Noncommutative Localization and Sheaves

D. C. Murdoch

University of British Columbia, Vancouver VGT IW5, Canada

AND

F. Van Oystaeyen

University of Antwerp, 2610 Wilrijk, Belgium

Communicated by A. W. Goldie

Received March 12, 1974

1. Introduction

In [1], Oscar Goldman has introduced the concept of a kernel functor on the category of left $R$-modules, where $R$ is an arbitrary ring. A kernel functor $\sigma$ is characterized by a filter $\mathcal{F}_{\sigma}$ in $R$, consisting of all left ideals $A$ of $R$ for which $\sigma(R/A) = R/A$. If $\sigma$ is idempotent, that is $\sigma(M/\sigma(M)) = 0$ for all left $R$-modules $M$, then for each $M$ there exists a module of quotients $Q_{\sigma}(M)$ and a canonical homomorphism $i: M \to Q_{\sigma}(M)$, the composite of the canonical map $M \to M/\sigma(M)$ and the injection $M/\sigma(M) \to Q_{\sigma}(M)$. For a $\sigma$-torsion free module $M$, the module $Q_{\sigma}(M)$ can be obtained by suitably defining the action of elements of $R$ on the direct limit of the directed system $\{\text{Hom}_R(A_i, M), A_i \in \mathcal{F}_{\sigma}\}$ in which, for $A_i \supset A_j$, $\pi_{ij}: \text{Hom}_R(A_i, M) \to \text{Hom}_R(A_j, M)$ is defined by $\pi_{ij}f = f$ restricted to $A_j$. It follows at once that $Q_{\sigma}$ is a functor on the category of left $R$-modules and is left exact. It is in general not right exact. Right exactness is assured if $\sigma$ has property (T) discussed by Goldman. The results in Goldman's paper will be assumed as needed and the reader is referred to [1] for details. In this paper we first introduce a special class of idempotent kernel functors which we call symmetric. These are characterized by the fact that every open left ideal contains an open ideal of $R$. In Section 4, we study the correspondence between left ideals of $R$ not in $\mathcal{F}(\sigma)$ and left ideals of $Q_{\sigma}(R)$. Under additional conditions on $R$ and $\sigma$, prime ideals correspond one-to-one to prime ideals under this correspondence. Then we take $R$ to be a prime left-Noetherian ring and let $X := \text{Spec } R$ be the set of proper prime ideals of $R$ with the usual Zariski topology, that is if for any ideal $A$
we set $V(A) = \{P \in X, P \supset A\}$ and the open sets in $X$ have the form $X_A = X - V(A)$. To every open set $X_A$ there is a topology $\mathcal{T}_A$ in $R$ such that $\mathcal{T}_A$ is the filter corresponding to a symmetric kernel functor $\sigma_A$. The quotient rings $Q_A(R)$ associated with the $\sigma_A$ form a structure sheaf on $X$. A set $X_A$ with the property that $\sigma_A$ has property (T) will be called a $T$-set. Using $T$-sets we deduce some properties analogous to the properties of affine schemes, although Spec does not enjoy the same functorial properties as in the commutative case.

2. Symmetric Kernel Functors

We assume throughout that ring means ring with unity, $R$-module means unital left $R$-module and ideal means two-sided ideal unless qualified by left or right.

**Definition.** A kernel functor $\sigma$ is said to be **bilateral** if every left ideal in $\mathcal{T}_\sigma$ contains an ideal in $\mathcal{T}_\sigma$.

**Proposition 1.** If $R$ is left-Noetherian, a bilateral kernel functor is idempotent if and only if, when $A_1, A_2 \in \mathcal{T}_\sigma$, then $A_1A_2 \in \mathcal{T}_\sigma$.

**Proof.** Goldman proved that idempotence implies that $\mathcal{T}_\sigma$ is multiplicatively closed. Conversely suppose $\sigma$ is bilateral and $\mathcal{T}_\sigma$ is multiplicatively closed. If $A \in \mathcal{T}_\sigma$ and $\sigma(A/B) = A/B$ for some $B \subseteq A$, then it is sufficient to show that $B \in \mathcal{T}_\sigma$. Since $\sigma(A/B) = A/B$, for every $a \in A$ there is a $C \in \mathcal{T}_\sigma$ such that $Ca \subseteq B$. Since $R$ is left-Noetherian $A = Ra_1 + \cdots + Ra_n$. Choose $C_i \in \mathcal{T}_\sigma$ such that $C_i a_i \subseteq B$. Because $\sigma$ is bilateral, the $C_i$ may be chosen to be ideals. Then if $C = \cap_i C_i$ we have $CA \subseteq C a_1 + \cdots + C a_n \subseteq B$ and since $CA \in \mathcal{T}_\sigma$, $B \in \mathcal{T}_\sigma$ and $\sigma$ is idempotent.

**Definition.** A kernel functor $\sigma$ is said to be **symmetric** if it is bilateral and idempotent.

Thus if $R$ is left-Noetherian symmetric kernel functors correspond to multiplicatively closed topologies.

Kernel functors are partially ordered by writing $\sigma \leq \tau$ if $\sigma(M) \subseteq \tau(M)$ for every left $R$-module $M$ or equivalently, if $\mathcal{T}_\sigma \subseteq \mathcal{T}_\tau$. Let $\{\sigma_\alpha, \alpha \in I\}$ be any set of kernel functors. It is clear (see [1]) that $\sigma = \inf \sigma_\alpha$ is given by $\sigma(M) = \cap_{\alpha \in I} \sigma_\alpha(M)$, $\alpha \in I$. If we assume that each $\sigma_\alpha$ is symmetric, we can easily characterize $\rho = \sup \sigma_\alpha$ by identifying $\mathcal{T}_\rho$. To do this let $\mathcal{T}$ be the set of all left ideals in $\bigcup_{\alpha \in I} \mathcal{T}_\alpha$, together with all finite products $\prod_{A_\alpha \in \mathcal{T}_\alpha}$, $A_\alpha \in \mathcal{T}_\alpha$, and all left ideals containing such products. It is easy to check that $\mathcal{T}$ defines a topology in $R$, and a basis for the open sets is given exactly by all
finite products $\prod A_i$, $A_i \in \mathcal{F}_a$, $A_i$ ideals. Hence the associated kernel functor $\rho$ defined by $\rho(M) = \{m \in M \mid Bm = 0 \text{ for some } B \in \mathcal{F}\}$ is symmetric and has the properties required of $\sup \sigma_s$. By construction $\mathcal{F}_\rho = \mathcal{F}$.

3. Kernel Functors Associated with Prime Ideals

We associate a kernel functor $\sigma_s$ with an arbitrary $m$-system $S$ by defining

$$\sigma_s(M) = \{m \in M \mid sRm = 0 \text{ for some } s \in S\}.$$  

Suppose $s_1Rm_1 = 0$ and $s_2Rm_2 = 0$. Since $S$ is an $m$-system we can choose $s = s_1x_2 \in S$ and $sR(m_1 + m_2) = 0$. It easily follows that $\sigma_s(M)$ is a submodule of $M$ and that $\sigma_s$ is a kernel functor.

Since $sRm = 0$ if and only if $m$ is annihilated by the principal ideal $(s)$, the topology $\mathcal{F}_s$ corresponding to $S$ consists of all left ideals that contain a principal ideal generated by an element of $S$. Because $S$ is an $m$-system $\mathcal{F}_s$ is multiplicatively closed. Proposition 1 therefore yields the following.

**Proposition 2.** Let $S$ be an $m$-system in a left-Noetherian ring $R$. Then $\sigma_s$, as defined above, is a symmetric kernel functor whose topology $\mathcal{F}_s$ consists of all left ideals $A$ in $R$ such that $A \supset (s)$ for some $s \in S$.

It should be observed that in the absence of the left-Noetherian condition, $\sigma_s$ is still bilateral but not necessarily idempotent. In this case $\tilde{\sigma}_s$ can be defined by

$$\tilde{\sigma}_s(M) = \{\bigcap N \mid N \subseteq M \text{ and } sRm \subseteq N, s \in S \Rightarrow m \in N\}$$

and $\tilde{\sigma}_s$ is then idempotent but presumably not necessarily symmetric. We remark also that $\tilde{\sigma}_s(R)$ and $\sigma_s(R)$ are respectively the upper and lower $S$-components of the zero ideal as defined in [3].

If $S$ is the complement $R - P$ of a prime ideal $P$ we write $\sigma_S = \sigma_{R - P}$ and call it the kernel functor associated with $P$. Its relationship to the torsion theory at $P$ studied by Lambek and Michler [2] will be investigated elsewhere [4].

4. Localization

Assume that $R$ is left-Noetherian and let $\sigma$ be any symmetric kernel functor with property $(T)$. We denote the ring of quotients $Q_\sigma(R)$ by $Q_\sigma$ and let $i: R \to Q_\sigma$ be the canonical homomorphism. If $A$ is a left ideal in $R$ we denote
by $A^e$ the extension $Q_\sigma i(A)$ to $Q_\sigma$, and if $B$ is a left ideal in $Q_\sigma$ we let $B^e = i^{-1}B$ be the contraction of $B$ to $R$.

**Theorem 3.** Let $R$ be left-Noetherian, $\sigma$ a symmetric kernel functor with property (T).

(a) For every left ideal $B$ of $Q_\sigma$, $B^e = B$,

(b) For every left ideal $A$ of $R$, $A^{ee} = A_\sigma = \{x \in R \mid Cx \subseteq A \text{ for some } C \in \mathcal{T}_\sigma\}$.

(Since $C$ may be taken to be an ideal, $A_\sigma$ is a left ideal.)

**Proof.**

(a) If $b \in B$, there is a $C \in \mathcal{T}_\sigma$ such that $Cb \subseteq i(R)$. By property (T), $Q_\sigma C = Q_\sigma$ hence $Q_\sigma b \subseteq B^e$, or $B = B^e$.

(b) To prove this we note that $A^{ee} = i^{-1}[Q_\sigma i(A)]$. If $x \in A_\sigma$ then $Cx \subseteq A$ with $C \in \mathcal{T}_\sigma$ and $C$ may be taken to be an ideal since $\sigma$ is symmetric. Then $Q_\sigma i(C) i(x) \subseteq Q_\sigma i(A)$ and by property (T) $i(x) \in Q_\sigma i(A) = A^e$ or $x \in A^{ee}$ and $A_\sigma \subseteq A^{ee}$.

Conversely, if $x \in A^{ee}$, $i(x) \in Q_\sigma i(A)$, so $i(x) = \sum q_j a_j$, where $q_j \in Q_\sigma$ and $a_j \in i(A)$. Since $Q_\sigma i(R)$ is a $\sigma$-torsion module, there is an ideal $C$ in $\mathcal{T}_\sigma$ such that $Cq_j \subseteq i(R)$ for all $j$, so $Ci(x) \subseteq i(A)$ and $Cx \subseteq A + \alpha(R)$. But, again by the left-Noetherian property we can find an ideal $C'$ in $T_\sigma$ such that $C'\sigma(R) = 0$ and hence $C'Cx \subseteq C'A \subseteq A$. Since $\sigma$ is idempotent $C'C \in \mathcal{T}_\sigma$ and $x \in A_\sigma$, so $A^{ee} = A_\sigma$.

If $A$ is an ideal in $R$ then $A_\sigma$ is also an ideal in $R$ but $A^e$ is not necessarily and ideal in $Q_\sigma$. A counter-example, pointed out to the authors by A. G. Heinicke, is provided by Example 8 of Goldman's paper ([1, p. 47]). However this property (that each ideal $A$ of $R$ extends to an ideal $A^e$ of $Q_\sigma$) does hold for a large class of left-Noetherian prime rings, for example, for all those in which every ideal is generated by central elements of $R$, and in particular for any complete matrix ring over a commutative Noetherian prime ring with unity. This follows because in a prime ring every ideal is essential as a left ideal and so if $\sigma$ is symmetric every left ideal in $\mathcal{T}_\sigma$ is essential. Hence $Q_\sigma$ is a subring of the classical full ring of left quotients and central elements of $R$ are also central in $Q_\sigma$.

In view of this it seems worthwhile to study further the correspondence between the ideals of $R$ and the ideals of $Q_\sigma$ under the additional assumption that $R$ is "$\sigma$-perfect" in the sense of the following.

**Definition.** A left-Noetherian ring $R$ is said to be $\sigma$-perfect with respect
to a symmetric kernel functor $\sigma$ having property (T) if every ideal $A$ of $R$ extends to an ideal $A^e$ of $Q_\sigma$.

Although it seems too much to expect that a left-Noetherian prime ring is $\sigma$-perfect for every symmetric kernel functor $\sigma$ having property (T), we have not been able to find a counter-example.

**Proposition 4.** Suppose $\sigma$ has property (T) and $R$ is $\sigma$-perfect. If $P \notin T_\sigma$ is a prime ideal of $R$ then $P^e$ is a proper prime ideal of $Q_\sigma$.

**Proof.** Since $R$ is $\sigma$-perfect, $P^e$ is an ideal. If $A, B$ are ideals in $Q_\sigma$ such that $AB \subseteq P^e$ then $A^eB^e \subseteq (AB)^e \subseteq P^{ee} = P_\sigma$.

For every $x \in P_\sigma$ there is an ideal $C' \in T_\sigma$ such that $C'x \subseteq P$ and since $R$ is left-Noetherian there is an ideal $C$ in $T_\sigma$ such that $CP_\sigma \subseteq P$. But $C \subseteq P$ since $P \notin T_\sigma$. Hence $P_\sigma = P$ and $A^eB^e \subseteq P$. Hence either $A^e$ or $B^e$ is in $P$ and either $A = A^e$ or $B = B^e$ is contained in $P^e$. Finally if $P^e = Q_\sigma$ then $P = P^{ee} = R$ contrary to $P \notin T_\sigma$.

**Theorem 5.** If $\sigma$ has property (T) and $R$ is $\sigma$-perfect there is a one-to-one correspondence between the prime ideals of $R$ not in $T_\sigma$, and the proper prime ideals of $Q_\sigma$.

**Proof.** The mapping $P \mapsto P^e$ maps the set of primes in $R$ which are not in $T_\sigma$ into the set of proper primes of $Q_\sigma$. Since $P^e = P^{ee}$ for every prime $P$ in $Q_\sigma$, it will be sufficient to show that if $P'$ is a proper ideal in $Q_\sigma$, $P'^e$ is a prime ideal not in $T_\sigma$. Let $A, B$ be ideals in $R$ such that $AB \subseteq P'$ then $A^eB^e \subseteq (AB)^e \subseteq P^{ee} = P'$. But since $Q_\sigma A$ is an ideal in $Q_\sigma$, $(AB)^e = Q_\sigma A^eB^e = Q_\sigma A^e B^e$, and hence $A^e \subseteq P'$ or $B^e \subseteq P'$. Thus $A \subseteq A_\sigma \subseteq P^e$ or $B \subseteq B_\sigma \subseteq P^e$ and $P^e$ is prime. If $P \in T_\sigma$ then $P^e = Q_\sigma j(P) = Q_\sigma$, contrary to the hypothesis that the extended ideal was proper, and if $P_1, P_2$ are distinct primes in $R$ not in $T_\sigma$ then $P_1^e \neq P_2^e$.

**Proposition 6.** If $R$ is $\sigma$-perfect and $A$ is an ideal in $R$, then $\text{rad } A^e = (\text{rad } A)^e$.

**Proof.** By the foregoing theorem, $\text{rad } A^e = \bigcap P^e$ when the intersection is over prime ideals $P \supseteq A$, $P \notin T_\sigma$. Hence

$$(\text{rad } A^e)^e = \bigcap P, P \supseteq A, P \notin T_\sigma$$

and $(\text{rad } A^e)^e \supseteq \text{rad } A$ or $\text{rad } A^e \supseteq (\text{rad } A)^e$. We show now that $(\text{rad } A^e)^e \supseteq (\text{rad } A)^e$ and therefore $\text{rad } A^e = (\text{rad } A^e)^e \supseteq (\text{rad } A)^e \supseteq (\text{rad } A)^e$ thus giving the required result.

Suppose $x \in (\text{rad } A^e)^e$, i.e., $x \in P$ for all $P \supseteq A, P \notin T_\sigma$. Let $P_0$ be an arbitrary prime in $T_\sigma$ and $P_0 \supseteq A$. Then if $x \notin P_0$, we can find an ideal $C_0 \in T_\sigma$ such
that $C \subseteq P\) \subseteq P\). By the left-Noetherian hypothesis, there is only a finite number of minimal ideals containing $A$, hence we can choose an ideal $C \in T_\sigma$ such that $Cx \subseteq P$ for every minimal prime $P$ containing $A$ and $P$ in $T_\sigma$. But, since $x$ and therefore $Cx$ is also contained in all minimal prime ideals containing $A$ which are not in $T_\sigma$ it follows that $Cx \subseteq \text{rad } A$ and $x \in (\text{rad } A)_\sigma$.

**Proposition 7.** If $R$ is $\sigma$-perfect let $P$ be a prime ideal of $R$, $P \notin T_\sigma$, and $P^\sigma$ the corresponding prime ideal of $Q_\sigma$. Then there is a one-to-one correspondence between the left $P$-primary ideals of $R$ and the left $P^\sigma$-primary ideals of $Q_\sigma$.

**Proof.** Recall that $T$ is left primary if $AB \subseteq T$ implies $B \subseteq T$ or $A \subseteq \text{rad } T$. The left-Noetherian condition then implies that $\text{rad } T$ is a prime ideal $P$ and $T$ is said to be left $P$-primary. As before we can find an ideal $C$ in $T_\sigma$ such that $CT_\sigma \subseteq T$. Since $P \notin C$ because $P \notin T_\sigma$, we have $T_\sigma \subseteq T$ so $T_\sigma = T$. Since the extension of the radical of an ideal is equal to the radical of the extended ideal, the proof becomes easy and it follows the same lines as that of Theorem 5.

**Proposition 8.** If $\sigma$ is an arbitrary idempotent kernel functor, $P$ a maximal ideal not in $T_\sigma$ then $P$ is a prime ideal.

**Proof.** If $A, B$ are ideals such that $AB \subseteq P$, $A \subseteq P$, $B \subseteq P$ then $A + P$ and $B + P$ are in $T_\sigma$. Since $\sigma$ is idempotent $(A + P)(B + P) \in T_\sigma$. But also $(A + P)(B + P) \subseteq P$, contrary to $P \notin T_\sigma$.

## 5. The Zariski Topology

Let $R$ be an arbitrary left-Noetherian ring with unit. Put $X = \text{Spec } R = \text{(proper prime ideals of } R\text{)}$. For any ideal $A$ of $R$, let $V(A) = \{P \in X, P \ni A\}$. It is clear that $V(A)$ depends only on the radical $\text{rad } A$ of $A$. One easily checks the following:

**Lemma 9.**

1. If $A, B$ are ideals, $A \subseteq B$, then $V(A) \supseteq V(B)$.
2. For any set of ideals $\{A_\alpha, \alpha \in I\}$ we have

$$V\left(\sum A_\alpha\right) = \bigcap V(A_\alpha).$$

3. For ideals $A, B$ in $R$, $V(A \cap B) = V(AB) = V(A) \cup V(B)$. It follows that the $X_A = X - V(A)$ are the open sets for a topology on $X$, called the Zariski topology.
A point \( P \) is closed if and only if \( I' \) is a maximal ideal \( R \). A subset \( S \subseteq X \) is called irreducible if \( S \) is not the union of closed sets different from \( S \). A generic point for an irreducible set \( S \) is a \( P \in X \) such that \( V(P) = S \).

**Proposition 10.** If \( P \in X \) then \( V(P) \) is irreducible. Conversely, every irreducible closed subset \( S \subseteq X \) is equal to \( V(P) \) for some \( P \in X \) and \( P \) is the unique generic point for \( S \).

**Proof.** If \( V(P) = W_1 \cup W_2 \), \( W_1 \) and \( W_2 \) closed, then \( P \) has to be in one of these closed sets, say, \( W_1 \) hence \( W_1 = V(P) \). Conversely, suppose that \( V(A) \) is irreducible. Since \( V(A) = V(\operatorname{rad} A) \) we assume \( A = \operatorname{rad} A \). If \( A \) is not prime then there are ideals \( B', C' \subseteq A \) such that \( B'C' \subseteq A \). Put \( B = A + B', \ C = A + C' \). Then \( A = B \cap C \); indeed, if \( x \in B \cap C \), \( x = a_1 + b = a_2 + c \) with \( a_1, a_2 \in A, b \in B', c \in C' \). Hence \( xR = xRc + A \subseteq A \), and \( x \in A \) since \( A = \operatorname{rad} A \). We have thus \( V(A) = V(B) \cup V(C) \) and for \( x \notin A \) there is a \( P \in V(A) \) such that \( P \not\supseteq (x) \); therefore, taking \( x \in B - A \) we get \( V(A) - V(B) \neq \emptyset \), entailing a contradiction with the irreducibility of \( V(A) \). If \( P \) is another generic point of \( V(A) \) then \( A \subset P \) but since \( A \in V(P) \) also \( A \supseteq P \) so \( A = P \).

As in the case of a commutative ring one easily verifies that a set \( (A_\alpha, \alpha \in I) \) of ideals of \( R \) gives rise to a covering of \( X \) by means of the open sets \( X_\alpha = X - V(A_\alpha) \), if and only if \( 1 \in \sum A_\alpha \). It follows that \( X \) is compact but not necessarily Hausdorff; for any ideal \( A \) the open set \( X_A \) is compact. We will now use the localization techniques of Section 4 to construct a structure sheaf on \( \operatorname{Spec} R \).

### 6. The Structure Sheaf

Let \( R \) be a left-Noetherian ring. Then it is well-known that if \( A, B \) are ideals of \( R \), \( A \subseteq \operatorname{rad} B \) is equivalent to \( A^n \subseteq B \) for some positive integer \( n \). To an ideal \( A \) of \( R \) we associate:

\[
\mathcal{F}(A) = \{ \text{left ideals } L \text{ of } R \text{ containing an ideal } B \text{ such that } \operatorname{rad} B \supseteq A \}
\]

\[
= \{ \text{left ideals } L \text{ of } R \text{ such that } L \supseteq A^n \text{ for some } n \}.
\]

Obviously \( \mathcal{F}(A) \) defines a symmetric kernel functor by

\[
\sigma_A(M) = \{ m \in M, Lm = 0 \text{ for some } L \in \mathcal{F}(A) \}.
\]

For any symmetric kernel functor \( \sigma \), let \( C(\sigma) \) denote the set of ideals of \( R \) maximal in the set of ideals not in \( \mathcal{F}(\sigma) \). By Proposition 8, \( C(\sigma) \) consists of prime ideals hence \( C(\sigma) \subseteq \operatorname{Spec} R \); \( C(\sigma_A) \) may be looked upon as being the set of "tops" of \( X_A \), i.e., the maximal elements of \( X_A \). The topology \( \mathcal{F}_\sigma \) corre-
sponding to \( \sigma \) will frequently be written \( T(\sigma) \) and \( T(\sigma_A) \) will be abbreviated to \( T(A) \).

To a non-empty open set \( X_A \), i.e., \( A \not\subseteq \text{rad}(0) \), we associate the ring of quotients \( Q_A(R) \) with respect to the kernel functor \( \sigma_A \); this is well defined since both \( X_A \) and \( T(A) \) depend only on \( \text{rad} A \). The canonical map of \( R \) into \( Q_A(R) \) will be denoted by \( i_A \). It is in general not injective.

**Proposition 11.** Assigning \( Q_A(R) \) to \( X_A \), for every ideal \( A \not\subseteq \text{rad}(0) \), defines a presheaf on \( \text{Spec} \, R \).

**Proof.** (1) If \( X_B \subseteq X_A \) then \( \text{rad} B \subseteq \text{rad} A \) and \( X_B \not= \emptyset \) if and only if \( B \not\subseteq \text{rad}(0) \). For an element \( L \in T(A) \) we have \( L \supseteq A^n \) but \( A \supseteq B^m \) since \( B \subseteq \text{rad} B \subseteq \text{rad} A \), so \( L \supseteq B^m \) or \( L \in T(B) \). This proves \( T(A) \subseteq T(B) \) and \( \sigma_A \leq \sigma_B \). Hence we have the canonical projection

\[
\pi: i_A(R) = R/\sigma_A(R) \rightarrow R/\sigma_B(R) = i_B(R).
\]

Consider the diagram

\[
\begin{array}{ccc}
\pi & \downarrow & i_A(R) \\
\downarrow & \rightarrow & Q_A(R) \rightarrow Q_A(R)/i_A(R) \\
R & \downarrow \pi & \rho(A,B) \\
i_B(R) & \rightarrow & Q_B(R).
\end{array}
\]

Since, \( Q_A(R)/i_A(R) \) is \( \sigma_A \)-torsion, a fortiori \( \sigma_B \)-torsion, it follows from the faithful \( \sigma_B \)-injectivity of \( Q_B(R) \) that \( \pi \) extends uniquely to a morphism \( \rho(A, B) \), and since \( Q_A(R) \) is an essential extension of \( i_A(R) \) it also shows that \( \text{Ker} \, \rho(A, B) \cap i_A(R) = \text{Ker} \, \pi \). Moreover the uniqueness of the extended map implies that \( \rho(A, A) \) is the identity on \( Q_A(R) \). One may verify that \( \rho(A, B) \) is even a ring homomorphism.

(2) If \( \emptyset \not\subseteq X_C \subseteq X_B \subseteq X_A \), then we have the following diagram:

\[
\begin{array}{ccc}
i_A(R) \rightarrow Q_A(R) \\
\downarrow \pi(A,C) & \rightarrow & Q_C(R) \rightarrow Q_C(R) \\
\downarrow \pi(A,B) & \rightarrow & \rho(A,C) \rightarrow \rho(A,C) \\
i_B(R) \rightarrow Q_B(R) \rightarrow Q_B(R) \rightarrow Q_B(R) \\
\downarrow \pi(B,C) & \rightarrow & \rho(A,B) \rightarrow \rho(B,C).
\end{array}
\]
Since \( \pi(B, C) \pi(A, B) = \pi(A, C) \), \( \rho(A, C) \) and \( \rho(B, C) \rho(A, B) \) are two extensions of \( \pi(A, C) \) to \( Q_c(R) \), hence they are equal, since \( Q_c(R) \) is faithfully \( \sigma_c \)-injective, and the diagram is commutative.

**Lemma 12.** For a finite set \( \{A_i, i \in I\} \) of ideals of \( R \), \( \text{rad} \sum A_i = \text{rad} (\text{rad} \sum A_i) \).

**Proof.** That \( \text{rad} \sum A_i \subseteq \text{rad} (\text{rad} \sum A_i) \) is obvious. Also, if \( P \supset \sum A_i \), then \( P \supset A_i \) for all \( i \), \( P \supset \text{rad} A_i \), and hence \( \text{rad} (\text{rad} \sum A_i) \subseteq \text{rad} \sum A_i \).

**Theorem 13.** Assigning \( Q_d(R) \) to \( X_d \), defines a monopresheaf on \( X \), i.e., for an arbitrary covering \( X_d = \bigcup X_i \), \( X_i = X - V(A_i) \), the only element \( g \in Q_d(R) \) such that \( \rho(A, A_i)g = 0 \) for all \( i \), is \( g = 0 \).

**Proof.** Since \( X_d \) is compact, we may suppose that we are given a finite covering of \( X_d \). We have the following commutative diagram (where \( \sigma_i = \sigma_{A_i} \) etc.)

\[
\begin{array}{ccc}
R & \xrightarrow{\pi(A, A_i)} & R/\sigma_i(R) \\
\downarrow t_A & & \downarrow t_i \\
R/\sigma_i(R) & \xrightarrow{n(A, A_i)} & Q_d(R) \\
\end{array}
\]

Since \( g \in Q_d(R) \), there is an ideal \( B \in \mathcal{F}(A) \) such that \( Bg \subseteq i_d(R) \). If \( g \in \ker \rho(A, A_i) \), then \( Bg \subseteq i_d(n(A, A_i)) = \ker n(A, A_i) \) since \( n(A, A_i) \) is the unique extension of \( \pi(A, A_i) \). Hence \( Bg \subseteq \sigma_d(R)/\sigma_i(R) \) for all \( j \).

If we prove that \( \bigcap_j (\sigma_j(R)/\sigma_i(R)) = 0 \) then \( Bg = 0 \) and \( g \in \sigma_d(Q_d(R)) = 0 \). So it remains to prove that \( \sigma_d(R) = \bigcap \sigma_j(R) \). Since \( \sigma_j(R) \supset \sigma_i(R) \) for all \( j \), \( \bigcap \sigma_j(R) \supset \sigma_i(R) \) is trivial.

Conversely, let \( x \in \bigcap \sigma_j(R) \). Then, for each \( j \), there is an open ideal \( C_j \) such that \( C_jx = 0 \), \( C_j \in \mathcal{F}(A_j) \). By definition \( \text{rad} C_j \supset A_j \). Take \( C = \sum C_j \). Then, using the lemma, \( \text{rad} C = \text{rad}(\sum \text{rad} C_j) \supset \sum A_j \). But \( V(A) = \bigcap V(A_j) \) implies that if \( P \supset \sum \text{rad} A_j \), then \( P \supset \text{rad} A_j \) for all \( j \), or \( P \in V(A_j) \) for all \( j \), and \( P \in \bigcap V(A_j) = V(A) \), entailing \( P \supset \text{rad} A \). Hence \( \text{rad} A \subseteq \text{rad} \sum A_j \).
This proves $\text{rad } C \supset \sum A_j \supset A$, or $C \in \mathcal{F}(A)$. Since $C x = 0$, it follows that $x \in \sigma_A(R)$.

At this point we could use classical sheafification methods to get a sheaf of sections out of this monopresheaf. This would probably lead to a theory similar to the theory of schemes and preschemes over commutative rings. However we contented ourselves here with the close generalization of affine varieties. Therefore we make the further assumption that $R$ be a prime ring, and prove that in this case the monopresheaf actually is a sheaf.

**Theorem 14.** Let $R$ be a left-Noetherian prime ring, then the monoresheaf on $\text{Spec } R$ is a sheaf, i.e., if we have a covering $X_A = \bigcup X_a$, $X_a = X - V(A_a)$, of an open set $X_A$, such that there are elements $g_a \in Q_A(R)$ for which

$$\rho(A_a, A_x A_y) g_a = \rho(A_x, A_y) g_y,$$

then there exists an element $g \in Q_A(R)$ such that $\rho(A, A_x) g = g_a$ for every $a$.

**Proof.** Note first that it will be sufficient to prove this for a finite covering. Indeed if $X_A = \bigcup X_i$ is a finite covering for which the property holds, then an open set $X_a$ is covered by $X_a \cap X_i = X - V(A_a A_i)$. Let $g$ be the element of $Q_A(R)$ mapped onto the $g_i \in Q_i(R)$, and let $h_a$ be the image of $g$ in $Q_A(R)$. Then $g_a$ and $h_a$ have the same image under every map $\rho(A_a, A_x A_y)$ and the foregoing theorem implies $h_a = g_a = 0$ and hence the required property holds for the arbitrary covering too.

Suppose that $X_A = \bigcup X_i$, a finite covering, with $X_i = X - V(A_i)$. If the elements $g_i$ have the prescribed property then it follows from the presheaf axioms that

$$\rho \left( A_i, \prod A_i \right) g_i = \rho \left( A_j, \prod A_i \right) g_j$$

for all $i, j$.

Note that $\prod A_i$, being a finite product of ideals in a prime ring, is a nonzero ideal. Note also, that although the product of ideals is not necessarily commutative, the order of the ideals in $\prod A_i$ is not important since the kernel functor associated with $\prod A_i$ depends only on the radical of the product and $V(\prod A_i) = \bigcup V(A_i)$. Let $B = \prod A_i$. We now prove that if $g_1 \in Q_1(R)$, $g_2 \in Q_1(R)$ such that $\rho(A_1, B) g_1 = \rho(A_2, B) g_2$, then there is an element $g \in Q_B(R)$, $C = A_1 + A_2$, such that $\rho(C, A_i) g = g_1$, $\rho(C, A_2) g = g_2$. The proposition will then follow, by repeating this process a finite number of times, ending up with an element $g$ in $Q_A(R)$ having the required property (since $\sigma_A$ is exactly the kernel functor associated with $\sum A_i$ by the covering property). Elements of $Q_1(R)$ are defined to be equivalence classes $[L_1, f_1]$ of pairs $(L_1, f_1)$ with $L_1 \in \mathcal{F}(A_i)$, $f_1 \in \text{Hom}_R(L_1, R)$ the relation being given by
(L_1, f_1) \sim (L_1', f_1') \text{ if and only if there exists a } L'_1 \in \mathcal{F}(A_1) \text{ such that } f_1 \text{ and } f_2 \text{ coincide on } L'_1 \subset L_1 \cap L_1'.

Let \( g_1 \in Q_1(R) \), \( g_1 = [L_1, f_1]_1 \) and similarly \( g_2 \in Q_2(R) \), \( g_2 = [L_2, f_2]_2 \) with \( L_2 \in \mathcal{F}(A_2) \), \( f_2 \in \text{Hom}_R(L_2, R) \). Since \( R \) is a prime ring, it is obviously \( \sigma_1 \)- and \( \sigma_2 \)-torsion-free, so the maps \( \pi(A_1, B) \) are all equal to the identity on \( R \) hence injective and thus the maps \( \rho(A_1, B) \) are all injections. They are in fact easily seen to be defined as follows:

\[
\rho(A_1, B)[L_1, f_1]_1 = [L_1, f_1]_B,
\]
similarly,

\[
\rho(A_2, B)[L_2, f_2]_2 = [L_2, f_2]_B,
\]

the right-hand sides denoting the classes defined by the respective pairs for the \( \mathcal{F}(B) \)-equivalence relation.

Since \( g_1, g_2 \) are mapped onto the same element of \( Q_2(R) \) this means that \( (L_1, f_1) \) is equivalent to \( (L_2, f_2) \) for the \( \mathcal{F}(B) \)-equivalence, i.e., there is a \( L' \in \mathcal{F}(B) \), \( L' \subset L_1 \cap L_2 \) and \( f_1 \mid L' = f_2 \mid L' \). Since \( \sigma_B \) is symmetric, we may assume that \( L' \) is two-sided. Take \( x \in L_1 \cap L_2 \), since \( L'x \subset L' \) we have that

\[
L'(f_1(x) - f_2(x)) = 0 \text{ yielding } f_1(x) - f_2(x) \in \sigma_B(R) = 0.
\]

Hence \( f_1 \) and \( f_2 \) coincide on the intersection \( L_1 \cap L_2 \). We define \( f : L_1 + L_2 \to R \) by

\[
f(a_1 + a_2) = f_1(a_1) + f_2(a_2) \quad a_1 \in L_1, \quad a_2 \in L_2.
\]

That \( f \) is well-defined follows from \( f_1 \mid L_1 \cap L_2 = f_2 \mid L_1 \cap L_2 \), and \( f \) is clearly an \( R \)-morphism. Take \( g = [L_1 + L_2, f]_C \), \( C = A_1 + A_2 \). It is easily verified that \( g \) is \( \mathcal{F}(A_1) \)-equivalent to \( [L_1, f_1]_1 \) and \( \mathcal{F}(A_2) \)-equivalent to \( [L_2, f_2]_2 \), hence \( g \) maps onto \( g_1, g_2 \) under \( \rho(A_1 + A_1, A_2) \) and \( \rho(A_1 + A_2, A_2) \), respectively.

**Definition.** Let \( R \) be a left-Noetherian prime ring. \( \text{Spec } R \) equipped with its Zariski topology and the corresponding sheaf of quotient rings will be called an affine variety.

**Remark.** Let \( R \) be a left-Noetherian prime ring, and let \( \sigma^* \) be the symmetric kernel functor defined by \( C(\sigma^*) = \{(0)\} \), i.e., \( \sigma^* = \sup \{\sigma_A, A \text{ a nonzero ideal of } R\} \). Let \( M \) be any \( \sigma^* \)-torsion free left \( R \)-module, i.e., \( M \) is \( \sigma_A \)-torsion free for every \( \sigma_A \), or every nonzero submodule of \( M \) is faithful. Assigning \( Q_A(M) \) to \( X_A \) yields a sheaf of modules. The proof of this fact follows exactly the same lines as in the case of the structure sheaf. A sheaf of modules is called a Module, (with capital M).
7. Affine Varieties

Let $\text{Spec } R$ be an affine variety. One easily verifies that, if $A_1$ and $A_2$ are ideals of $R$, then $\mathcal{F}(A_2A_2) = \text{sup}\{\mathcal{F}(A_1), \mathcal{F}(A_2)\}$ hence if $C = A_1A_2$ then $\sigma_C = \text{sup}\{\sigma_1, \sigma_2\}$. Furthermore, $Q_{\sigma^*}(R)$ will be called the function ring of $\text{Spec } R$. Since every $\sigma_A < \sigma^*$ it follows that we have injections $Q_A(R) \rightarrow Q_{\sigma^*}(R)$ for every ideal $A$ of $R$.

It follows that $Q_{\sigma^*}(R)$ can be looked upon as being the direct limit of the direct limit of the system

$$\{Q_A(R), \rho(A, B)\}$$

$\rho(A, B)$ being the inclusion of $Q_A(R)$ in $Q_B(R)$ when $B \subset A$, $B \neq 0$.

Our next step is to describe the stalks of the structure sheaf. Recall that we can associate a kernel functor $\sigma_{R-P}$ to a prime ideal $P$ of $R$ by taking

$$\mathcal{F}(\sigma_{R-P}) = \{\text{left ideals } L \text{ of } R \mid L \supset \langle s \rangle \text{ for some } s \notin P\}$$

and $\sigma_{R-P}$ is obviously a symmetric kernel functor. The quotient ring with respect to $\sigma_{R-P}$ will be denoted by $Q_{\sigma_{R-P}}(R)$ to avoid ambiguity with $Q_{\sigma}(R)$ the quotient ring associated with the Zariski open set $X_P$.

**Proposition 15.** (a) $\sigma_{R-P} = \text{sup}\{\sigma_A \mid P \in X_A\}$

(b) $Q_{\sigma_{R-P}} = \lim_{P \in X_A} Q_A(R)$,

*Proof.* (a) If $P \in X_A$, then $P \nsubseteq A$ and thus $A \supset \langle s \rangle$ for some $s \in R - P$.

If $L \in \mathcal{F}(A)$, then $L \supset A^n \supset \langle s^n \rangle$, $s \in R - P$, and since $\langle s^n \rangle \nsubseteq P$, $\langle s^n \rangle \supset \langle s' \rangle$ with $s' \in R - P$ or $L \supset \langle s' \rangle$ and $L \in \mathcal{F}(\sigma_{R-P})$ follows. Conversely if $B \in \mathcal{F}(\sigma_{R-P})$ then $B \supset \langle s \rangle$ for some $s \notin P$, hence, putting $A = \langle s \rangle$ we have $P \in X_A$ and $B \supset A$, i.e., $B \subset \mathcal{F}(A)$ for some ideal $A$ with $P \in X_A$. This proves that $\mathcal{F}(\sigma_{R-P})$ is the idempotent topology generated by the union of the topologies $\mathcal{F}(A), P \in X_A$, and we have proved (a).

(b) This follows from (a) since if $P \in X_A$, $P \in X_B$ then $P \in X_{AB}$ and $\sigma_{AB} = \text{sup}\{\sigma_A, \sigma_B\}$ or $Q_{AB}(R)$ is the quotient ring associated to $\text{sup}\{\sigma_A, \sigma_B\}$. Furthermore if $Q_A(R)$ and $Q_B(R)$ are injected in some $Q_C(R)$ then $Q_{AB}(R)$ is injected into $Q_{C}(R)$. It follows that the direct limit of the directed system $(Q_A(R), \rho(A, B))$, $B \subset A$, $P \in X_B$ is obtained as the quotient ring for the functor $\text{sup}\{\sigma_A \mid P \in X_A\} = \sigma_{R-P}$.

Note that, since $\sigma_{R-P} \leq \sigma^*$, the stalks $Q_{\sigma_{R-P}}(R)$ also inject into the function ring $Q_{\sigma^*}(R)$. The fact that both $\sigma_{R-P}$ and $\sigma^*$ may be defined as sups suggests the importance of investigating the properties of kernel functors which are inherited by the sup of these kernel functors. See for example the corollary to Proposition 22.
**Definition.** Let $R_1, R_2$ be left-Noetherian prime rings, $X = \text{Spec } R_1$, $Y = \text{Spec } R_2$ the corresponding affine varieties. A morphism from $X$ to $Y$ is given by

(a) A Zariski continuous map $\tilde{\phi}: X \to Y$, such that

(b) For every open set $U$ in $Y$, if we let $\tilde{\phi}^{-1}(U) = V$ and let $\sigma_U, \sigma_V$ be the associated kernel functors on left $R_2$-modules, left $R_1$-modules respectively, then there exists a ring homomorphism $\phi_U: Q_U(R_2) \to Q_V(R_1)$ compatible with restrictions, i.e. we have commutative diagrams, for $U' \subseteq U, V' = \tilde{\phi}^{-1}(U')$

\[
\begin{array}{ccc}
Q_U(R_2) & \xrightarrow{\phi_U} & Q_V(R_1) \\
\downarrow{\rho(U, U')} & & \downarrow{\rho(V, V')}
\end{array}
\]

A morphism of affine varieties $X, Y$ is called an isomorphism if and only if

(a) $\tilde{\phi}$ is a homeomorphism of Zariski topological spaces,

(b) all the induced maps $\phi_U$ are ring isomorphisms.

**Remark.** Let $V(K)$ be an irreducible closed subset of an affine variety, and let $\pi: R \to R/K$ be the canonical epimorphism. Put $X = \text{Spec } R$, $X' = \text{Spec } R/K$. A map $\pi^*: X' \to X$ may be defined by $\pi^*(P') = \pi^{-1}(P') = P$. Obviously, $\text{Im } \pi = V(K)$ and $\pi$ clearly is an injective map. From

\[
\pi^{-1}(X_A) = \{P' \in X', \pi^{-1}(P') \in X_A\}
\]

\[
= \{\pi(P), P \in X_A \cap V(K)\}
\]

\[
= \{P', P' \not\in A'\} = X_A' \quad \text{(with } A' = \pi(A)).
\]

It follows that $V(K)$ is homeomorphic to $\text{Spec } R/K$. To an open set $X_A \cap V(K)$ in $V(K)$ we may associate $Q_{\pi^*}(R/K)$ and this makes $V(K)$ into an affine variety.

**Remark.** It seems to be impossible to obtain more functorial properties for $\text{Spec}$. However in the sequel we show that the structure sheaf has some of the properties which are characteristic for $\text{Spec}$ in the commutative case; we will have to restrict our attention to some special open sets and their associated kernel functors. Therefore we introduce the following.

**Definition.** A symmetric kernel functor $\sigma$ having property (T) will be called a T-functor. A Zariski open set $X_A$ will be called a T-set if $\sigma_A$ is a
T-functor. The stalk $Q_{R,p}(R)$ will be called a $T$-stalk when $\sigma_{R,p}$ is a $T$-functor.

When $R$ is a commutative Noetherian integral domain, all basic open sets $X_f$, $f \in R$, are $T$-sets and all stalks are $T$-stalks. If $R$ is not a hereditary ring however, not every open $X_A$ is a $T$-set. For example, if $R$ is Noetherian, integrally closed but not a Dedekind domain, then taking $M$ to be a non-invertible maximal ideal in $R$ we find that $\sigma_M$ is not a $T$-functor (cf. [1], Example 2, p. 45). It will sometimes be useful to have enough $T$-sets on $\text{Spec } R$. If $R$ is left-Noetherian and prime then $\text{Spec } R$ is said to have a $T$-basis if there exists a basis for the Zariski topology consisting of $T$-sets.

Let $M$ be a $\sigma^*$-torsion free left $R$-module denote by $\hat{M}$ the sheaf of modules given by sticking $Q_A(M)$ onto $X_A$. Associating $Q_A(R) \otimes_R M$ to $X_A$ defines a presheaf denoted by $\hat{R} \otimes M$, we have the following.

**PROPOSITION 16.** If $X$ has a $T$-basis $\overline{M} \cong \Gamma(\hat{R} \otimes M)$, $\Gamma(\hat{R} \otimes M)$ being the sheaf of sections associated with the presheaf $\hat{R} \otimes M$.

**Proof.** If $X_A$ is a $T$-set then $Q_A(M) \cong Q_A(R) \otimes_R M$. A section $s \in \Gamma(X_A, \hat{R} \otimes M)$ may thus be identified with a section in $\Gamma(X_A, \hat{M})$. Since $\Gamma(X_A, \hat{M}) \cong Q_A(M)$ we see that $\hat{M}$ and $\hat{R} \otimes M$ agree on a basis for the topology in $X_A$; hence $\hat{M}$ is isomorphic to the sheafification of the presheaf $\hat{R} \otimes M$ or $\hat{M} \cong \Gamma(\hat{R} \otimes M)$.

Note that, since $M$ is assumed to be $\sigma^*$-torsion free, it is not immediate that $\sigma_{R,p}$ is also a $T$-functor even if it is true that $X$ has a $T$-basis, however see Proposition 21. Applying Theorem 5 we easily deduce the following:

**PROPOSITION 17.** A $T$-stalk $Q_{R,p}(R)$ such that $R$ is $\sigma_{R,p}$-perfect is a "local ring," i.e., it is a left-Noetherian prime ring with a unique maximal ideal.

If $\sigma^*$ is a $T$-functor then $Q_{\sigma^*}(R)$ has $(0)$ for a maximal ideal and hence it is a simple ring generalizing the function field of a variety over a commutative ring. If $\pi: R \to R'$ is a ring epimorphism with kernel $K \neq 0$ then $K$ is in $\mathcal{T}(\sigma^*)$ and hence $(0)$ is open in $R'$ viewed as a left $R$-module via the morphism $\pi$.

It follows that $R'$ is $\sigma^*$-torsion and $Q_{\sigma^*}(R') = 0$. Hence $\pi$ does not extend to a morphism on $Q_{\sigma^*}(R)$ unless $K = 0$, then $R \cong R'$.

**THEOREM 18.** Let $X_A$ be a $T$-set such that $R$ is $\sigma_A$-perfect, then $X_A$ is an affine variety, in fact $X_A = \text{Spec } Q_A(R) = X'$.

**Proof.** The prime ideals $P \nmid A$ are exactly the primes $P \notin \mathcal{T}(A)$. From Theorem 5 it follows that these prime ideals are in one-to-one correspondence with proper prime ideals in $Q_A(R)$. An open set in $X_A$ is of the form
\[ X_A \cap X_B = X_{AB}, \] hence of the form \( X_C \) with \( C \subseteq A \). Since the operators \( c, e \) on the ideals of \( R \) and \( Q_A(R) \) respect inclusion it follows that \( X_C \) corresponds one-to-one with the set \( \{ P' \subseteq X' \mid P' \not\subseteq C' \} = X_{C'e} \). Thus \( e \) defines a homeomorphism of topological spaces \( X_A \) and \( X' \).

Looking at the sheaves, we have \( Q_C(R) \) associated with \( X_C \), and \( Q_{C'e}[Q_A(R)] \) associated with \( X_{C'e} \). We show that \( \sigma_{C'e} \) and \( \sigma_C \) coincide on \( Q_A(R) \)-modules. Let \( M \) be any left \( Q_A(R) \)-module. Then \( x \in \sigma_{C'e}(M) \) if and only if \( (C'e)^n_x = 0 \) but then \( Cnx = 0 \) and \( x \in \sigma_c(M) \) or \( \sigma_{C'e} \leq \sigma_c \).

On the other hand, \( x \in \sigma_c(M) \) if and only if \( C^n x = 0 \); since \( M \) is a left \( Q_A(R) \)-module we have \( Q_A(R) C^n = 0 \) but \( Q_A(R) C^n = (C^n) \) since \( C^n \) is an ideal of \( Q_A(R) \), hence \( x \in \sigma_{C'e}(M) \) and \( \sigma_c \leq \sigma_{C'e} \), proving that \( \sigma_{C'e} \) and \( \sigma_{C'} \) coincide on \( Q_A(R) \)-modules. It is then easily seen that \( Q_{C'(Q_A(R))} = Q_C(Q_A(R)) \) and since \( X_C \subseteq X_A \) we have that \( Q_{C}(Q_A(R)) \cong Q_A(R) \) proving \( X_A \cong \text{Spec} Q_A(R) \) is an affine variety.

**Theorem 19.** Let \( X_C, X_A \) be T-sets in \( X \) such that \( X_C \subseteq X_A \), \( X_C \not= X_A \) and \( R \) is \( \sigma_A \)-perfect, then \( X_C \) is a T-set in \( \text{Spec} Q_A(R) \).

**Proof.** We have to show that \( \sigma_{C'} \) has property \( (T) \) and since \( R \) is left-Noetherian it will be sufficient to prove that every left ideal \( L \in \mathcal{F}(C') \) is a \( \sigma_{C'} \)-projective \( Q_A(R) \)-module. Let \( L \in \mathcal{F}(C') \) and let \( M' \rightarrow M \rightarrow 0 \) be an exact sequence of \( \sigma_{C'} \)-torsion free \( Q_A(R) \)-modules. Suppose we have given a left \( Q_A(R) \)-module morphism \( h : L \rightarrow M \). Since \( L \supset (C')^n \) it follows that \( L^c \supset C^n \) and \( L^c \in \mathcal{F}(C) \). The restriction \( h : L^c \rightarrow M \) is a left \( R \)-module morphism. Now, \( M, M' \) are left \( R \)-modules via the inclusion \( R \rightarrow Q_A(R) \) and since \( \sigma_{C'} = \sigma_C \) on \( Q_A(R) \)-modules, \( M \) and \( M' \) are \( \sigma_{C'} \)- but also \( \sigma_C \)-torsion free. By the \( \sigma_{C'} \)-projectivity of \( L^c \) we can find a left ideal \( B \) in \( \mathcal{F}(C) \), \( B \subseteq L^c \) such that we have the following commutative diagram of left \( R \)-module morphisms

\[
\begin{array}{ccc}
B & \longrightarrow & L^c \\
\downarrow \quad f' & & \quad \downarrow h_c \\
M' & \longrightarrow & M & \longrightarrow & 0.
\end{array}
\]

If \( L \) is a proper left ideal of \( Q_A(R) \) then by property \( (T) \), \( L^c \not\subseteq \mathcal{F}(A) \) hence also \( B \not\subseteq \mathcal{F}(A) \). Since \( h \) is a \( Q_A(R) \)-module morphism it is defined by \( h_c \) as follows:

\[
h(qa) = qh_c(a), \quad a \in L^c, \quad q \in Q_A(R).
\]

We extend \( f' \) in the same way to a left \( Q_A(R) \)-module homomorphism \( f \) on \( B^c = Q_A(R)B \), i.e., \( f_gb = gf'(b) \). Since \( B \in \mathcal{F}(C), \ B \not\subseteq \mathcal{F}(A) \) we get
\( B^e \in \mathcal{F}(C^e), B^e \neq Q_A(R) \) and obviously the following diagram is commutative, (all maps being \( Q_A(A) \)-module morphisms)

\[
\begin{array}{ccc}
B^e & \xrightarrow{f} & L \\
\downarrow & & \downarrow \\
M' & \xrightarrow{h} & M & \longrightarrow & 0
\end{array}
\]

thus proving that \( L \) is \( \sigma_{\mathcal{F}} \)-projective.

**Proposition 20.** Let \( X_A \) and \( X_B \) be \( T \)-sets in \( \text{Spec } R \), then \( X_A \cap X_B = X_{AB} \) is a \( T \)-set.

**Proof.** We have to show that \( \sigma = \sup \{ \sigma_A, \sigma_B \} \) is a \( T \)-functor. The topology \( \mathcal{F}(\sigma) \) has a basis for the open sets given by the finite products of ideals in \( \mathcal{F}(A) \) or \( \mathcal{F}(B) \); let \( C = C_1 \cap \cdots \cap C_r \) be such a product. Knowing that \( Q_\sigma(R) \) contains \( Q_A(R) \) and \( Q_B(R) \) as subrings we easily deduce that \( 1 \in Q_\sigma(R) \) from the fact that \( Q_\sigma(R) \) is a \( T \)-functor. Thus, if \( C \in \mathcal{F}(\sigma) \), then \( Q_\sigma(R) \) is a \( T \)-functor and this implies that \( \sigma \) is a \( T \)-functor.

**Corollary.** The sup of a set of \( T \)-functors is a \( T \)-functor.

**Proposition 21.** If \( \text{Spec } R \) has a \( T \)-basis then each stalk is a \( T \)-stalk.

**Proof.** Let \( X_A \) be a Zariski open set in \( X \) and let \( P \in X_A \). The existence of \( T \)-basis entails that \( X_A \) is union of \( T \)-sets and hence \( P \in X_A \subseteq X_A' \), where \( X_A \) is a \( T \)-set. Since \( \sigma_A \geq \sigma_A' \), we have (cf., Proposition 15)

\[
\sigma_{R-P} = \sup \sigma_A , \quad (P \in X_A)
\]

\[
= \sup \sigma_A , \quad (P \in X_A \text{ and } X_A \text{ is a } T\text{-set})
\]

and, by the corollary to Proposition 20, \( \sigma_{R-P} \) is a \( T \)-functor.

**References**