{1}{2}s > s + n - 1 - \tfrac{1}{2}s = n - 1 + \tfrac{1}{2}s \ge n.$$

So

$$\begin{aligned}d_B(x,y) &\le d(x,y) + b + r_0\\ &\le n - r_0 - 1 + b + r_0\\ &= n + b - 1 \le 2n - s - 1,\\ \gamma(x,y) &\le 2n - s - 1 + \tfrac{1}{2}s(n-3)\\ &= n + s(n-3) - \tfrac{1}{2}(s-2)(n-3) - s + 2\\ &\le n + s(n-3) - \tfrac{1}{2}(s-2)(2s-2) - s + 2\\ &= n + s(n-3) - (s-1)^2 + 1\\ &\le n + s(n-3) - \tfrac{1}{2}s^2 \qquad (*)\\ &= W_0(s,n) \le W_1(s,n)\end{aligned}$$

[where $(*)$ holds because $(s-1)^2 \ge \frac{1}{2}s^2 + 1$ for $s \ge 4$].

Combining cases 1, 2, and 3, we have

$$\gamma(D) \le W_1(s,n), \quad \text{and} \quad \gamma(D) < W_1(s,n) \quad \text{if} \quad s \ne 6.$$

The lemma is proved. ■

Lemmas 4.3, 4.5, and 4.6 imply that $\gamma(D) \le W_1(s,n)$ for any $D \in \mathrm{MD}_s(n)\backslash\psi_s(n)$ if $s \ge 3$; furthermore $\gamma(D) < W_1(s,n)$ if $s \ne 2, 6$. In the following we discuss the cases for $D \in \psi_s(n)$.

First we have

$$\begin{aligned}W_1(s,n) - \Phi(s,r_0) &= n + s(r_0 - 2) - (s-1)(r_0 - 1)\\ &= n + sr_0 - 2s - sr_0 + s + r_0 - 1\\ &= n - 1 + r_0 - s.\end{aligned}$$

Therefore, if $d_B(x,y) \le n-1+r_0-s$ [or $d_B(x,y) < n-1+r_0-s$], then $\gamma(x,y) \le W_1(s,n)$ [or $\gamma(x,y) < W_1(s,n)$], where $D \in \psi_s(n)$, $x,y \in V(D)$, $B=\{r_0,s\}$.

LEMMA 4.7. *Let $D \in \psi_s(n)$, $\gcd(n-1,s) > 1$, $x,y \in V(D)$. If $d(x,y) \ge n-s$, then $\gamma(x,y) < W_1(s,n)$.*

PROOF. Let $B=\{r_0,s\}$; then by using $d(x,y) \ge n-s$ we have $d_B(x,y)= d(x,y) \le n-1 < n-1+r_0-s$, so $\gamma(x,y) < W_1(s,n)$. ■

LEMMA 4.8. *Let $D \in \psi_s(n)$ with $s \ge 3$, $\gcd(s,n-1) > 1$, $x,y \in V(D)$. If $n-r_0 \le d(x,y) \le n-s-1$, then $\gamma(x,y) \le W_1(s,n)$.*

PROOF. Take $B=\{r_0,s\}$. We know $s \ge 3$, so $r_0+s > n$. We have $d_B(x,y) \le d(x,y)+r_0 \le n-s-1+r_0 = n-1+r_0-s$, and $\gamma(x,y) \le W_1(s,n)$. ■

The following lemma gives a property of primitive ministrong digraphs. It will be used in Lemma 4.10.

LEMMA 4.9. *Let $D \in \mathrm{MD}_s(n)$, $s \ge 3$, $\gcd(n-1,s) > 1$. If D has an elementary cycle C of length $n-2$, x,y are two distinct vertices which don't belong to C, and no s-cycle contains both x and y, then there exist no arcs which join x and y.*

PROOF. Suppose there is an arc which joins x and y, say, $(x,y) \in E(D)$; then $(y,x) \notin E(D)$ (by $s \ge 3$).

Let W be the shortest path from the cycle C to the vertex x. Since $|C| = n-2$, we have $|W| \le 2$. On the other hand, from $(y,x) \notin E(D)$, we know the vertex y does not belong to the path W. So $|W| = 1$, and there exists a vertex $i \in V(C)$ such that $(i,x) \in E(D)$.

Similarly, there is a vertex $j \in V(C)$ such that $(y,j) \in E(D)$. Thus D has a spanning subgraph $D' = C \cup \{(x,y),(i,x),(y,j)\}$ which is strong, and we obtain $D=D'$ because $D \in \mathrm{MD}_s(n)$ is minimally strong. Let $C' = xyjCix$; then C and C' are all the cycles of D. From hypothesis we have $|C'| \ne s$. Since $\gcd(n-1,s) > 1$, we have $|C| = n-2 \ne s$. So D has no cycles of length s; namely, $s \notin L(D)$. This contradicts the fact that $D \in \mathrm{MD}_s(n)$. Hence, the lemma is proved. ■

LEMMA 4.10. *Let $D \in \psi_s(n)$, $\gcd(n-1,s) > 1$, $s \ge 3$, $x,y \in V(D)$, $B=\{r_0,s\}$. Let b be the length of one of the cycles which contains*

x or y. If $d(x,y) \le n - r_0 - 1$, then we have:

(1) $b + s \neq n$.
(2) $\gamma(x,y) < W_1(s,n)$ *when* $b + s > n$.
(3) $\gamma(x,y) < W_1(s,n)$ *when* $b + s < n - 1$.
(4) $\gamma(x,y) < W_1(s,n)$ *when* $b + s = n - 1$ *and* $s \nmid (n-1)$.
(5) $\gamma(x,y) \le W_1(s,n)$ *when* $b + s = n - 1$ *and* $s \mid (n-1)$.
(6) $\gamma(x,y) < W_1(s,n)$ *when* $b+s = n-1$, $n-1 = ks$ *and* $(k-1) \mid (s-2)$.

PROOF. (1): Suppose $b+s=n$. From $b+s=n$ and $\gcd(n-1,s)>1$ we obtain that b is not a multiple of s, and so $b = r_0$ by $D \in \psi_s(n)$, so $r_0 + s = n$. On the other hand, by $s \ge 3$ we have $r_0 + s \ge n - 1 - \frac{1}{2}s + s = n - 1 + \frac{1}{2}s > n$. This is a contradiction. Hence $b + s \neq n$.

(2): Since $b + s > n$, we have

$$d_B(x,y) \le d(x,y) + b \le n - r_0 - 1 + b.$$

Also,

$$r_0 \ge b - 1 \tag{4.1}$$

[for otherwise $b \ge r_0 + 2$, $b \in L(D)$, and $D \in \psi_s(n)$ means that b is a multiple of s; hence $\gcd(b-1,s) = 1$ with $b - 1 \ge r_0 + 1$. This contradicts the definition of r_0] and

$$r_0 \ge s + 1. \tag{4.2}$$

It follows that

$$2r_0 \ge b + s. \tag{4.3}$$

But if $2r_0 = b + s$, then equalities also hold in (4.1) and (4.2), so $r_0 = b - 1 = s + 1$ and $b = s + 2$, so $s = 2$ (since $r_0 = b - 1 \Rightarrow b \neq r_0 \Rightarrow s \mid b$). This contradicts the fact that $s \ge 3$. Therefore, the strict inequality in (4.3) holds, namely $b < 2r_0 - s$. So $d_B(x,y) \le n - r_0 - 1 + b < n - 1 + r_0 - s$, and $\gamma(x,y) < W_1(s,n)$.

(3): Because $b + s < n - 1$, we have $b < n - 1 - s < n - 1 - \frac{1}{2}s \le r_0$, so $s \mid b$. By using $b + s + 1 \le n - 1$ and $\gcd(s + b + 1, s) = 1$, we obtain that $r_0 \ge b + s + 1$, so $b < r_0 - s$. Hence $d_B(x,y) \le d(x,y) + b + r_0 \le n + b - 1 < n - 1 + r_0 - s$, and $\gamma(x,y) < W_1(s,n)$.

(4): Since $b + s = n - 1$ and $s \nmid n - 1$, we have $s \nmid b$ and thus $b = r_0$ by $D \in \psi_s(n)$. So $d_B(x,y) \le d(x,y) + r_0 \le n - r_0 - 1 + r_0 = n - 1 < n - 1 + r_0 - s$, and $\gamma(x,y) < W_1(s,n)$.

(5): First, we have that $b = n - 1 - s$, $r_0 = n - 2$ and $d(x,y) \le n - r_0 - 1 = 1$. In the following we consider three different cases.

Case 1. x or y belongs to some s-cycle. Then $d_B(x,y) \leq d(x,y)+s \leq s+1 < n-1+r_0-s$. So $\gamma(x,y) < W_1(s,n)$.

Case 2. x or y belongs to some r_0-cycle. Then $d_B(x,y) \leq d(x,y) + r_0 \leq r_0+1 < n-1+r_0-s$, and $\gamma(x,y) < W_1(s,y) < W_1(s,n)$.

Case 3. neither x nor y belongs to any s-cycle or r_0-cycle. Then by using Lemma 4.9 we obtain that $(x,y) \notin E(D)$. But $d(x,y) \leq 1$, so $d(x,y) = 0$. Hence $d_B(x,y) \leq b+r_0 = n-1+r_0-s$, and $\gamma(x,y) \leq W_1(s,n)$.

(6): Since $n-1 = ks$ and $\gcd(ks-1,s) = 1$, then $r_0 = ks-1 = n-2$. We have

$$\begin{aligned}
\left(\frac{s-2}{k-1} + s + 1\right) b &= b + \left(s + \frac{s-2}{k-1}\right)(k-1)s \\
&= b + (ks-2)s \\
&= b + (ks-1) + (s-1)(ks-2) - 1 \\
&= b + r_0 + \Phi(r_0, s) - 1, && \text{(a)} \\
b + r_0 + (k(s-1)-2)s &= b + r_0 + (s-1)(ks-2) - 2 \\
&= b + r_0 + \Phi(r_0, s) - 2. && \text{(b)}
\end{aligned}$$

We also have:

(I) $\gamma(x,y) \leq d(x,y) + b + r_0 + \Phi(r_0,s)$.
(II) By (a) there is a walk from x to y of length $d(x,y) + b + r_0 + \Phi(s, r_0) - 1$.
(III) By (b) there is a walk from x to y of length $d(x,y) + b + r_0 + \Phi(r_0, s) - 2$.

So

$$\begin{aligned}
\gamma(x,y) &\leq d(x,y) + b + r_0 + \Phi(r_0,s) - 2 \\
&\leq n - r_0 - 1 + b + r_0 + \Phi(r_0,s) - 2 \\
&= n - 1 + r_0 - s + \Phi(r_0,s) - 1 \\
&= W_1(s,n) - 1 < W_1(s,n).
\end{aligned}$$

■

Lemmas 4.7, 4.8, and 4.10, imply that $\gamma(D) \leq W_1(s,n)$ for any $D \in \psi_s(n)$ for $s \geq 3$ [so $\gamma(D) \leq W_1(s,n)$ for any $D \in \mathrm{MD}_s(n)$ with $s \geq 3$], and when $s \nmid (n-1)$ or $n-1 = ks$, but $(k-1) \mid (s-2)$, if $\gamma(x,y) = W_1(s,n)$, then $n - r_0 \leq d(x,y) \leq n-s-1$ for $s \neq 2, 6$.

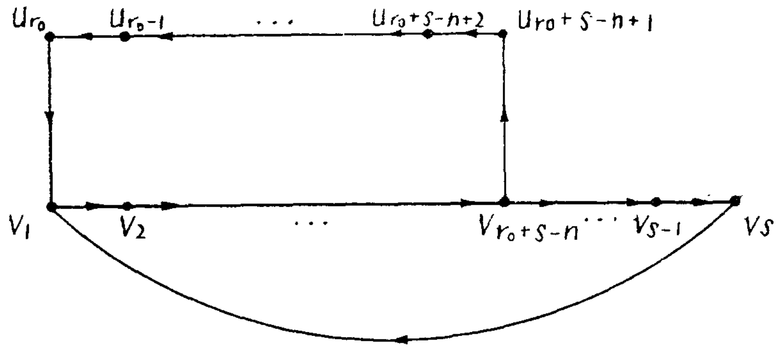

FIG. 2.

THEOREM 4.1. *Let* $\gcd(n-1, s) > 1$, $s \geq 3$. *Then:*

(1) $b_s(n) = W_1(s, n)$.
(2) *If* $D \in \mathrm{MD}_s(n)$ *and* $\gamma(D) = W_1(s, n)$, *then* $D \in \psi_s(n)$ *when* $s \neq 6$.

PROOF. (1): We already know that $\gamma(D) \leq W_1(s, n)$ for any $D \in \mathrm{MD}_s(n)$ with $s \geq 3$. Now we construct the digraph D_0 shown in Figure 2. Clearly, $D_0 \in \mathrm{MD}_s(n)$ and $L(D_0) = \{r_0, s\}$.

It is easy to compute that $\gamma(D_0) = \gamma(U_{r_0} + s - n + 1, U_{r_0}) = r_0 + n - s - 1 + (r_0 - 1)(s - 1) = W_1(s, n)$ (see [5, Lemma 4.1]). So we have $b_s(n) = W_1(s, n)$.

(2): This follows from Lemma 4.3, Lemma 4.5, and Lemma 4.6. ■

For the case $s = 2$, the following theorem gives the expression for $b_2(n)$ when $\gcd(n-1, 2) = 2$.

THEOREM 4.2. *If* $s = 2$ *and* $\gcd(n-1, 2) = 2$, *then* $b_2(n) = W_1(2, n) + 1 = 3n - 7$.

PROOF. Firstly, we can easily see that $r_0 = n - 2 \geq s + 1$, so $n \geq 5$. Let $D \in \mathrm{MD}_2(n)$. From Lemmas 4.3 and 4.5 we have $\gamma(D) \leq W_1(2, n)$ when $r_0 \notin L(D)$. In the following, we discuss the case $r_0 \in L(D)$.

Take $B = \{r_0, s\} = \{n-2, 2\}$; then $\Phi(B) = n - 3$. For any $x, y \in V(D)$ we now estimate the upper bounds of $\gamma(x, y)$ by classifying $d_2(x)$ into the following two cases.

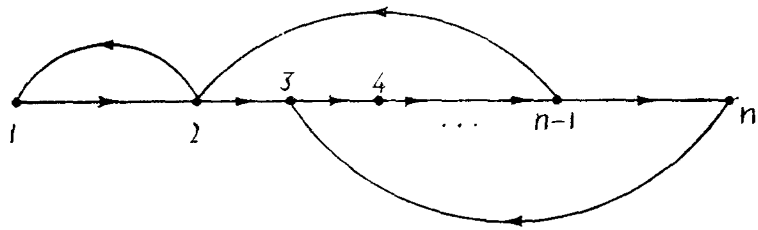

FIG. 3.

Case 1. $d_2(x) \geq n - r_0 = 2$. Then from Lemma 3.1 we have $d_B(x, y) \leq n - 2 + n - 2 = 2n - 4$. So

$$\gamma(x, y) \leq 2n - 4 + n - 3 = 3n - 7.$$

Case 2. $d_2(x) \leq n - r_0 - 1 = 1$. Then again from Lemma 3.1 we have

$$d_B(x, y) \leq 1 + n - r_0 + n - 2 = n + 1 \leq 2n - 4 \qquad (\text{by} \quad n \geq 5).$$

So $\gamma(x, y) \leq 2n - 4 + n - 3 = 3n - 7$.

Combining cases 1 and 2, we get $\gamma(D) \leq W_1(2, n) + 1 = 3n - 7$ for any $D \in \mathrm{MD}_2(n)$ when $r_0 = n - 2 \in L(D)$. Therefore, $\gamma(D) \leq W_1(2, n) + 1 = 3n - 7$ for any $D \in \mathrm{MD}_2(n)$ with $\gcd(2, n - 1) = 2$.

Next, we consider the digraph D_1 shown in Figure 3. Clearly $D_1 \in \mathrm{MD}_2(n)$ and $L(D_1) = \{2, n-2\}$. It is easy to prove that $\gamma(D_1) = \gamma(n, n) = W_1(2, n) + 1 = 3n - 7$. Hence $b_2(n) = W_1(2, n) + 1 = 3n - 7$. The theorem is proved. ∎

5. CHARACTERIZATION OF THE DIGRAPHS $D \in \mathrm{MD}_s(n)$ WITH $\gamma(D) = b_s(n)$ WHEN $s \neq 2, 6$

In this section, we always assume $s \neq 2, 6$ and $\gcd(n - 1, s) > 1$. In Section 4, we got the expressions for $b_s(n)$. Here, we characterize those digraphs $D \in \mathrm{MD}_s(n)$ with $\gamma(D) = b_s(n)$ and $s \neq 2, 6$.

LEMMA 5.1. *Let* $D \in \mathrm{MD}_s(n)$, $\gcd(n - 1, s) > 1$, $s \neq 2, 6$, $B = \{r_0, s\}$, $x, y \in V(D)$. *If* $\gamma(x, y) = W_1(s, n)$ *and* $n - r_0 \leq d(x, y) \leq n - s - 1$, *then*:

(1) $D \in \psi_s(n)$.
(2) $d_B(x, y) = n - 1 + r_0 - s$ *and* $d(x, y) = n - s - 1$.
(3) *There is a unique elementary path* $P(x, y)$ *from* x *to* y, *where* $|P(x, y)| = n - s - 1$, *and* $P(x, y)$ *does not meet any* s*-cycle.*
(4) D *contains a unique* s*-cycle.*

PROOF. (1): Using Theorem 4.1.
(2): By using $n - r_0 \leq d(x, y) \leq n - s - 1$, we have

$$d_B(x, y) \leq d(x, y) + r_0 \leq n - 1 + r_0 - s.$$

On the other hand, since $\gamma(x, y) = W_1(s, n)$, we have

$$d_B(x, y) \geq W_1(s, n) - \Phi(s, n) = n - 1 + r_0 - s.$$

So $d_B(x, y) = n - 1 + r_0 - s$, and $d(x, y) = n - s - 1$.

(3): Let $P(x, y)$ be a shortest elementary path from x to y of length $n - s - 1$. Since $d_B(x, y) = n - 1 + r_0 - s$, $P(x, y)$ does not meet any s-cycle. Suppose there exists another elementary path $P'(x, y) \neq P(x, y)$ from x to y. Then $P'(x, y)$ must meet some s-cycle and some r_0-cycle, because $|V(P'(x, y))| \geq |V(P(x, y))| = n - s > n - r_0$ and $V(P'(x, y)) \neq V(P(x, y))$. So $d_B(x, y) \leq |P'(x, y)| \leq n - 1 < n - 1 + r_0 - s$. This contradicts the fact that $d_B(x, y) = n - 1 + r_0 - s$. Hence there is a unique elementary path $P(x, y)$ from x to y, and $P(x, y)$ does not meet any s-cycle.

(4): Suppose there exist two different cycles C_1 and C_2 of length s. Then $|V(C_1) \cup V(C_2)| \geq s + 1$, since s is the shortest cycle length of D, so $P(x, y)$ as a path of length $n - s - 1$ will meet the cycle C_1 or C_2. This contradicts (3). So D has a unique s-cycle. ■

It is easy to prove the following property of primitive ministrong digraphs.

LEMMA 5.2. *If* $D \in \mathrm{MD}_s(n)$ *and* $\gcd(n-1, s) > 1$, *then* $n - 1 \notin L(D)$.

PROOF. If $n - 1 \in L(D)$, then we must have $L(D) = \{n - 1, s\}$, (since D is ministrong), and $\gcd(n - 1, s) = 1$, a contradiction. ■

COROLLARY 5.1. *Let* $D \in \mathrm{MD}_s(D)$ *with* $s \neq 2, 6$, $\gcd(n - 1, s) > 1$, $b > r_0$. *If* $\gamma(D) = W_1(s, n)$, *then* $b \notin L(D)$.

PROOF. Suppose $b \in L(D)$. By $\gamma(D) = W_1(s,n)$ we have $D \in \psi_s(n)$, so $s \mid b$ (since $b > r_0$). Thus $\gcd(b+1, s) = 1$ and $b + 1 > n - 1$, so $b = n - 1 \in L(D)$. But from Lemma 5.2 we have $n - 1 \notin L(D)$. This is a contradiction. Hence $b \notin L(D)$: the corollary is proved. ■

Denote by $T_{s,n}$ the set of digraphs D which satisfy the following five conditions:

(5.1) $V(D) = V(C_s) \dot{\cup} V(P_{n-s-1})$, where C_s is a cycle of length s, and P_{n-s-1} is a elementary path of length $n-s-1$ with $V(C_s) \cap V(P_{n-s-1}) = \emptyset$. Furthermore, $E(D) = E(C_s) \dot{\cup} E(P_{n-s-1}) \dot{\cup} E'$, where E' is an arc subset of D such that P_{n-s-1} is a unique path from the starting vertex (say, x) of P_{n-s-1} to the end vertex (say, y) of P_{n-s-1}.
(5.2) $D \in \psi_s(n)$.
(5.3) D contains a unique cycle of length s.
(5.4) $V(C_s) \cap V(C_b) = \emptyset$, where C_b is any cycle of length b which is not equal to s or r_0.
(5.5) $r_0 - 1 \neq \sum_{i=1}^{\lambda} a_i r_i / s$, where $L(D) = \{s, r_0, r_1, r_2, \ldots, r_\lambda\}$, and a_i is a nonnegative integer, $i = 1, 2, \ldots, \lambda$ (i.e., $r_0 - 1$ is not a nonnegative integral combination of $r_1/s, r_2/s, \ldots, r_\lambda/s$).

Clearly the digraph D_0 in Figure 2 belongs to $T_{s,n}$, so $T_{s,n} \neq \emptyset$. If $D \in T_{s,n}$ and $b \in L(D)$ with $b \neq s$, then P_{n-s-1} meets a b-cycle of D, because $|P_{n-s-1}| = n - s - 1 \geq n - b$.

THEOREM 5.1. *Let $D \in \mathrm{MD}_s(n)$ with $s \neq 2, 6$, $\gcd(n-1, s) > 1$, $s \nmid (n-1)$, or $n - 1 = ks$, but $(k-1) \mid (s-2)$. Then $\gamma(D) = b_s(n) = W_1(s,n)$ if and only if $D \in T_{s,n}$.*

PROOF. Necessity:

(1) By Theorem 4.1 we see that $D \in \psi_s(n)$. So the condition 5.2 is satisfied.
(2) Let $\gamma(D) = \gamma(x,y) = W_1(s,n)$, where $x, y \in V(D)$. By using Lemmas 4.7 and 4.10 we obtain that $n - r_0 \leq d(x,y) \leq n - s - 1$. So from Lemma 5.1 we see that the conditions (5.1) and (5.3) are satisfied, and $d(x,y) = n - 1 - s$, so the shortest path from x to y meets every cycle of D whose length is not equal to s.
(3) Suppose $V(C_s) \cap V(C_b) \neq \emptyset$ for some cycle C_b of length b ($b \neq s$, $b \neq r_0$). Take $B = \{r_0, s\}$, and add the cycle C_b to a shortest path $P(x,y)$ from x to y; we obtain a walk from x to y of length $d(x,y) + b$ which meets some s-cycle [while $P(x,y)$ already meets

any r_0-cycle]. By using Corollary 5.1, we have $b < r_0$. So $d_B(x,y) \leq d(x,y) + b < n - 1 + r_0 - s$. On the other hand, from Lemma 5.1 we get $d_B(x,y) = n - 1 + r_0 - s$. This is a contradiction. Hence $V(C_s) \cap V(C_b) = \emptyset$, namely, the condition (5.4) is satisfied.

(4) Suppose $r_0 - 1 = \sum_{i=1}^{\lambda} a_i r_i / s$ for some nonnegative integers $a_1, a_2, \ldots, a_\lambda$, where $L(D) = \{r_0, s, r_1, r_2, \ldots, r_\lambda\}$. Then $s(r_0 - 1) + n - s - 1 = n - s - 1 + \sum_{i=1}^{\lambda} a_i r_i$, i.e.,

$$\begin{aligned} W_1(s,n) - 1 &= \Phi(r_0, s) - 1 + n - 1 + r_0 - s \\ &= n - s - 1 + \sum_{i=1}^{\lambda} a_i r_i = d(x,y) + \sum_{i=1}^{\lambda} a_i r_i. \end{aligned}$$

Since the shortest path $P(x,y)$ from x to y has length $n - s - 1$, it already meets all cycles of length r_i $(i = 1, 2, \ldots, \lambda)$. So there exists a walk from x to y of length $d(x,y) + \sum_{i=1}^{\lambda} a_i r_i = W_1(s,n) - 1$, and $\gamma(x,y) < W_1(s,n)$. This contradicts the fact that $\gamma(x,y) = W_1(s,n)$. Hence $r_0 - 1 \neq \sum_{i=1}^{\lambda} a_i r_i / s$, namely, the condition (5.5) is satisfied. Thus the necessity is proved.

Sufficiency: Since $D \in T_{s,n}$, we have $V(D) = V(C_s) \,\dot\cup\, V(P_{n-s-1})$ and $E(D) = E(C_s) \,\dot\cup\, E(P_{n-s-1}) \,\dot\cup\, E'$. In the following we want to show $\gamma(x,y) = \gamma(D) = W_1(s,n)$, where x is the starting vertex of P_{n-s-1}, and y is the end vertex of P_{n-s-1}.

Since $\gamma(x,y) \leq W_1(s,n)$, we need only to prove that there doesn't exist a walk of length $W_1(s,n) - 1$ from x to y. Otherwise, $W_1(s,n) - 1$ can be expressed as follows:

$$W_1(s,n) - 1 = n - s - 1 + \sum_{i=1}^{\lambda} a_i r_i + a_0 r_0 + a_{\lambda+1} s, \tag{5.6}$$

where a_i is a nonnegative integer $(i = 0, 1, 2, \ldots, \lambda + 1)$, and by using the condition (5.4), if $a_0 = 0$, then $a_{\lambda+1} = 0$.

Case 1. $a_0 = 0$. Then $a_{\lambda+1} = 0$, and from the equality (5.6) we have $n + s(r_0 - 2) = n - s + \sum_{i=1}^{\lambda} a_i r_i$. Hence $s(r_0 - 1) = \sum_{i=1}^{\lambda} a_i r_i$ and $(r_0 - 1) = \sum_{i=1}^{\lambda} a_i r_i / s$. This is in contradiction with (5.5).

Case 2. $a_0 > 0$. Then by the equality (5.6) we have

$$s(r_0 - 1) = \sum_{i=1}^{\lambda} a_i r_i + a_0 r_0 + a_{\lambda+1} s. \tag{5.7}$$

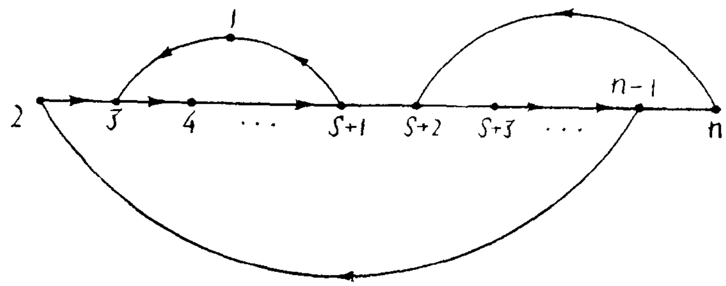

FIG. 4.

Since $s \mid r_i$ $(i = 1, 2, \ldots, \lambda)$, we have $s \mid a_0 r_0$. Again because $\gcd(s, r_0) = 1$, we get that $s \mid a_0$, so $a_0 \geq s$, and

$$s(r_0 - 1) < sr_0 \leq a_0 r_0 \leq \sum_{i=1}^{\lambda} a_i r_i + a_0 r_0 + a_{\lambda+1} s,$$

which contradicts (5.7).

Combining cases 1 and 2, we obtain that there does not exist a walk of length $W_1(s, n) - 1$ from x to y, and $\gamma(D) = \gamma(x, y) = W_1(s, n)$. This completes the proof of the sufficiency part. ■

By an argument similar to the proof of Theorem 5.1, we can obtain the following theorem.

THEOREM 5.2. *Let* $D \in \mathrm{MD}_s(n)$, $\gcd(n-1, s) > 1$, $s \neq 2, 6$, $n - 1 = ks$, $(k-1) \nmid (s-2)$. *Then* $\gamma(D) = W_1(s, n)$ *if and only if* $D \in T_{s,n}$ *or* $D \cong D_2$, *where* D_2 *is the digraph in Figure* 4.

REFERENCES

1 Wei-Quan Dong, Jia-Yu Shao, and Chun-Fei Dong, On the exponents of primitive digraphs with the shortest elementary circuit length s, *Linear Algebra Appl.* 104:1–27 (1988).

2 Y. Vitck, Bounds for a linear diophantine problem of Frobenius, *J. London Math. Soc.* 10:79–85 (1975).

3 Jia-Yu Shao, On the exponent of a primitive digraph, *Linear Algebra Appl.* 64:21–31 (1985).

4 J. A. Ross, On the exponent of a primitive nearly reducible matrix. *II, SIAM J. Algebraic Discrete Methods*, 3:395–410 (1982).

5 Jia-Yu Shao, On a conjecture about the exponent set of primitive matrices, *Linear Algebra Appl.* 65:91–123 (1985).
6 R. A. Brualdi and J. A. Ross, On the exponent of a primitive, nearly reducible matrix, *Math. Oper. Res.* 5:229–241 (1980).

Received 16 June 1993; final manuscript accepted 27 September 1993