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A Generalized Mean

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A class of means is defined and an inequality is established for them. Some standard inequalities, as well as new ones, can be obtained as special cases.

1. INTRODUCTION

Let $f(T_1,..., T_n)$ be an n-parameter function and $g_i(T_i)$ be n one-parameter functions. Then a mean, T, of the T_i with respect to f may be defined by

(a) $f(T_1,..., T_n) = f(T,..., T)$.

This is the definition of Chisini [1]. A somewhat more specialized definition occurs if one chooses

$$f(T_1,..., T_n) = \sum_{i=1}^{n} g_i(T_i)$$

in the above definition. Then the equation for T is

(b) $\sum_{i=1}^{n} g_i(T_i) = \sum_{i=1}^{n} g_i(T)$.

This is still a little more general than the Hardy *et al.* definition in Ref. [2, Chap. 3], which utilizes, in effect,

$$f(T_1,..., T_n) = \sum_{i=1}^{n} p_i g(T_i),$$

the p_i being probabilities. The equation for T is then

(c) $\sum_{i=1}^{n} p_i g(T_i) = g(T)$.

We shall here pursue the intermediate case (b).

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Let $C_i(T)$ $(i = 1,\dots, n)$, $\phi(T)$ be positive functions of a real positive variable T, and let $\phi(T)$ be non-decreasing. Define

$$k_\phi \equiv T[\{C_i\}, \phi; \{T_i\}] \qquad \text{by} \quad \sum_{i=1}^{n} \int_{T_i}^{T} \frac{C_i(t)}{\phi(t)}\, dt = 0. \tag{1}$$

This definition is of type (b) with

$$g_i(T_i) = \int_A^{T_i} \frac{C_i(t)}{\phi(t)}\, dt,$$

A being a constant.

2. An Inequality

We prove next

$$k_1 \equiv T[\{C_i\}, 1; \{T_i\}] \geqslant T[\{C_i\}, \phi; \{T_i\}] \equiv k_\phi\,. \tag{2}$$

Clearly for general $k > 0$

$$X \equiv \sum_{i=1}^{n} \int_{T_i}^{k} \frac{C_i(t)}{\phi(t)}\, dt \geqslant \sum_{\substack{i \\ (T_i \leqslant k)}} \frac{1}{\phi(k)} \int_{T_i}^{k} C_i(t)\, dt - \sum_{\substack{i \\ (T_i > k)}} \frac{1}{\phi(k)} \int_{k}^{T_i} C_i(t)\, dt$$

$$= \frac{1}{\phi(k)} \sum_{i=1}^{n} \int_{T_i}^{k} C_i(t)\, dt \equiv Y. \tag{3}$$

If we chose $k = k_1$ then $Y = 0$ by (1) so that $X \geqslant 0$ by (3). If we chose next $k = k_\phi$, then $X = 0$ by (1). This reduction in X is produced by a reduction in k since $dX/dk > 0$. It follows that $k_\phi \leqslant k_1$ as stated in (2).

3. Special Cases

As usual, let

$$M_r(T, p) \equiv \left[\left(\sum_{i=1}^{n} p_i T_i^{\,r}\right)\Big/ \sum_{i=1}^{n} p_i\right]^{1/r}$$

$$M_r(T) \equiv \left[n^{-1} \sum_{i=1}^{n} T_i^{\,r}\right]^{1/r}$$

$$G(T, p) \equiv [T_1^{p_1} \cdots T_n^{p_n}]^{1/\Sigma p_i}$$

$$G(T) \equiv [T_1 \cdots T_n]^{1/n}$$

Then $M(T)$, $G(T)$, $M_{-1}(T)$ are the usual arithmetic, geometric and harmonic means; $M_1(T, p)$, $G(T, p)$ and $M_{-1}(T, p)$ are the corresponding weighted means. Standard means can be obtained from (1) by appropriate choices of functions C_i and ϕ. By chosing $\phi = 1$ as well, inequality (2) can be used, as seen from Table I.

Another special case is the following. If $l > 0$ is fixed and $k > l$ is a variable upper limit, the function ϕ can be used to define a monotonic function of k,

$$\Phi(k) \equiv \int_l^k \frac{dt}{\phi(t)}.$$

This function is concave provided only ϕ is differentiable and $\phi \geqslant 0$ in the range of integration. Using $C_i(T) = p_i A$, the equation for k_ϕ and k_1 give

$$\Phi(k_\phi) = \sum_i p_i \Phi(T_i) \Big/ \sum_i p_i ,$$

$$k_1 = \sum_i p_i T_i \Big/ \sum_i p_i .$$

Since $k_1 \geqslant k_\phi$ and $\Phi'(k) > 0$ it follows that

$$\Phi\left(\frac{\sum p_i T_i}{\sum p_i}\right) \geqslant \frac{\sum p_i \Phi(T_i)}{\sum p_i} .$$

This result corresponds to Eq. (3.4.3), p. 70, of Ref. [1] (stated there for convex functions).

TABLE I

Inequalities Implied by (2)[a]

	$C_i(T)$	$\phi(T)$	Constraints	$k_1 \geqslant k_\phi$	See Ref. [1] page
1.	$Ap_i T_i^{r+s-1}$	T^s	$r \neq 0$ $r + s \neq 0$ $s > 0$	$M_{r+s}(T, p) \geqslant M_r(T, p)$	26
2.	$\dfrac{Ap_i}{T}$	T	case 1 with $r = -1$, $s = 1$	$G(T, p) \geqslant M_{-1}(T, p)$	13
3.	Ap_i	T	case 1 with $r = 0$, $s = 1$	$M_1(T, p) \geqslant G(T, p)$	13

[a] A and $p_1, \ldots, p_n$ are positive constants.

4. Conclusion

One of many new inequalities which can be otained by this method is the following. Consider, for example, k_1 given by

$$\left(\frac{k_1}{e}\right)^{k_1} = \left[\left(\frac{T_1}{e}\right)^{T_1}\left(\frac{T_2}{e}\right)^{T_2}\cdots\left(\frac{T_n}{e}\right)^{T_n}\right]^{1/n}$$

and

$$k_\phi = \exp\left\{\left[\frac{1}{n}\sum_{i=1}^{n}(\ln T_i)^2\right]^{1/2}\right\}.$$

Then $k_1 \geqslant k_\phi$, and to prove it one may use (2) with $C_i(T) = \ln T$ and $\phi(T) = T$.

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References

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4. P. T. Landsberg, *Phys. Lett. A* **67** (1978), 1; S. S. Sidhu, *Phys. Lett. A* **76** (1980), 107.