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Rank in Noetherian Rings*

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INTRODUCTION

Let R be a commutative Noetherian ring with 1. Nagata has shown that it is possible for there to exist primes $P \subset Q$ in R such that rank $Q >$ rank $(Q/P) +$ rank P. If we call such a pair $P \subset Q$ abnormal, the main result of this paper is as follows: for a fixed prime P there is a strictly increasing chain of ideals $P = I_0 \subset I_1 \subset \cdots \subset I_n$ with the property that for any prime Q containing P, $P \subset Q$ is abnormal if and only if the largest $j = 0, 1,..., n$, with $I_j \subset Q$, is odd. The second part of this paper investigates when in a finitely generated extension of Noetherian rings, $R \subset T$, a prime of T contracts to a prime of R having larger rank.

1. THE BEHAVIOR OF RANK

DEFINITIONS. A containment $P_1 \subset P_2$ of prime ideals in some ring will be said to be *normal* if rank P_2 = rank(P_2/P_1) + rank P_1. Otherwise it will be said to be *abnormal*. More specifically, if rank P_2 = rank P_2/P_1 + rank $P_1 + k$ then we will say that $P_1 \subset P_2$ is *k-abnormal*. Alternatively we will call k the *degree of abnormality*. We allow the case $k = 0$, thus equating 0-abnormality with normality. The phrase "*almost all*" will mean all but finitely many.

The proof of the next lemma is straightforward and is left to the reader.

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Lemma 1.1. *Let $P_1 \subset P_2 \subset P_3$ be prime ideals in some ring. If the degrees of abnormality of $P_1 \subset P_2$, $P_2 \subset P_3$, $P_1 \subset P_3$ and $P_2/P_1 \subset P_3/P_1$ are respectively k_1, k_2, k_3, and k_4 then $k_1 + k_2 = k_3 + k_4$.*

Proposition 1.2. *Let P be a prime in a Noetherian ring R. Let $W = \{Q \mid Q$ is prime, $P \subset Q$ and $P \subset Q$ is abnormal$\}$. Then P is a proper subset of $\cap \{Q \in W\}$.*

Proof. Suppose that rank $P = r$ and that $a_1, \ldots, a_r$ are elements of R such that P is minimal over $(a_1, \ldots, a_r)$, [1, Theorem 153]. Let $P = P_1, P_2, \ldots, P_n$ be all the primes of R minimal over $(a_1, \ldots, a_r)$. We will show that each $Q \in W$ contains one of $P_2, \ldots, P_n$. This will show that $\cap \{Q \in W\}$ contains not only P but also $P_2 \cap \cdots \cap P_n$, and so prove the proposition.

For $Q \in W$, rank $Q >$ rank(Q/P) + rank P = rank$(Q/P) + r$. However by [1, Theorem 154], rank $Q \leqslant$ rank$[Q/(a_1, \ldots, a_r)] + r$. Thus rank$[Q/(a_1, \ldots, a_r)] >$ rnak(Q/P). Therefore, Q must contain one of $P_2, \ldots, P_n$.

Notation. Let Q' be a prime and let W' be an infinite set of primes all of which contain Q'. If for every infinite subset W'' of W' we have $Q' = \cap \{Q \in W''\}$ then we will call (Q', W') a *conforming pair.*

Remark. Notice that if $P \subset Q'$ and (Q', W') is a conforming pair, then $(Q'/P, \{Q/P \mid Q \in W'\})$ is also a conforming pair.

Lemma 1.3. *Let I be an ideal in a Noetherian ring R and let W be an infinite set of primes all of which contain I. Then there is a conforming pair (Q', W') with $I \subset Q'$ and $W' \subset W$.*

Proof. Expand I to an ideal Q' maximal with respect to being contained in infinitely many members of W. That Q' is prime is straightforward.

Let $W' = \{Q \in W \mid Q' \subset Q\}$. If W'' is an infinite subset of W' then by the maximality of Q', $Q' = \cap \{Q \in W''\}$, showing that (Q', W') is a conforming pair.

Lemma 1.4. *Let (Q', W') be a conforming pair in a Noetherian ring R. Then for almost all $Q \in W'$ we have that $Q' \subset Q$ is normal.*

Proof. Let $W'' = \{Q \in W' \mid Q' \subset Q$ is abnormal$\}$. By Proposition 1.2, Q' is a proper subset of $\cap \{Q \in W''\}$. Since (Q', W') is a conforming pair, W'' must be a finite set.

Lemma 1.5. *Let (Q', W') be a conforming pair in a Noetherian ring R. Let P be a prime contained in Q' such that $P \subset Q'$ is l-abnormal. Then for almost all $Q \in W'$, $P \subset Q$ is l-abnormal.*

Proof. Applying Lemma 1.4 to the conforming pairs (Q', W') and $(Q'/P, \{Q/P \mid Q \in W'\})$ we see that for almost all $Q \in W'$ we have both $Q' \subset Q$ and $Q'/P \subset Q/P$ normal. By Lemma 1.1, it follows that since $P \subset Q'$ is l-abnormal, $P \subset Q$ is also l-abnormal for almost all $Q \in W'$.

THEOREM 1.6. *Let P be prime in the Noetherian ring R. Then $\{k \mid$ there is a prime Q containing P with $P \subseteq Q$ k-abnormal$\}$ is finite.*

Proof. Suppose not. Then for each of infinitely many distinct positive integers k, we may pick a prime Q_k containing P such that $P \subset Q_k$ is k-abnormal. Let W be the infinite set of Q_k thus chosen. By Lemma 1.3, there is a conforming pair (Q', W') with $P \subset Q'$ and $W' \subset W$. By Lemma 1.5, for almost all $Q_k \in W'$, the degree of abnormality of $P \subset Q_k$ equals the degree of abnormality of $P \subset Q'$. This contradicts the fact that for distinct k's, the degrees of abnormality of $P \subset Q_k$ are distinct.

LEMMA 1.7. *Let P be a prime in the Noetherian ring R and let V be a subset of $\{k \mid$ there is a prime Q containing P with $P \subset Q$ k-abnormal$\}$. Suppose that I is an ideal with $P \subset I$. Let $W = \{Q \mid Q$ is prime, $I \subset Q$, and $P \subset Q$ is k-abnormal with $k \in V\}$. Then W has only finitely many minimal members.*

Proof. Let W_1 be the set of minimal members of W. If W_1 is infinite, by Lemma 1.3, there is a conforming pair (Q', W') with $I \subset Q'$ and $W' \subset W_1$. By Lemma 1.5, for almost all $Q \in W'$ the degree of abnormality of $P \subset Q$ equals the degree of abnormality of $P \subset Q'$. Since for $Q \in W' \subset W_1 \subset W$ the degree of abnormality of $P \subset Q$ is in V, we have that the degree of abnormality of $P \subset Q'$ is in V. That is, $Q' \in W$. This contradicts the fact that Q' is contained in infinitely many minimal members of W.

COROLLARY 1.8. *Let P be prime in the Noetherian ring R. Let V be a subset of $\{k \mid$ there is a prime Q containing P such that $P \subset Q$ is k-abnormal$\}$. Then the set $\{Q \mid Q$ is prime, $P \subset Q$ and $P \subset Q$ is k-abnormal with $k \in V\}$ has only finitely minimal members.*

Proof. This is a special case of Lemma 1.7 with $I = P$.

COROLLARY 1.9. *Let P be prime in the Noetherian ring R. The set $\{Q \mid Q$ is prime, $P \subset Q$ and $P \subset Q$ is abnormal$\}$ has only finitely many minimal members.*

Proof. This is a special case of the last corollary with $V = \{k > 0 \mid$ there is a prime Q containing P with $P \subset Q$ k-abnormal$\}$.

COROLLARY 1.10. ([2, Theorem 1]). *Let P be prime in the Noetherian ring R. Then for almost all Q satisfying $P \subset Q$, and $\operatorname{rank}(Q/P) = 1$, we have $\operatorname{rank} Q = \operatorname{rank} P + 1$.*

Proof. For Q containing P with rank $Q/P = 1$, if rank $Q >$ rank $P + 1$ the Q is clearly a minimal member of the set $\{Q \mid Q$ is prime, $P \subset Q$ and $P \subset Q$ is abnormal$\}$.

THEOREM 1.11. *Let P be a prime in the Noetherian ring R and let $V_1 \cup V_2$ be a disjoint partition of $\{k \mid$ there is a prime Q containing P with $P \subset Q$ k-abnormal$\}$. Suppose further that $0 \in V_1$. Then there is a strictly increasing chain of ideals $P = I_0 \subset I_1 \subset I_2 \subset \cdots \subset I_n$ with the following property: For a prime Q containing P let j be the largest of $0, 1, 2, \ldots, n$ such that $I_j \subset Q$. If $P \subset Q$ is k-abnormal then $k \in V_1$ if and only if j is even and $k \in V_2$ if and only if j is odd.*

Remark. $P \subset P$ is 0-abnormal so 0 is in $V_1 \cup V_2$, and we may assume that it is in V_1.

Proof. Let $I_0 = P$. Suppose that I_j has been constructed. We will inductively construct I_{j+1}. Suppose that j is even (for j odd, do the following construction using V_1 rather than V_2). Let $W = \{Q \mid Q$ is prime, $I_j \subset Q$, and $P \subset Q$ is k-abnormal with $k \in V_2\}$. If W is empty the chain stops. If W is not empty, by Lemma 1.7, W has only finitely many minimal members. Let I_{j+1} be their intersection. Clearly $I_j \subset I_{j+1}$. In fact, since I_{j+1} is a finite intersection of primes ,Q, satisfying $P \subset Q$ is k-abnormal with $k \in V_2$, and (inductively) I_j is a finite intersection of primes, Q, satisfying $P \subset Q$ is k-abnormal with $k \in V_1$, we have that I_j is a proper subset of I_{j+1}. Since R is Noetherian our chain eventually stops. We now show that this chain has the stated property. Let Q be a prime containing P and let j be the largest of $1, 2, \ldots, n$ such that $I_j \in Q$. Let $P \subset Q$ be k-abnormal. We must show that if j is even then $k \in V_1$ and if j is odd then $k \in V_2$. Assume that j is even, the other case being symmetric. If $k \notin V_1$ then $k \in V_2$. Since $I_j \subset Q$ we may find a prime Q' with $I_j \subset Q' \subset Q$, $P \subset Q'$ is l-abnormal with $l \in V_2$ and Q' is minimal in the set $\{Q'' \mid Q''$ is prime, $I_j \subset Q''$ and $P \subset Q''$ is m-abnormal with $m \in V_2\}$. By construction, $I_{j+1} \subset Q'$. Thus $I_{j+1} \subset Q' \subset Q$, violating the maximality of j.

COROLLARY 1.12. *Let P be a prime in the Noetherian ring R. Then there is a strictly increasing chain of ideals $P = I_0 \subset I_1 \subset \cdots \subset I_n$ with the following property: If Q is any prime containing P and if j is the largest of $0, 1, 2, \ldots, n$ such that $I_j \subset Q$, then $P \subset Q$ is normal if and only if j is even.*

Proof. Immediate from 1.11 using $V_1 = \{0\}$.

Open Questions. (i) In a Noetherian ring, is $\{k \mid$ there are primes $P \subset Q$ with $P \subset Q$ k-abnormal$\}$ finite? (ii) Let P be a prime in a Noetherian ring. Let $W = \{Q \mid Q$ is prime, $P \subset Q$ and $P \subset Q$ is abnormal$\}$. Let W_1 be the set of minimal members of W. Is $\{k \mid \text{there is a } Q \in W \text{ with } P \subset Q \text{ } k\text{-abnormal}\}$ equal to $\{k \mid \text{there is a } Q \in W_1 \text{ with } P \subset Q \text{ } k\text{-abnormal}\}$?

THEOREM 1.13. *Let R be a Noetherian ring and let k be the degree of abnormality of some pair of primes of R. Then there is a G-ideal, Q, and a prime $P \subset Q$ with $\operatorname{rank}(Q/P) = 1$ and with $P \subset Q$ k-abnormal.*

Remark. By [1, Theorem 146], we see that in a Noetherian ring, G-ideals are large, being either maximal or at least submaximal. Thus our theorem says that all degrees of abnormality in R can be found in pairs of primes near the top of R.

Proof. Suppose that $P_1 \subset P_2$ is k-abnormal. Let Q be maximal with respect to containing P_1 and having $P_1 \subset Q$ k-abnormal. We claim that Q is a G-ideal. If not, by [1, Theorems 146 and 144] there would be infinitely many primes Q' all containing Q and with $\operatorname{rank}(Q'/Q) = 1$. By Corollary 1.10 we could find such a Q' with both $Q \subset Q'$ 0-abnormal and $Q/P_1 \subset Q'/P_1$ 0-abnormal. By Lemma 1.1, we would have $P_1 \subset Q'$ k-abnormal contradicting the maximality of Q. Thus Q is a G-ideal.

Now choose P to the maximal with respect to being contained in Q and with $P \subset Q$ k-abnormal. We claim that $\operatorname{rank}(Q/P) = 1$. If not by [1, Theorem 144] we can find infinitely many primes P' with $P \subset P' \subset Q$, $\operatorname{rank}(P'/P) = 1$ and $\operatorname{rank}(Q/P') = \operatorname{rank}(Q/P) - 1$. Thus $P'/P \subset Q/P$ is 0-abnormal. Furthermore by Corollary 1.10 we may choose such a P' satisfying $P \subset P'$ is 0-abnormal. By lemma 1.1, $P' \subset Q$ is k-abnormal contradicting the maximality of P. This shows that $\operatorname{rank}(P'/P) = 1$ and completes our proof.

THEOREM 1.14. *Let (R, M) be a local ring. Let $k > 0$ be the degree of abnormality of some pair of primes in R. Then there is a submaximal prime P such that $P \subset M$ is l-abnormal with $l \geqslant k$.*

Proof. By Theorem 1.13 there are primes $P_1 \subset P_2$ with P_2 a G-ideal, $\operatorname{rank}(P_2/P_1) = 1$ and $P_1 \subset P_2$ k-abnormal. If P_2 is in fact M, then let $P = P_1$ and we are done. If $P_2 \neq M$, then by [1, Theorem 146] $\operatorname{rank}(M/P_2) = 1$. Let the degrees of abnormality of $P_2 \subset M$, $P_1 \subset M$ and $P_2/P_1 \subset M/P_1$ be, respectively, b, c and d. By Lemma 1.1, $k + b = c + d$. Since $P_1 \subset P_2 \subset M$ with $\operatorname{rank}(P_2/P_1) = 1 = \operatorname{rank}(M/P_2)$, by [5, Proposition 2.2] and Corollary 1.10, there is a prime P satisfying $P_1 \subset P \subset M$, $\operatorname{rank}(P/P_1) = 1 = \operatorname{rank}(M/P)$ and $P_1 \subset P$ is 0-abnormal. It is also clear that $P/P_1 \subset M/P_1$, like $P_2/P_1 \subset M/P_1$, is d-abnormal. Let $P \subset M$ be l-abnormal. Our goal is to show that $l \geqslant k$. However by Lemma 1.1, $0 + l = c + d$. Thus $0 + l = c + d = k + b \geqslant k$.

We close this section by giving an alternate proof of [3, Proposition 7].

THEOREM 1.15. *Let (R, M) be a local ring satisfying rank $P +$ corank $P = \dim R$ for all primes P of R. Then R satisfies the first chain condition.*

Proof. The assumption on R implies that for any prime P, $P \subset M$ is normal. Theorem 1.14 now easily shows that any pair of primes $Q_1 \subset Q_2$ must be normal. Suppose that R does not satisfy the first chain condition. Then there is a saturated chain of primes $P_0 \subset P_1 \subset \cdots \subset P_{n-1} \subset M$ with P_0 minimal and $n < \dim R = \operatorname{rank} M$. By [2, Theorem 5] we may assume that $\operatorname{rank}(P_{n-1}/P_0) = n - 1$. Since $n < \dim R = \operatorname{rank} P_{n-1} + \operatorname{corank} P_{n-1} = \operatorname{rank} P_{n-1} + 1$, $\operatorname{rank} P_{n-1} > n - 1$. This shows that $P_0 \subset P_{n-1}$ is abnormal, a contradiction.

2. BEHAVIOR OF PRIME CONTRACTIONS IN $R \subset T$

PROPOSITION 2.1. *Let $A \subset B$ be a finitely generated extension of Noetherian integral domains. Let $W = \{Q$ prime in $B \mid \operatorname{rank}(Q \cap A) > \operatorname{rank} Q\}$. If $W \neq \varnothing$ then $\cap \{Q \in W\} \neq 0$.*

Proof. It is easy to see that by induction we may assume that B is generated over A by a single element. If that element is transcendental over A, it is well known that W is empty. Therefore, assume that $B = A[u]$ with u algebraic over A and let K be the kernel of the map from $A[x]$ to $A[u] = B$ sending the indeterminate x to u. Since B is a domain and u is algebraic over A, K is a nonzero prime in $A[x]$. Also, $K \cap A = 0$. We may identify $A[u]$ with $A[x]/K$. For a prime Q in W (assume $W \neq \varnothing$) suppose that under the above identification, Q identifies with Q'/K in $A[x]/K$ where Q' is a prime of $A[x]$. Since Q is in W we must have $\operatorname{rank}(Q' \cap A) > \operatorname{rank}(Q'/K)$. Our task is thus, to show that if $W' = \{Q'$ prime in $A[x] \mid K \subset Q'$ and $\operatorname{rank}(Q' \cap A) > \operatorname{rank}(Q'/K)\}$, then $\cap \{Q' \in W'\}$ is strictly larger than K. We partition W' into two disjoint subsets $W_1' = \{Q' \in W' \mid Q' = (Q' \cap A)\, A[x]\}$ and $W_2' = \{Q' \in W' \mid Q' \neq (Q' \cap A)\, A[x]\}$. Certainly since K is prime, it will be enough to show that for $i = 1, 2$, K is properly contained in $\cap \{Q' \in W_i'\}$. Consider any nonzero polynomial $f(x)$ in K and let c be a nonzero coefficient of $f(x)$. For $Q' \in W_1'$, since $f(x) \in K \subset Q'$ and $Q' = (Q' \cap A)\, A[x]$ we have $c \in Q'$. Thus $c \in \cap \{Q' \in W_1'\}$. However, $K \cap A = 0$ so that $c \notin K$, and so $\cap \{Q' \in W_1'\}$ properly contains K. For $Q' \in W_2'$, let $P = Q' \cap A$. We then know that $PA[x] \neq Q'$ so that $\operatorname{rank} Q' = \operatorname{rank} P + 1$ [1, Theorem 149] and $\operatorname{rank} P = \operatorname{rank}(Q' \cap A) > Q'/K$. However, since $K \neq 0$ but $K \cap A = 0$, $\operatorname{rank} K = 1$. Thus $\operatorname{rank} Q' = \operatorname{rank} P + 1 > \operatorname{rank} Q'/K + 1 = \operatorname{rank} Q'/K + \operatorname{rank} K$, showing that for any $Q' \in W_2'$, $K \subset Q'$ is abnormal. By Proposition 1.2, $\cap \{Q' \subset W_2'\}$ properly contains K and we are done.

LEMMA 2.2. *Let $R \subset T$ be a finitely generated extension of Noetherian rings. Let (Q', W') be a conforming pair in T. Then for almost all $Q \in W'$ we have* $\operatorname{rank}((Q \cap R)/(Q' \cap R)) \leqslant \operatorname{rank}(Q/Q')$. *If $R \subset T$ satisfies incomparability the inequality can be replaced with equality.*

Proof. Let $W'' = \{Q \in W' \mid \operatorname{rank}((Q \cap R)/(Q' \cap R) > \operatorname{rank}(Q/Q')\}$. By Proposition 2.1. applied to $R/(Q' \cap R) \subset T/Q'$, we see that Q' is properly contained in $\cap \{Q \in W''\}$. Because (Q', W') is a conforming pair, W'' must be finite. This proves the first part of the lemma. If $R \subset T$ has incomparability then $\operatorname{rank}(Q \cap R)/(Q' \cap R) \nless \operatorname{rank}(Q/Q')$ and the inequality becomes equality.

LEMMA 2.3. *Let $R \subset T$ be a finitely generated extension of Noetherian rings. Let (Q', W') be a conforming pair in T. Then for almost all $Q \in W'$ we have $Q' \cap R \subset Q \cap R$ normal.*

Proof. Let $W'' = \{Q \in W' \mid Q' \cap R \subset Q \cap R$ is abnormal$\}$. By Proposition 1.2, $Q' \cap R$ is properly contained in $\cap \{Q \cap R \mid Q \in W''\}$. Thus Q' is properly contained in $\cap \{Q \in W''\}$. Since (Q', W') is a conforming pair, W'' must be finite.

LEMMA 2.4. *Let $R \subset T$ be a finitely generated extension of Noetherian rings. Let (Q', W') be a conforming pair in T. If $\operatorname{rank}(Q' \cap R) = \operatorname{rank} Q' + l$ then for almost all $Q \in W'$, $\operatorname{rank}(Q \cap R) = \operatorname{rank} Q + k$ with $k \leqslant l$. If $R \subset T$ satisfies incomparability then for almost all $Q \in W'$, $\operatorname{rank}(Q \cap R) = \operatorname{rank} Q + l$.*

Proof. By Lemmas 1.4, 2.3 and 2.2, for almost all $Q \in W'$ we have $Q' \subset Q$ normal, $Q' \cap R \subset Q \cap R$ normal and $\operatorname{rank}((Q \cap R)/(Q' \cap R)) \leqslant \operatorname{rank} Q/Q'$. Thus for almost all $Q \in W'$ we have $\operatorname{rank}(Q \cap R) = \operatorname{rank}((Q \cap R)/(Q' \cap R)) + \operatorname{rank}(Q' \cap R) \leqslant \operatorname{rank}(Q/Q') + \operatorname{rank}(Q' \cap R) = \operatorname{rank}(Q/Q') + \operatorname{rank} Q' + l = \operatorname{rank} Q + l$. Therefore $\operatorname{rank}(Q \cap R) = \operatorname{rank} Q + k$ for some $k \leqslant l$.

If $R \subset T$ satisfies incomparability, by Lemma 2.2, the only inequality in the last paragraph becomes an equality, which proves the last statement in the lemma.

THEOREM 2.5. *Let $R \subset T$ be a finitely generated extension of Noetherian rings. Then $\{k \geqslant 0 \mid$ there is a prime Q of T with $\operatorname{rank}(Q \cap R) = \operatorname{rank} Q + k\}$ is finite.*

Proof. Suppose not. Then for infinitely many distinct positive integers k we can find a prime Q_k of T such that $\operatorname{rank}(Q_k \cap R) = \operatorname{rank} Q_k + k$. Let W be the infinite set of the Q_k so chosen. By Lemma 1.3, with $I = 0$, there is a conforming pair (Q', W') with $W' \subset W$. If $\operatorname{rank}(Q' \cap R) = \operatorname{rank} Q' + l$ then by Lemma 2.4, for almost all $Q_k \in W'$ we have $k \leqslant l$. This contradicts that there are infinitely many distinct positive integers k with $Q_k \in W'$.

Remark. With $R \subset T$ as in 2.5, it is also true that $\{k < 0 \mid$ there is a prime Q of T with $\operatorname{rank}(Q \cap R) = \operatorname{rank} Q + k\}$ is finite. In fact, this set is bounded below by $-n$, where n is the number of generators of T over R. This follows easily from [1, Theorem 149] and induction. We, however, are only concerned with the positive k's. If $R \subset T$ satisfies incomparability, there are no negative k's.

THEOREM 2.6. *Let $R \subset T$ be a finitely generated extension of Noetherian rings. Let $W = \{Q \mid Q$ is prime in T and $\operatorname{rank}(Q \cap R) > \operatorname{rank} Q\}$. Then W has only finitely many minimal members.*

Proof. Suppose that W_1 is the set of minimal members of W. If W_1 is infinite then by Lemma 1.3. there is a conforming pair of T, (Q', W') with $W' \subset W_1 \subset W$. By Lemma 2.4, for almost all $Q \in W'$ we have $\operatorname{rank}(Q' \cap R) - \operatorname{rank} Q' \geqslant \operatorname{rank}(Q \cap R) - \operatorname{rank} Q > 0$ showing that $Q' \in W$ and therefore contradicting that Q' is contained in infinitely many minimal members of W.

The next corollary extends [2, Theorem 7].

COROLLARY 2.7. *Let $R \subset T$ be a finitely generated extension of Noetherian domains. Then almost all rank 1 primes of T contract to rank 1 primes of R.*

Proof. Since 0 contracts to 0, any rank 1 prime of T contracting to a larger rank is minimal in the set given in 2.6.

Remark. Theorem 2.6 fails for

$$\{Q \mid Q \text{ is prime in } T \text{ and } \operatorname{rank}(Q \cap R) < \operatorname{rank} Q\}$$

as is shown by $F \subset F[x]$ with F a field. More subtly, it fails if we do not consider all of $\{Q \mid Q$ is prime in T and $\operatorname{rank}(Q \cap R) > \operatorname{rank} Q\}$, but only consider $\{Q \mid Q$ is prime in T and $\operatorname{rank}(Q \cap R) = \operatorname{rank} Q + k\}$ for some fixed positive integer k. However, if $R \subset T$ also satisfies incomparability, then even this last set has only finitely many minimal members, as we shall show. First, however, we demonstrate why incomparability is needed.

EXAMPLE 2.8. Let (R, M) be a 3-dimensional local domain which has a prime having rank 1 and corank 1 [4, Example 2, pp. 202–205]. By [2, Theorem 4], there is a prime K in the polynomial ring $R[x]$ such that $K \neq 0$, $K \cap R = 0$, $K \subset M^* = MR[x]$ and $\operatorname{Rank}(M^*/K) = 1$. If $T = R[x]/K$ then T is a finitely generated algebraic extension of R. By [1, Section 1–5] there are infinitely many primes Q of $R[x]$ containing M^* and satisfying rank$(Q/M^*) = 1$. By Corollary 1.10 applied to M^*/K, we have that infinitely many of those Q satisfy rank$(Q/K) = 2$. Of course, those Q also satisfy $Q \cap R = M$ so that the rank 2 primes Q/K of T contract to the rank 3 prime M of R. That is, we have produced infinitely many primes Q of $R[x]$ containing M^* for which rank$(Q/K) \cap R =$ rank$(Q/K) + 1$. Suppose that the set of primes of T which increase in rank by 1 upon contraction to R has only finitely many minimal members. Then there must be a prime Q' of $R[x]$ containing K such that rank$(Q'/K) \cap R) =$ rank$(Q'/K) + 1$ and which is contained in infinitely many of the Q/K described above. In particular Q' is contained in infinitely many prime Q of $R[x]$ which also contain M^*. It follows from [1, Section 1–5] that $Q' \subset M^*$. We now have $K \subset Q' \subset M^*$. Since rank$(Q'/K) \cap R) =$ rank$(Q'/K) + 1, Q' \neq K$. Since rank$(M^*/K) = 1$, we have $Q' = M^*$. However $(M^*/K) \cap R = M$ so that rank$((Q'/K) \cap R) =$ rank$((M^*/K) \cap R) =$ rank $M = 3$ while rank$(Q'/K) + 1 =$ rank$(M^*/K) + 1 = 1 + 1 = 2$, a contradiction.

LEMMA 2.9. *Let $R \subset T$ be a finitely generated extension of Noetherian rings which satisfies incomparability. Let V be a subset of $\{k \mid$ there is a prime Q of T with rank$(Q \cap R) =$ rank $Q + k.\}$. Suppose that I is an ideal of T and that $W = \{Q$ prime in $T \mid I \subset Q$ and rank$(Q \cap R) =$ rank $Q + k$ with $k \in V\}$. Then W has only finitely many minimal members.*

Remark. Incomparability insures that all k involved are nonnegative.

Proof. Let W_1 be the set of minimal members of W. Assume that W_1 is infinite. By Lemma 1.3, T has a conforming pair (Q', W') with $I \subset Q'$ and $W' \subset W_1$. By Lemma 2.4, for almost all $Q \in W'$ we have rank$(Q' \cap R) -$ rank $Q' =$ rank$(Q \cap R) -$ rank Q. As the right-hand side of this equation is in V, Q' is in W, contradicting that it is contained in infinitely many minimal members of W.

THEOREM 2.10. *Let $R \subset T$ be a finitely generated extension of Noetherian rings which satisfies incomparability. Let $V_1 \cup V_2$ be a disjoint partition of $\{k \mid$ there is a prime Q of T with rank$(Q \cap R) =$ rank $Q + k\}$. Assume that $0 \in V_1$. Then there is a strictly increasing chain of ideals $0 = I_0 \subset I_1 \subset \cdots \subset I_n$ with the following property: For any prime Q of T let j be the largest of $1, 2, \ldots, n$ such that $I_j \subset Q$. If rank$(Q \cap R) =$ rank $Q + k$, then $k \in V_1$ if and only if j is even while $k \in V_2$ if and only if j is odd.*

Remark. By [1, Exercise 1, p. 41] there is a minimal prime of T which contracts to a minimal prime of R. Thus $0 \in V_1 \cup V_2$ and we may assume that $0 \in V_1$.

Proof. The proof is completely analogous to that of Theorem 1.11.

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