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S-Shaped Global Bifurcation Curve and Hopf Bifurcation of Positive Solutions to a Predator–Prey Model

Yihong Du

Department of Mathematical and Computer Science, University of New England, Armidale, NSW 2351, Australia

and

Yuan Lou

Department of Mathematics, University of Chicago, Chicago, Illinois 60637

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1. INTRODUCTION

In this paper we study the following elliptic system

$$\begin{cases} \Delta u + u(a - u - bv/(1+mu)) = 0 & \text{in } D, \quad u|_{\partial D} = 0, \\ \Delta v + v(d - v + cu/(1+mu)) = 0 & \text{in } D, \quad v|_{\partial D} = 0, \end{cases} \tag{1.0}$$

where D is a bounded domain in $R^N (N \geqslant 1)$ with smooth boundary ∂D, and a, b, c, d, m are positive constants. Problem (1.0) is known as the predator-prey model with Holling-Tanner type interactions, where u and v represent the densities of the prey and predator respectively. Hence we are only interested in positive solutions by which we mean that both u and v are positive in D. It is easy to see that $a > \lambda_1$ is necessary for the existence of positive solutions to (1.0), and we shall also assume that $d > \lambda_1$. The case $d \leqslant \lambda_1$ is rather different and will not be discussed here.

We are mainly interested in studying the number of positive solutions of (1.0) and the stability of these solutions. In particular, we shall show that when b, d, c, and m fall into certain range, the solution set $\{(u, v, a)\}$ of (1.0) forms a S-shaped smooth curve, and that Hopf bifurcation occurs along this curve. This not only confirms rigorously similar numerical observations on (1.0) made in [4], but also shows that the corresponding parabolic system

$$\begin{cases} u_t = \Delta u + u(a - u - bv/(1+mu)), & x \in D, \quad t > 0, \\ \sigma v_t = \Delta v + v(d - v + cu/(1+mu)), & x \in D, \quad t > 0, \\ u = v = 0, & x \in \partial D, \quad t > 0, \end{cases}$$

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has a quite complicated dynamical behavior. Most of our results hold true for general domains in R^N, though the numerical observations in [4] are for the special case where D is an interval.

In the extreme case $m=0$, (1.0) is reduced to the classical Lotka–Volterra predator–prey model which has been studied extensively in the last decade, see, e.g., [2, 6, 12, 13, 15, 23–30].

For $m>0$, existence and nonexistence of positive solutions to (1.0) are first investigated by Blat and Brown [3]. Later in [4], A. Casal, J. C. Eilbeck, and J. Lopez-Gomez improve the results of [3]; among other things, they find the exact range of parameters (a, b, c, d) where (1.0) has a positive solution when m is small. On the other hand, when m is large, our work [19] gives the exact parameter range where (1.0) has a positive solution. In many cases, the exact number of the positive solutions and their stability are also determined when m is large (see [19]). Nevertheless, two interesting numerical observations made in [4] are left unconfirmed in a rigorous way: Numerical experiment in [4] reveals that sometimes the global positive solution curve $\{(u, v, d)\}$ of (1.0) is S-shaped (see Fig. 3 in [4]) with two stable positive solutions for each d in a certain range, and also Hopf bifurcation may occur (see Fig. 4 in [4]). This contrasts sharply with previous results on this model: For the case where m is non-negative and small, it has been proved that (1.0) has at most one positive solution for the case where D is an interval (see [4, 27]); for the case where m is large, results from [19] show that when $d>\lambda_1$, at most one stable positive solution exists for any given a. Moreover, it is unclear from the previous studies whether Hopf bifurcation can occur in this model.

As mentioned before, the main purpose of this paper is to determine when the numerical results in [4] hold and to confirm them rigorously. We should point out that, for technical reasons, we use a instead of d as the main bifurcation parameter in this paper.

To state our main result, a few notations are in order. Let $\lambda_1(q)$ denote the least eigenvalue for the linear eigenvalue problem

$$-\Delta u + qu = \lambda u \qquad \text{in } D, \qquad u|_{\partial D} = 0,$$

where q is Holder continuous in $\bar{D}$. We shall simply denote $\lambda_1(0)$ by λ_1, and let Φ_1 be the corresponding positive eigenfunction uniquely determined by the normalization $\max_{\bar{D}} \Phi_1 = 1$. It is well known that if $d>\lambda_1$, the following problem

$$\Delta u + u(d-u) = 0 \qquad \text{in } D, \qquad u|_{\partial D} = 0$$

has a unique positive solution, which we denote by θ_d; furthermore, $d \to \theta_d$ and $q \to \lambda_1(q)$ are continuous and increasing functions.

It is convenient for our discussions to write c in the form $c=\tau m$, where τ is a positive constant. Then (1.0) becomes

$$\begin{cases} \Delta u+u(a-u-bv/(1+mu))=0 & \text{in } D, \quad u|_{\partial D}=0, \\ \Delta v+v(d-v+\tau mu/(1+mu))=0 & \text{in } D, \quad v|_{\partial D}=0. \end{cases} \tag{1.1}$$

From now on, we will use the form (1.1) instead of (1.0). In the framework of this paper (also in view of results from [19]), it seems crucial that we choose c to be this form in order to observe the S-shaped bifurcation curve and Hopf bifurcation phenomenon.

Now we are ready to state the following result which is a rather special case of Theorem 4.1, the main result of this paper.

THEOREM 1.1. *For any fixed* $b>0$, *there exists a nonempty open set* $O=O(b)\subset(0,\infty)\times(0,\infty)$, *such that for any* $(\tau, d)\in O$, *we can find* $M=M(b, d, \tau)$ *large, so that for each* $m\geqslant M$:

(i) *All positive solutions* (u, v, a) *of* (1.1) *lie on an unbounded smooth curve* Γ *which bifurcates from the semi-trivial solution curve* $\{(0, \theta_d, a): a>\lambda_1\}$ *at the point* $(0, \theta_d, \lambda_1(b\theta_d))$. *Moreover*, Γ *is roughly* S-*shaped*: *There exist two positive constants* $a_*\in((\lambda_1, \lambda_1(b\theta_d))$ *and* $a^*>\lambda_1(b\theta_d)$, *such that* (1.1) *has a positive solution if and only if* $a\geqslant a_*$; (1.1) *has exactly one positive solution for* $a=a_*$ *and* $a>a^*$, *at least two positive solutions for* $a\in(a_*, \lambda_1(b\theta_d))\cup\{a^*\}$, *and at least three positive solutions for* $a\in(\lambda_1(b\theta_d), a^*)$;

(ii) *there exist* $\bar{b}>\underline{b}>0$, *such that if* $b\leqslant\underline{b}$, *then Hopf bifurcation does not occur along* Γ; *if* $b\geqslant\bar{b}$, *Hopf bifurcation does occur along* Γ *at some* $a_0\in(\lambda_1(b\theta_d), a^*)$.

(iii) *if* D *is a ball in* R^N *with* $N\leqslant 3$ *and* $b\leqslant\underline{b}$, *then* Γ *is exactly* S-*shaped*: (1.1) *has exactly one positive solution for* $a=a_*$ (*neutrally stable*), *exactly two positive solutions for* $a\in(a_*, \lambda_1(b\theta_d)]$ (*one stable, one unstable*), *exactly three positive solutions for* $a\in(\lambda_1(b\theta_d), a^*)$ (*two stable, one unstable*), *exactly two positive solutions for* $a=a^*$ (*one stable, one neutrally stable*), *and exactly one positive solution for* $a\in(a^*, \infty)$ (*stable*).

Following [19], our strategy in proving Theorem 1.1 is to make use of the limiting equations of (1.1) which are obtained by letting $m\to\infty$ formally in (1.1). It turns out that one of the limiting problem differs significantly from the corresponding one in [19], and it is exactly this difference that enables us to observe the S-shaped solution curve (with two stable positive solutions) and Hopf bifurcation phenomenon. Accordingly, new techniques have to be explored.

First of all, we observe that if a is bounded away from λ_1 and m is large, positive solutions of (1.1) are of two types. More precisely, let (u, v) be any positive solution of (1.1), then either (u, v) is close to a positive solution of the problem

$$\begin{cases} \Delta u + u(a-u) = 0 & \text{in } D, \quad u|_{\partial D} = 0, \\ \Delta v + v(d+\tau - v) = 0 & \text{in } D, \quad v|_{\partial D} = 0, \end{cases} \tag{1.2}$$

or (mu, v) is close to a positive solution of the problem

$$\begin{cases} \Delta w + w(a - bv/(1+w)) = 0 & \text{in } D, \quad w|_{\partial D} = 0, \\ \Delta v + v(d + \tau w/(1+w) - v) = 0 & \text{in } D, \quad v|_{\partial D} = 0. \end{cases} \tag{1.3}$$

As in [19], our idea is to use solutions of (1.2) and (1.3) to construct two pieces of solution curves of (1.1), and then to piece them together to obtain one global solution curve of (1.1). Since (1.2) has a unique stable positive solution $(\theta_a, \theta_{d+\tau})$, thus it is easy to show that this solution of (1.2) induces a stable positive solution of (1.1) close to $(\theta_a, \theta_{d+\tau})$. In contrast, (1.3) turns out to be rather complicated. In order to obtain detailed information about (1.3), we restrict to the case that d is close to λ_1 and τ is small. In this case, by a Lyapunov–Schmidt reduction procedure and some perturbation arguments, we are able to completely understand the solution set $\{(w, v, a)\}$ of (1.3), which is in fact a smooth curve characterized by the function $s^{1/2} \int_D \Phi_1^3/(1+s\Phi_1)$ for $s \in [0, +\infty)$. This enables us to gain a rather complete understanding of the structure of positive solutions to (1.3) and the stability of these solutions. The perturbation arguments here come from some abstract perturbation results based on ideas of E. N. Dancer in [10], but they are proved by different method here and are improvements of the results in [10]. These abstract results have their own interests and are presented in the appendix.

Secondly, the two limit Eqs. (1.2) and (1.3) cease to induce solutions to the original system (1.1) when a is close to λ_1, although global bifurcation theory implies that these two pieces of solution curves can be extended towards $a = \lambda_1$ to form one connected solution set of (1.1). It turns out to be a difficult problem to understand the structure of this extended part of solution set which has to be near $a = \lambda_1$. A key ingredient to overcome this difficulty is to show that near any degenerate positive solution of (1.1) with m large and a close to λ_1, all positive solutions of (1.1) form a smooth curve which bends to the right of this degenerate solution. This relies crucially on a priori estimates on degenerate positive solutions of (1.1). This result, together with some other facts, implies that there exists a unique degenerate solution to (1.1) for a close to λ_1 and m large, and thus the two pieces of solution curves are connected by a third piece of solution curve with a unique turning point on it. This trick was used in [18] and [19], but the techniques here are very different. We call this turning point the left turning point as the curve bends to the right at this point. There is at least another turning point on the global solution curve of (1.1) which is inherited from the global solution curve of (1.3). Therefore the global solution curve of (1.1) is of S-shaped.

Analysis on (1.3) shows that Hopf bifurcation occurs along the global solution curve of (1.3) if the parameters are chosen suitably. This is used to show that Hopf bifurcation can also occur to (1.1) for certain parameter ranges.

This paper is organized as the follows: In Section 2, we show how to match the two pieces of solution curves of (1.1) which are obtained from the limit Eqs. (1.2) and (1.3) respectively. The limit Eq. (1.3) will be studied in detail in Section 3. In Section 4, we combine results from Sections 2 and 3 to prove the main results of the paper, Theorem 4.1, from which Theorem 1.1 follows. Some abstract perturbation results, which are needed in Sections 2–4, are presented in the Appendix, which consists of Section 5.

2. THE LEFT TURNING POINT FOR THE SOLUTION CURVE OF (1.1)

2.1. *The Main Result.*

Throughout this section we let $b>0, d>\lambda_1$ and $\tau>0$ be fixed. M_i, ε_i $(i=1, 2, \ldots)$ always denote generic positive constants depending only on b, d and τ unless otherwise specified. The main result of this section is

THEOREM 2.1. *There exists M_1 large such that for each $m \geqslant M_1$, there exists a unique a_* depending on m, b, d and τ, such that* (1.1) *has a positive solution if and only if $a \geqslant a_*$, where $a_* = \lambda_1 + O(1/\sqrt{m})$. Furthermore, there exists ε_1 small such that if $m \geqslant M_1$ and $a \in (a_*, \lambda_1+\varepsilon_1]$, then* (1.1) *has exactly two positive solutions, one stable and one unstable. When $a=a_*$,* (1.1) *has exactly one positive solution.*

The following two lemmas are the main ingredients for the proof of Theorem 2.1.

LEMMA 2.2. *There exists ε_2 small such that for any $\varepsilon \in (0, \varepsilon_2)$, there exists $M_2 = M_2(\varepsilon)$ large such that if $m \geqslant M_2(\varepsilon)$, then* (1.1) *has at least one positive solution for $a \geqslant \lambda_1+\varepsilon$, exactly two positive solutions for $a \in [\lambda_1+\varepsilon, \lambda_1+\varepsilon_2]$, of which one is stable while the other is unstable.*

LEMMA 2.3. *There exist ε_3 small and M_3 large such that if $m \geqslant M_3$ and $(\hat{a}, \hat{u}, \hat{v})$ is a degenerate positive solution of* (1.1) *with $\hat{a} \in (\lambda_1, \lambda_1+\varepsilon_3]$ (i.e., the linearization of* (1.1) *with respect to (u, v) at $(a, u, v) = (\hat{a}, \hat{u}, \hat{v})$ has non-trivial solutions), then the solutions of* (1.1) *close to $(\hat{a}, \hat{u}, \hat{v})$ lie on a smooth* *curve given by* $(a(s), u(s), v(s)) = (\hat{a} + s\eta(s),\ \hat{u} + O(s),\ \hat{v} + O(s))$, *where* $\eta(0) = 0$, $\eta'(0) > 0$; *in particular*, (1.1) *has no positive solution* (a, u, v) *near but to the left* (*i.e.*, $a < \hat{a}$) *of* $(\hat{a}, \hat{u}, \hat{v})$, *while there are exactly two solutions near but to the right of this point. Here* $O(s)$ *denotes functions defined on D with* L^∞ *norm of the order at most s as* $s \to 0$.

Remark 2.1. By using Theorem 3.6 of [9], one sees that $\eta(0) = 0$, $\eta'(0) > 0$ imply that for $a > \hat{a}$ but close to $\hat{a}$, one of the solutions on the smooth curve in Lemma 2.3 is stable and the other is unstable.

Proof of Theorem 2.1 (assuming Lemmas 2.2 and 2.3). Let $\varepsilon_1 = \min\{\varepsilon_2, \varepsilon_3, \lambda_1(b\theta_d) - \lambda_1\}$ and $M_1 = \max\{M_2(\varepsilon_1/2), M_3\}$. Fix $m \geqslant M_1$ and set $a_* = \inf\{a > \lambda_1 : (1.1)$ has at least a positive solution$\}$. It follows from Lemma 2.2 that $a_* < \lambda_1 + \varepsilon_1 \leqslant \lambda_1(b\theta_d)$. By the definition of a_*, there exist $a_i \to a_*$ and (u_i, v_i) which are positive solutions to (1.1) with $a = a_i$. Since the right hand side of (1.1) with $(a, u, v) = (a_i, u_i, v_i)$ is L^∞ bounded uniformly in i, by standard elliptic regularity and by passing to a subsequence, we may assume that (u_i, v_i) converges in C^1 to some (u_*, v_*). A simple comparison argument shows that $0 \leqslant u_i \leqslant \theta_{a_i}$, $\theta_d \leqslant v_i \leqslant \theta_{d+\tau}$. Hence $v_* \geqslant \theta_d$. If $u_* = 0$, then we necessarily have $v_* = \theta_d$. Set $\tilde{u}_i = u_i / \|u_i\|_\infty$. By using the equation of $\tilde{u}_i$ and elliptic regularity, we may assume that, subject to a subsequence, $\tilde{u}_i \to \tilde{u}$, where

$$\Delta\tilde{u} + (a_* - b\theta_d)\,\tilde{u} = 0, \qquad \tilde{u} \geqslant 0 \qquad \text{in } D, \quad \|\tilde{u}\|_\infty = 1, \quad \tilde{u}|_{\partial D} = 0.$$

Therefore $a_* = \lambda_1(b\theta_d)$, a contradiction. This shows that (1.1) has at least a positive solution (u_*, v_*) when $a = a_*$. It follows from the Implicit Function Theorem that (u_*, v_*) must be degenerate. Thus Lemma 2.3 implies that (a_*, u_*, v_*) lies on a smooth solution curve Γ of (1.1) which bends to the right at $a = a_*$. We can think of Γ as two branches of smooth curves joining smoothly at (a_*, u_*, v_*). Again by the Implicit Function Theorem, both branches of Γ can be extended smoothly rightward till $a = \lambda_1 + \varepsilon_1$, because by Lemma 2.3, Γ can not have a second degenerate solution on it for $a \in [a_*, \lambda_1 + \varepsilon_1]$. To save notations we denote the extension still by Γ. We show that (1.1) has no other positive solutions besides those on Γ for $a \in [a_*, \lambda_1 + \varepsilon_1]$. To this end we again argue by contradiction. Set $\hat{a} = \inf\{a > \lambda_1 : (1.1)$ has at least one positive soloution not on $\Gamma\}$. Repeating the above argument we see that there exists another smooth solution curve $\hat{\Gamma}$ which bends to the right at $a = \hat{a}$, and $\hat{\Gamma}$ can also be extended till $a = \lambda_1 + \varepsilon_1$. Note that these two curves cannot intersect each other for $a \in [\hat{a}, \lambda_1 + \varepsilon_1]$, due to the non-degeneracy of the solutions. This contradicts Lemma 2.2 as (1.1) has only two non-degenerate solutions when $a \in [\lambda_1 + \varepsilon_1/2, \lambda_1 + \varepsilon_1]$ and $m \geqslant M_1$. This establishes our assertion on the number of solutions.

The stability properties of the solutions follow from Lemma 2.2 and Remark 2.1. The assertion $a_* = \lambda_1 + O(1/\sqrt{m})$ follows from (2.16) below. This establishes Theorem 2.1. ∎

2.2. *Proof of Lemma* 2.2.

The proof of Lemma 2.2 relies on the following three lemmas.

LEMMA 2.4. *$\forall \varepsilon$ small, there exists $M_4 = M_4(\varepsilon)$ large such that if $a \geqslant \lambda_1 + \varepsilon$ and $m \geqslant M_4$, then* (1.1) *has a positive solution $(\tilde{u}, \tilde{v})$ which satisfies*

$$\theta_{a-\varepsilon/2} \leqslant \tilde{u} \leqslant \theta_a, \qquad \theta_{d+\tau\Phi_1/2} \leqslant \tilde{v} \leqslant \theta_{d+\tau}. \tag{2.1}$$

LEMMA 2.5. *Let ε small and $A > 0$ large be fixed. $\forall \delta > 0$ small, there exists $M_5 = M_5(\delta)$ large such that if $a \in [\lambda_1 + \varepsilon, A]$ and $m \geqslant M_5$, and if (u, v) is a positive solution of* (1.1), *then either $\|u - \theta_a\|_\infty + \|v - \theta_{d+\tau}\|_\infty \leqslant \delta$ or $\|mu - \tilde{w}\|_\infty + \|v - \tilde{v}\|_\infty + |a - \tilde{a}| \leqslant \delta$, where $(\tilde{w}, \tilde{v})$ is a positive solution of* (1.3) *with $a = \tilde{a}$.*

LEMMA 2.6. *There exists ε_4 small such that if $\lambda_1 < a \leqslant \lambda_1 + \varepsilon_4$, then* (1.3) *has a unique positive solution; furthermore, this solution is non-degenerate and unstable.*

Proof of Lemma 2.2 (assuming Lemmas 2.4–2.6). It follows from Lemma 2.4 that (1.1) has at least one solution for $a \geqslant \lambda_1 + \varepsilon$ if $m > M_4(\varepsilon)$. It remains to show that we can find some $\varepsilon_2 > 0$ and $M_2(\varepsilon) > M_4(\varepsilon)$ such that (1.1) has exactly two positive solutions for $a \in [\lambda_1 + \varepsilon, \lambda_1 + \varepsilon_2]$ whenever $\varepsilon \in (0, \varepsilon_2)$ and $m > M_2(\varepsilon)$.

Let $\varepsilon_2 = \varepsilon_4$ and fix any $\varepsilon \in (0, \varepsilon_2)$. Then by Lemma 2.6, (1.3) has a unique positive solution (w_a, v_a) for $a \in [\lambda_1 + \varepsilon, \lambda_1 + \varepsilon_2]$. Since it is non-degenerate,

$$C_1 \equiv \{(a, w_a, v_a) : a \in [\lambda_1 + \varepsilon, \lambda_1 + \varepsilon_2]\}$$

is a piece of smooth curve in $R \times C(D) \times C(D)$. It is easily checked that

$$C_2 \equiv \{(a, \theta_a, \theta_{d+\tau}) : a \in [\lambda_1 + \varepsilon, \lambda_1 + \varepsilon_2]\}$$

is a piece of smooth solution curve of problem (1.2) and the solutions on C_2 are non-degenerate and linearly stable.

By the Implicit Function Theorem, perturbation theory of linear operators [22] and the compactness of C_1, one easily sees that there exists a small neighborhood N_1 of C_1 such that the solutions of any regular perturbation of (1.3) in N_1 form a smooth curve near C_1, they are non-degenerate and linearly unstable. Similarly, there is a neighborhood N_2 of C_2 such that the solutions of any regular perturbation of (1.2) in N_2 form a smooth curve near C_2, and they are non-degenerate and linearly stable.

On the other hand, if m is large, it is easy to see that near N_2, (1.1) is a regular perturbation of (1.2); near N_1, (1.1) with the first equation multiplied by m and u replaced by w/m is a regular perturbation of (1.3). Hence by our previous discussion, for any $a \in [\lambda_1+\varepsilon, \lambda_1+\varepsilon_2]$ and large m, (1.1) has exactly one solution (u_1, v_1) with $(a, mu_1, v_1) \in N_1$ and it is linearly unstable; (1.1) has exactly one solution (u_2, v_2) with $(a, u_2, v_2) \in N_2$ and it is linearly stable. By Lemma 2.5, (1.1) has no other positive solutions for large m. Hence, for large m, (1.1) has exactly two positive solutions for each $a \in [\lambda_1+\varepsilon, \lambda_1+\varepsilon_2]$, one stable and one unstable. ∎

Proof of Lemma 2.4. By super and sub-solution method for predator–prey systems (see, e.g., [29] or [31]), it suffices to show that $(\bar{u}, \bar{v}) = (\theta_a, \theta_{d+\tau})$ and $(\underline{u}, \underline{v}) = (\theta_{a-\varepsilon/2}, \theta_{d+\tau\Phi_1/2})$ are pairs of super-sub solutions of (1.1) for large m. It is trivial to check the inequalities for $(\bar{u}, \bar{v})$. For $\underline{u}$, it suffices to have

$$m \geqslant M_6 = (2b/\varepsilon) \sup_D (\theta_{d+\tau}/\theta_{\lambda_1+\varepsilon/2}). \tag{2.2}$$

For $\underline{v}$ to satisfy the corresponding equation, we need

$$m\theta_{\lambda_1+\varepsilon/2}/(1+m\theta_{\lambda_1+\varepsilon/2}) \geqslant \Phi_1/2. \tag{2.3}$$

It is easy to see that as $m \to \infty$, $m\theta_{\lambda_1+\varepsilon/2}/(1+m\theta_{\lambda_1+\varepsilon/2}) \to 1$ uniformly in any compact subset of D, and that

$$\frac{\partial}{\partial v}\left(\frac{m\theta_{\lambda_1+\varepsilon/2}}{1+m\theta_{\lambda_1+\varepsilon/2}}\right)\bigg|_{\partial D} = m\,\frac{\partial\theta_{\lambda_1+\varepsilon/2}}{\partial v}\bigg|_{\partial D} \to -\infty \tag{2.4}$$

uniformly on ∂D. Therefore there exists $M_7 = M_7(\varepsilon)$ such that (2.3) holds provided $m \geqslant M_7$. It suffices to choose $M_4 = \max\{M_6, M_7\}$. ∎

Proof of Lemma 2.5. We argue by contradiction. Suppose that there exist $\delta_0 > 0$, $a_i \to a \in [\lambda_1+\varepsilon, A]$, $m_i \to \infty$, and positive solution (u_i, v_i) of (1.1) with $(a, m) = (a_i, m_i)$, such that $\|u_i - \theta_{a_i}\|_\infty + \|v_i - \theta_{d+\tau}\|_\infty \geqslant \delta_0$ and $\|mu_i - \tilde{w}\|_\infty + \|v_i - \tilde{v}\|_\infty + |a_i - \tilde{a}| \geqslant \delta_0$ for any positive solution $(\tilde{a}, \tilde{w}, \tilde{v})$ of (1.3). By passing to a subsequence, we have two possibilities:

Case a. $m_i \|u_i\|_\infty \to \infty$. By the equations of u_i and v_i, we can easily show that, subject to a subsequence, $(u_i, v_i) \to (u, v)$ in C^1 for some $v \geqslant \theta_d$. We may assume that $1/(1+m_i u_i) \to h$ weakly in L^2 with $0 \leqslant h \leqslant 1$ a.e. in D. Thus u satisfies the following equation weakly.

$$\Delta u + u(a - u - bvh) = 0, \qquad u|_{\partial D} = 0. \tag{2.5}$$

We show next that $u>0$ in D. Suppose that $u=0$. Set $\tilde{u}_i=(u_i/\|u_i\|_\infty)$. Using the equation of $\tilde{u}_i$, we may assume that $\tilde{u}_i\to\tilde{u}$ in C^1, where $\tilde{u}$ satisfies the following equation weakly.

$$\Delta\tilde{u}+\tilde{u}(a-bvh)=0, \qquad \|\tilde{u}\|_\infty=1, \qquad \tilde{u}|_{\partial D}=0, \qquad \tilde{u}\geqslant 0. \tag{2.6}$$

By Harnack inequality, $\tilde{u}>0$ in D. Hence $1/(1+m_iu_i)=1/(1+m_i\,\|u_i\|_\infty\,\tilde{u}_i)\to 0$ pointwisely in D. Therefore $h=0$ a.e., and then by (2.6), $a=\lambda_1$, which contradicts $a\geqslant\lambda_1+\varepsilon$. Thus $u\geqslant\neq 0$, and again by Harnack inequality, $u>0$ in D. This implies $h=0$, and hence by (2.5), $u=\theta_a$. It follows then $v=\theta_{d+\tau}$. However, this contradicts our assumption that (u_i, v_i) is uniformly bounded away from $(\theta_{a_i}, \theta_{d+\tau})$ which converges to $(\theta_a, \theta_{d+\tau})$.

Case b. $m_i\|u_i\|_\infty$ *is uniformly bounded.* In this case, set $w_i=m_iu_i$. Then

$$\begin{cases}\Delta w_i+w_i(a_i-u_i-bv_i/(1+w_i))=0, & w_i|_{\partial D}=0\\ \Delta v_i+v_i(d+\tau w_i/(1+w_i)-v_i)=0, & v_i|_{\partial D}=0.\end{cases} \tag{2.7}$$

Since $\|w_i\|_\infty$ is uniformly bounded, using (2.7), we may assume that $(w_i, v_i)\to(w, v)$ in C^1. As $u_i\to 0$, thus (a, w, v) is a non-negative solution of (1.3). If $w\geqslant\neq 0$, by Harnack inequality we know that $w>0$. This implies that (a, w, v) is a positive solution of (1.3), which contradicts our assumption that (a_i, w_i, v_i) is bounded away from any positive solution of (1.3). Therefore, we must have $w\equiv 0$. It follows that $(w_i, v_i)\to(0, \theta_d)$, and hence $a_i=\lambda_1(u_i+bv_i/(1+w_i))\to\lambda_1(b\theta_d)$.

By standard local bifurcation analysis we can show that (1.3) has a positive solution branch bifurcating from $(a, w, v)=(\lambda_1(b\theta_d), 0, \theta_d)$. Hence, we can find $a=\tilde{a}_i\to\lambda_1(b\theta_d)$ such that (1.3) with $a=\tilde{a}_i$ has a positive solution $(\tilde{w}_i, \tilde{v}_i)$ converging in L^∞ to $(0, \theta_d)$. Thus (a_i, m_iu_i, v_i) is close to $(\tilde{a}_i, \tilde{w}_i, \tilde{v}_i)$ for i large. This again contradicts our assumption. The proof is now complete. ∎

Proof of Lemma 2.6. We first claim that there exists $\varepsilon_4>0$ small such that for $a\in(\lambda_1, \lambda_1+\varepsilon_4]$, any positive solution (w, v) to (1.3) is non-degenerate, and the linearized eigenvalue problem

$$\begin{cases}-\Delta h+h(-a+bv/(1+w)^2))+bwk/(1+w)=\eta h, & h|_{\partial D}=0,\\ -\Delta k-\tau vh/(1+w)^2+k(-d-\tau w/(1+w)+2v)=\eta k, & k|_{\partial D}=0.\end{cases} \tag{2.8}$$

has a unique eigenvalue η_0 such that $Re\eta_0<0$; furthermore, η_0 is of multiplicity one.

For any sequence $a_i\to\lambda_1+$, let (w_i, v_i) be a solution of (1.3) with $a=a_i$. We first show that $\|w_i\|_\infty\to\infty$, $w_i/\|w_i\|_\infty\to\Phi_1$ and $v_i\to\theta_{d+\tau}$ in C^1.

Suppose that $\|w_i\|_\infty \leqslant C$. Then by using the equations of w_i and v_i, we may assume that $(w_i, v_i) \to (w, v)$ in C^1, where $v \geqslant \theta_d$ and w satisfies

$$\Delta w + w(\lambda_1 - bv/(1+w)) = 0, \qquad w|_{\partial\Omega} = 0, \qquad w \geqslant 0. \tag{2.9}$$

Hence $w = 0$: otherwise, $\lambda_1 = \lambda_1(bv/(1+w)) > \lambda(0) = \lambda_1$. Then, by the same reasoning we may assume that $w_i/\|w_i\|_\infty \to \tilde{w}$ in C^1, where $\tilde{w}$ satisfies

$$\Delta \tilde{w} + \tilde{w}(\lambda_1 - bv) = 0, \qquad \tilde{w}|_{\partial\Omega} = 0, \qquad \|\tilde{w}\|_\infty = 1, \qquad \tilde{w} \geqslant 0. \tag{2.10}$$

It follows that $\lambda_1 = \lambda_1(bv) > \lambda_1$. This contradiction shows that we must have $\|w_i\|_\infty \to \infty$. As before, we may assume that $1/(1+w_i)$ converges to some function h weakly in L^2, where $0 \leqslant h \leqslant 1$. Also, by the equation of w_i and elliptic regularity, we may assume that $w_i/\|w_i\|_\infty$ converges in C^1 to $\tilde{w}$, where $\tilde{w}$ satisfies the following equation weakly.

$$\Delta \tilde{w} + \tilde{w}(\lambda_1 - bvh) = 0, \qquad \tilde{w}|_{\partial\Omega} = 0, \qquad \|\tilde{w}\|_\infty = 1, \qquad \tilde{w} \geqslant 0. \tag{2.11}$$

Now by Harnack inequality $\tilde{w} > 0$ in D, which implies $h = 0$. Therefore by (2.11), we necessarily have $\tilde{w} = \Phi_1$. This implies that the whole sequence $w_i/\|w_i\|_\infty$ converges to Φ_1 in C^1. Using this and the equation of v_i, we deduce easily that $v_i \to \theta_{d+\tau}$ in C^1 by employing elliptic regularity.

Define T_i: $(W^{2,2} \cap H_0^1)^2 \to (L^2)^2$ by

$$T_i \begin{pmatrix} h \\ k \end{pmatrix} = \begin{pmatrix} \Delta h + \left(a_i - \dfrac{bv_i}{(1+w_i)^2}\right) h - \dfrac{bw_i k}{1+w_i} \\ \Delta k + \dfrac{\tau v_i h}{(1+w_i)^2} + \left(d + \dfrac{\tau w_i}{1+w_i} - 2v_i\right) k \end{pmatrix}.$$

It is easy to see that $T_i \to T_0$ in the operator norm, where T_0 is given by

$$T_0 \begin{pmatrix} h \\ k \end{pmatrix} = \begin{pmatrix} \Delta h + \lambda_1 h - bk \\ \Delta k + (d + \tau - 2\theta_{d+\tau})\, k \end{pmatrix}. \tag{2.12}$$

It is also easy to see that T_0 has 0 as an isolated eigenvalue and it is simple, with eigenfunction $\binom{h}{k} = \binom{\Phi_1}{0}$. Moreover, all the other eigenvalues are positive and bounded away from 0. Therefore by [22] we know that for large i, T_i must have a unique eigenvalue η_i close to zero, and all the other eigenvalues of T_i have positive real parts and are bounded away from 0. Moreover, η_i is simple and we can choose the corresponding eigenfunction $\binom{h_i}{k_i}$ in such a way that $\binom{h_i}{k_i} \to \binom{\Phi_1}{0}$ in, e.g., L^2. Since complex eigenvalues of T_i must come in conjugate pairs, it is easy to see that η_i is real and $\eta_i \to 0$. We want to further show that $\eta_i < 0$ for large i, which implies our claim.

Multiplying the equation of h_i by $\|w_i\|_\infty \Phi_1$ and integrating, after some rearrangements we have

$$\eta_i \|w_i\|_\infty \int h_i \Phi_1 = (\lambda_1 - a_i) \|w_i\|_\infty \int h_i \Phi_1 + \int \frac{b h_i \Phi_1 v_i}{\|w_i\|_\infty (1/\|w_i\|_\infty + \tilde{w}_i)^2} + \int \frac{b k_i \|w_i\|_\infty \tilde{w}_i \Phi_1}{1/\|w_i\|_\infty + \tilde{w}_i}. \tag{2.13}$$

It is not hard to show that

$$k_i \|w_i\|_\infty = \left(-\Delta - d - \frac{\tau w_i}{1 + w_i} + 2v_i - \eta_i \right)^{-1} \times \left\{ \frac{\tau h_i v_i}{\|w_i\|_\infty (1/\|w_i\|_\infty + \tilde{w}_i)^2} \right\} \to 0 \quad \text{in } H_0^1.$$

Multiplying the equation of w_i by Φ_1 and integrating, we obtain

$$\lim_{i \to \infty} (a_i - \lambda_1) \|w_i\|_\infty = b \int \theta_{d+\tau} \Phi_1 \Big/ \int \Phi_1^2. \tag{2.14}$$

Therefore passing to the limit in (2.13) we deduce

$$\lim_{i \to \infty} \eta_i \|w_i\|_\infty = -b \int \theta_{d+\tau} \Phi_1 \Big/ \int \Phi_1^2 < 0, \tag{2.15}$$

which implies that $\eta_i < 0$ for large i. This establishes our claim at the beginning.

By a simple contradiction argument, it is easy to show that $\forall \varepsilon > 0$ small, there exists $C = C(\varepsilon)$ such that if $a \geqslant \lambda_1 + \varepsilon$, then $\|w\|_\infty \leqslant C$ and $\|v\|_\infty \leqslant C$ for any positive solution (w, v) of (1.3). The uniqueness assertion now follows from a rather standard degree argument (see, e.g., the proof of Theorem 2.6 in [18] for a detailed treatment under a similar situation). We sketch it rather informally below. First note that (1.3) has no positive solution for $a > \lambda_1(b\theta_{d+\tau})$; thus $(w, v) = (0, 0)$ and $(0, \theta_d)$ are the only nonnegative solutions for such a. Combining these with the a priori estimates above, and using the homotopy property of the degree, one can show that the total degree for all nonnegative solutions of (1.3) is 0 for any $a > \lambda_1$. It is easy to show that $(w, v) = (0, 0)$ has degree 0 and $(w, v) = (0, \theta_d)$ has degree 1 when $\lambda_1 < a \leqslant \lambda_1 + \varepsilon < \lambda_1(b\theta_d)$. Hence by the additivity property of the degree, for a in this range, the total degree of positive solutions must be -1. On the other hand, we have already shown that any positive solution of (1.3) for $a \in (\lambda_1, \lambda_1 + \varepsilon]$ is non-degenerate and its linearization has exactly one negative eigenvalue. Thus the degree of any such solution is -1. As the total degree of the positive solutions is -1, there must be exactly one positive solution. This completes the proof. ∎

2.3. *Proof of Lemma* 2.3

To establish Lemma 2.3, we need the following technical result.

LEMMA 2.7. Suppose that (u_i, v_i) is a degenerate positive solution of (1.1) with $(a, m) = (a_i, m_i)$, $a_i \to \lambda_1+$, $m_i \to \infty$. Then $(u_i, v_i) \to (0, \theta_{d+\tau})$, $(u_i/\|u_i\|_\infty) \to \Phi_1$ in C^1 and

$$\lim_{i\to\infty} m_i \|u_i\|_\infty^2 = b \int \theta_{d+\tau}\Phi_1 \Big/ \int \Phi_1^3$$
$$\lim_{i\to\infty} m_i(a_i - \lambda_1)^2 = 4b \int \theta_{d+\tau}\Phi_1 \Big/ \left[\left(\int \Phi_1^2\right)^2 \int \Phi_1^3\right]. \tag{2.16}$$

Proof. Since the proof is quite lengthy, we separate it into several steps.

Step 1. $(u_i, v_i) \to (0, \theta_{d+\tau})$, $u_i/\|u_i\|_\infty \to \Phi_1$ in C^1 and $m_i\|u_i\|_\infty \to \infty$.

By elliptic regularity we may assume that $(u_i, v_i) \to (u, v)$ in C^1 with $v \geqslant \theta_d$. Since $u_i \leqslant \theta_{a_i}$, we see that $u \equiv 0$. We may also assume that $1/(1+m_iu_i) \to h$ weakly in L^2 and $0 \leqslant h \leqslant 1$ a.e. in D. Set $\tilde{u}_i = u_i/\|u_i\|_\infty$. Using the equations and elliptic regularity, we may assume that $\tilde{u}_i \to \tilde{u}$ in C^1, where $\tilde{u}$ satisfies the following equation weakly.

$$\Delta\tilde{u} + \tilde{u}(\lambda_1 - bvh) = 0, \qquad \|\tilde{u}\|_\infty = 1, \qquad \tilde{u} \geqslant 0, \qquad \tilde{u}|_{\partial D} = 0.$$

Harnack inequality implies that $\tilde{u} > 0$ in D. Multiplying the equation of $\tilde{u}$ by Φ_1 and integrating, we have $\int_D \tilde{u}vh = 0$. Hence $h = 0$ a.e., and thus $\tilde{u} = \Phi_1$. This implies that $m_i\|u_i\|_\infty \to \infty$. Thus $m_iu_i/(1+m_iu_i) \to 1$ in L^2, and then by the equation of v_i, $v \equiv \theta_{d+\tau}$.

Step 2. Since (u_i, v_i) is degenerate, there exists (h_i, k_i) with $\|h_i\|_2 + \|k_i\|_2 = 1$ and

$$\begin{cases} \Delta h_i + h_i(a_i - 2u_i - bv_i/(1+m_iu_i)^2) - bu_ik_i/(1+m_iu_i) = 0, \\ \Delta k_i + k_i(d + \tau m_iu_i/(1+m_iu_i) - 2v_i) + \tau m_iv_ih_i/(1+m_iu_i)^2 = 0, \\ h_i|_{\partial D} = k_i|_{\partial D} = 0. \end{cases} \tag{2.17}$$

Claim. Any subsequence of $\{h_i\}$ has a further subsequence, still denoted by $\{h_i\}$ for the sake of convenience, such that $h_i \to \mu\Phi_1$ in L^2 for some $\mu \neq 0$.

For the sake of late argument we collect here two useful identities. Multiplying the equations of u_i and h_i by Φ_1 and integrating respectively, we have

$$
\begin{aligned}
&m_i \|u_i\|_\infty (a_i - \lambda_1) \int \tilde{u}_i \Phi_1 = m_i \|u_i\|_\infty^2 \int \tilde{u}_i^2 \Phi_1 + b \int \frac{v_i \Phi_1 m_i u_i}{1+m_i u_i}, \\
&\int h_i \Phi_1 = \frac{\|u_i\|_\infty}{a_i - \lambda_1} \left(2 \int h_i \tilde{u}_i \Phi_1 + \int \frac{b k_i \tilde{u}_i \Phi_1}{1+m_i u_i} \right) + \int \frac{b h_i v_i \Phi_1}{(a_i - \lambda_1)(1+m_i u_i)^2}.
\end{aligned}
\tag{2.18}
$$

Now we set to prove our claim. By (2.17) and our a priori estimates in Step 1, we may assume that, passing to a subsequence if necessary, $(h_i, k_i) \to (h, k)$ in H_0^1. Then $\Delta h + \lambda_1 h = 0$ and hence $h = \mu \Phi_1$ for some real number μ. Suppose that $\mu = 0$, i.e., $h_i \to 0$. Set $(\tilde{h}_i, \tilde{k}_i) = (h_i / \|h_i\|_2, k_i / (\|h_i\|_2 m_i))$. Then $(\tilde{h}_i, \tilde{k}_i)$ satisfies

$$
\begin{cases}
\Delta \tilde{h}_i + \tilde{h}_i (a_i - 2u_i - b v_i / (1+m_i u_i)^2) - b m_i u_i \tilde{k}_i / (1+m_i u_i) = 0, \\
\qquad \tilde{h}_i|_{\partial D} = 0, \\
\Delta \tilde{k}_i + \tilde{k}_i (d + \tau m_i u_i / (1+m_i u_i) - 2v_i) + \tau v_i \tilde{h}_i / (1+m_i u_i)^2 = 0, \\
\qquad \tilde{k}_i|_{\partial D} = 0.
\end{cases}
$$

We first show that $\tilde{h}_i \to \Phi_1 / \|\Phi_1\|_2$ in $C^1(\bar{D})$. Since

$$
v_i / (1+m_i u_i)^2 < v_i / m_i u_i = (1/m_i \|u_i\|_\infty)(v_i / \tilde{u}_i) \to 0 \qquad \text{in } L^\infty,
$$

one sees from the equation of $\tilde{k_i}$ that $\tilde{k}_i \to 0$ in H_0^1. Similarly, $\tilde{h}_i \to \tilde{h}$ in H_0^1 for some $\tilde{h}$. Clearly $\|\tilde{h}\|_2 = 1$ and $\tilde{h}$ satisfies $\Delta \tilde{h} + \lambda_1 \tilde{h} = 0$, which implies that $\tilde{h} = \Phi_1 / \|\Phi_1\|_2$ if we change the signs of (h_i, k_i) if necessary. Hence the whole sequence $\tilde{h}_i \to \Phi_1 / \|\Phi_1\|_2$ strongly in H_0^1. By further pursuing the regularity we can show that $\tilde{h}_i \to \Phi_1 / \|\Phi_1\|_2$ in C^1. Now there are two possibilities for our consideration:

(i) Subject to choosing a subsequence, $m_i \|u_i\|_\infty^2 \geqslant C$ for some positive constant C. For this case we have

$$
m_i v_i h_i / (1+m_i u_i)^2 \leqslant v_i \|h_i\|_2 \tilde{h}_i / (m_i \|u_i\|_\infty^2 \tilde{u}_i^2) \leqslant \tilde{C} \|h_i\|_2 \to 0.
$$

Thus by (2.17) we obtain $k_i \to 0$ in L^2. This together with $h_i \to 0$ contradicts $\|h_i\|_2 + \|k_i\|_2 = 1$.

(ii) Subject to choosing a subsequence, $m_i \|u_i\|_\infty^2 \to 0$. For this case, set $\hat{k}_i = k_i \|u_i\|_\infty / \|h_i\|_2$. Then

$$
\begin{aligned}
&\Delta \hat{k}_i + \hat{k}_i (d + \tau m_i u_i / (1+m_i u_i) - 2v_i) + \tau m_i \|u_i\|_\infty \tilde{h}_i v_i / (1+m_i u_i)^2 = 0, \\
&\hat{k}_i|_{\partial D} = 0.
\end{aligned}
$$

Since $m_i \|u_i\|_\infty \to \infty$, we have

$$\frac{\tau m_i \|u_i\|_\infty \tilde{h}_i v_i}{(1+m_i u_i)^2} \leqslant \frac{\tau}{m_i \|u_i\|_\infty} \frac{\tilde{h}_i v_i}{\tilde{u}_i^2} \leqslant \frac{C}{m_i \|u_i\|_\infty} \to 0.$$

Thus by the equation for $\hat{k}_i$ we see $\hat{k}_i \to 0$ in L^2. Rewrite the second equation of (2.18) as

$$\begin{aligned}\int \tilde{h}_i \Phi_1 &= 2 \|u_i\|_\infty/(a_i-\lambda_1) \int \tilde{h}_i \tilde{u}_i \Phi_1 \\ &\quad + b/(a_i-\lambda_1) \int \tilde{h}_i v_i \Phi_1/(1+m_i u_i)^2 \\ &\quad + b \|u_i\|_\infty/(a_i-\lambda_1) \int (k_i/\|h_i\|_2)\, \tilde{u}_i \Phi_1/(1+m_i u_i). \qquad (2.19)\end{aligned}$$

Since $m_i \|u_i\|_\infty^2 \to 0$, passing to the limit in the first equation of (2.18) we find

$$\lim_{i\to\infty} m_i \|u_i\|_\infty (a_i-\lambda_1) \to b \int \theta_{d+\tau} \Phi_1 \bigg/ \int \Phi_1^2. \qquad (2.20)$$

Hence, using $m_i \|u_i\|_\infty^2 \to 0$ again,

$$\lim_{i\to\infty} \|u_i\|_\infty/(a_i-\lambda_1) = \lim_{n\to\infty} \frac{m_i \|u_i\|_\infty^2}{m_i \|u_i\|_\infty (a_i-\lambda_1)} = 0. \qquad (2.21)$$

Therefore the first term of the right hand side of (2.19) goes to zero as $i \to \infty$. For the other terms, using (2.20), we have

$$\begin{aligned}\frac{b}{a_i-\lambda_1} \int \frac{\tilde{h}_i v_i \Phi_1}{(1+m_i u_i)^2} &\leqslant \frac{b(a_i-\lambda_1)}{((a_i-\lambda_1)\, m_i \|u_i\|_\infty)^2} \int \frac{\tilde{h}_i v_i \Phi_1}{\tilde{u}_i^2} \\ &\leqslant C(a_i-\lambda_1) \to 0,\end{aligned}$$

$$\begin{aligned}\frac{b \|u_i\|_\infty}{a_i-\lambda_1} \int \frac{k_i}{\|h_i\|_2} \frac{\tilde{u}_i \Phi_1}{1+m_i u_i} &= \frac{b}{m_i(a_i-\lambda_1)} \int \frac{k_i}{\|h_i\|_2} \frac{\tilde{u}_i \Phi_1}{\tilde{u}_i + 1/(m_i \|u_i\|_\infty)} \\ &\leqslant \frac{b}{(a_i-\lambda_1)\, m_i \|u_i\|_\infty} \int \hat{k}_i \Phi_1 \\ &\leqslant C \|\hat{k}_i\|_2 \|\Phi_1\|_2 \to 0.\end{aligned}$$

Therefore by passing to the limit in (2.19) we have $\int \tilde{h}_i \Phi_1 \to 0$, i.e., $\int_\Omega \Phi_1^2 = 0$, which is a contradiction. This establishes our assertion that $\mu \neq 0$.

Step 3. $\{m_i \|u_i\|_\infty^2\}$ is bounded away from both ∞ and 0.

If this assertion is not true, then subject to passing to a subsequence, $m_i \|u_i\|_\infty^2 \to \infty$ or $m_i \|u_i\|_\infty^2 \to 0$.

If $m_i \|u_i\|_\infty^2 \to \infty$, by the first equation in (2.18) we see that

$$\lim_{i\to\infty} (a_i - \lambda_1)/\|u_i\|_\infty = \int \Phi_1^3 \Big/ \int \Phi_1^2.$$

On the other hand, we have

$$\begin{aligned} \int \frac{k_i \tilde{u}_i \Phi_1}{1+m_i u_i} &= \frac{1}{m_i \|u_i\|_\infty} \int \frac{k_i \tilde{u}_i \Phi_1}{\tilde{u}_i + 1/(m_i \|u_i\|_\infty)} \\ &\leqslant \frac{\|k_i\|_2 \|\Phi_1\|_2}{m_i \|u_i\|_\infty} \to 0, \end{aligned}$$

$$\begin{aligned} \int \frac{h_i v_i \Phi_1}{(a_i-\lambda_1)(1+m_i u_i)^2} &= \frac{1}{(a_i-\lambda_1)\, m_i^2 \|u_i\|_\infty^2} \int \frac{h_i v_i \Phi_1}{(\tilde{u}_i + 1/(m_i \|u_i\|_\infty))^2} \\ &\leqslant \frac{\|u_i\|_\infty}{a_i - \lambda_1} \frac{1}{m_i^2 \|u_i\|_\infty^3} \int \frac{h_i v_i \Phi_1}{\tilde{u}_i^2} \\ &\leqslant \frac{C}{m_i^2 \|u_i\|_\infty^3} \to 0. \end{aligned}$$

Therefore by Step 2, passing to the limit for a subsequence in the second equation of (2.18) we obtain

$$\mu \int \Phi_1^2 = 2 \lim_{i\to\infty} \|u_i\|_\infty/(a_i - \lambda_1) \int h_i \tilde{u}_i \Phi_1 = 2\mu \int \Phi_1^2,$$

which gives $\mu = 0$, contradicting step 2 above. It remains to consider the case where $m_i \|u_i\|_\infty^2 \to 0$. Note that for this case, both (2.20) and (2.21) hold. Therefore it is easy to check that

$$\lim_{i\to\infty} \|u_i\|_\infty/(a_i - \lambda_1) \left(2 \int h_i \tilde{u}_i \Phi_1 + b \int k_i \tilde{u}_i \Phi_1/(1+m_i u_i) \right) = 0.$$

For the last term in the second equation of (2.18), by (2.20) we have

$$\int \frac{h_i v_i \Phi_1}{(a_i-\lambda_1)(1+m_i u_i)^2}$$
$$=\frac{a_i-\lambda_1}{(m_i\,\|u_i\|_\infty(a_i-\lambda_1))^2}\int \frac{h_i v_i \Phi_1}{(\tilde{u}_i+1/m_i\,\|u_i\|_\infty)^2}\to 0.$$

Now passing to the limit in (2.18) we again have $\mu=0$, which contradicts the Step 2. This proves our assertion.

Step 4. We prove (2.16) holds and hence finish the proof.

By Steps 2 and 3, we may assume, passing to a subsequence if necessary, that $m_i\,\|u_i\|_\infty^2\to A\in(0,\infty)$ and $h_i\to\mu\Phi_1$, $\mu\neq 0$. After suitably rescaling we may assume that $h_i\to\Phi_1$ and then,

$$k_i\to(-\Delta-d-\tau+2\theta_{d+\tau})^{-1}\left(\frac{\tau\theta_{d+\tau}}{A\Phi_1}\right)\qquad\text{in } H_0^1.$$

Passing to the limit in the first equation in (2.18) we have

$$\lim_{i\to\infty} m_i\,\|u_i\|_\infty\,(a_i-\lambda_1)=\left(A\int\Phi_1^3+b\int\theta_{d+\tau}\Phi_1\right)\bigg/\int\Phi_1^2. \tag{2.22}$$

Then we can pass to the limit in the second equation in (2.18) to conclude that

$$\lim_{i\to\infty}\|u_i\|_\infty/(a_i-\lambda_1)=\int\Phi_1^2\bigg/\left(2\int\Phi_1^3\right). \tag{2.23}$$

On the other hand, by (2.22) we find

$$\lim_{i\to\infty}(a_i-\lambda_1)/\|u_i\|_\infty=\left(A\int\Phi_1^3+b\int\theta_{d+\tau}\Phi_1\right)\bigg/A\int\Phi_1^2. \tag{2.24}$$

Therefore from (2.23) and (2.24) it follows that $A=(b\int\theta_{d+\tau}\Phi_1)/\int\Phi_1^3$. This implies that the whole sequence $m_i\,\|u_i\|_\infty^2$ converges to $A=(b\int\theta_{d+\tau}\Phi_1)/\int\Phi_1^3$, and by (2.24),

$$(a_i-\lambda_1)\,m_i^{1/2}\to 2\left(b\int\theta_{d+\tau}\Phi_1\bigg/\int\Phi_1^3\right)^{1/2}\bigg/\int\Phi_1^2. \tag{2.25}$$

The proof is now complete. ∎

Finally we set to establish Lemma 2.3.

Proof of Lemma 2.3. Let (u_i, v_i) be a degenerate positive solution of (1.1) with $(a, m) = (a_i, m_i)$, where $a_i \to \lambda_1 +$ and $m_i \to \infty$. Define T_i: $[W^{2,p} \cap H_0^1]^2 \to [L^p]^2$, $p > N$ by

$$T_i \binom{h}{k} = \begin{pmatrix} \Delta h + (a_i - 2u_i - bv_i/(1+m_iu_i)^2)\, h - bu_ik/(1+m_iu_i) \\ \Delta k + (d + \tau m_iu_i/(1+m_iu_i) - 2v_i)\, k + \tau v_im_ih/(1+m_iu_i)^2 \end{pmatrix}.$$

Clearly 0 is an eigenvalue of T_i by the degeneracy of (u_i, v_i). To show that zero is the only eigenvalue of T_i close to zero and that it is a simple eigenvalue, the term $\tau m_iv_i/(1+m_iu_i)^2$ brings along trouble as it approaches the unbounded function $\tau\theta_{d+\tau}/\Phi_1^2$. However, this can be overcome by introducing another operator

$$\tilde{T}_i \binom{h}{k} = \begin{pmatrix} \Delta h + (a_i - 2u_i - bv_i/(1+m_iu_i)^2)\, h - bm_iu_ik/(1+m_iu_i) \\ \Delta k + (d + \tau m_iu_i/(1+m_iu_i) - 2v_i)\, k + \tau v_ih/(1+m_iu_i)^2 \end{pmatrix}.$$

We observe that T_i and $\tilde{T}_i$ have the same eigenvalues with the same multiplicity: $\binom{h}{k}$ is an eigenfunction of T_i if and only if $\binom{mh}{k}$ is an eigenfunction for $\tilde{T}_i$ corresponding to the same eigenvalue. Note that $\tilde{T}_i \to T_0$ in operator norm, where T_0 is given as in (2.14). Therefore as in the proof of Lemma 2.6 we see that zero is a simple eigenvalue of T_i for large i, and all other eigenvalues are uniformly bounded away from zero. Set $Ker\ T_i = \{\binom{h_i}{k_i}\}$, then, by Lemma 2.7 and its proof, we may assume that

$$(h_i, k_i) \to (h, k) \equiv \left(\Phi_1, \frac{\tau \int \Phi_1^3}{b \int \theta_{d+\tau}\Phi_1} (-\Delta - d - \tau + 2\theta_{d+\tau})^{-1} \left(\frac{\theta_{d+\tau}}{\Phi_1}\right)\right)$$

in H_0^1 and hence in C^1 by elliptic regularity.

Now we want to apply Theorem 3.2 of Crandall and Rabinowitz [9] to see that all solutions close to (a_i, u_i, v_i) must lie on a smooth curve given by

$$(a_i(s), u_i(s), v_i(s)) = (a_i + s\eta_i(s), u_i + sh_i + s\phi_i(s), v_i + sk_i + s\psi_i(s)), \qquad (2.26)$$

where $\eta_i(0) = 0$, $(\phi_i(0), \psi_i(0)) = (0, 0)$, and $\binom{\phi_i(s)}{\psi_i(s)}$ is in the complement of $\binom{h_i}{k_i}$.

To be able to use that result we still need to check that the condition $(u_i, 0) \notin$ Range of $\tilde{T}_i$ is satisfied for all large i. To this end, we define a functional l_0 by $l_0(u, v) = \int u\Phi_1$. Then clearly $l_0 \in N(T_0^*)$ and $N(l_0) = R(T_0)$. Choose $l_i \in N(\tilde{T}_i^*)$ satisfying $\|l_i\| = \|l_0\|$ and $l_i \to l_0$ in $[L^p]^2$. Then $(u_i, 0) \in R(\tilde{T}_i)$ would imply that

$$0 = l_i(\tilde{u}_i, 0) \to l_0(\Phi_1, 0) = \int \Phi_1^2 > 0.$$

This justifies the use of [9]. It remains to show that $\eta_i'(0) > 0$, i.e., $a_i''(0) > 0$ for all large i.

By substituting $(a, u, v)=(a_i(s), u_i(s), v_i(s))$ into (1.1) and differentiating the equations with respect to s twice at $s=0$, we obtain

$$T_i\begin{pmatrix}2\phi_i'(0)\\2\psi_i'(0)\end{pmatrix}=\begin{pmatrix}-a_i''(0)\,u_i+2h_i^2+\dfrac{2bh_ik_i}{(1+m_iu_i)^2}-\dfrac{2bm_ih_i^2v_i}{(1+m_iu_i)^3}\\2k_i^2+\dfrac{2\tau m_i^2v_ih_i^2}{(1+m_iu_i)^3}-\dfrac{2\tau m_ih_ik_i}{(1+m_iu_i)^2}\end{pmatrix},$$

which is equivalent to

$$\tilde{T}_i\begin{pmatrix}2\phi_i'(0)\\2\psi_i'(0)/m_i\end{pmatrix}=\begin{pmatrix}-a_i''(0)\,\|u_i\|_\infty\,\tilde{u}_i+2h_i^2+\dfrac{2bh_ik_i}{(1+m_iu_i)^2}-\dfrac{2bm_ih_i^2v_i}{(1+m_iu_i)^3}\\\dfrac{2k_i^2}{m_i}+\dfrac{2\tau m_iv_ih_i^2}{(1+m_iu_i)^3}-\dfrac{2\tau h_ik_i}{(1+m_iu_i)^2}\end{pmatrix}.$$

We first show that $a_i''(0)\,\|u_i\|_\infty$ is uniformly bounded. Suppose not: without loss of generality we may assume that $a_i''(0)\,\|u_i\|_\infty\to\infty$. Set

$$(\tilde{\phi}_i, \tilde{\psi}_i)=(2\phi_i'(0)/(a_i''(0)\,\|u_i\|_\infty), 2\psi_i'(0)/(a_i''(0)\,\|u_i\|_\infty)). \qquad (2.27)$$

By elliptic regularity we may assume that $(\tilde{\phi}_i, \tilde{\psi}_i)\to(\tilde{\phi}, \tilde{\psi})$ in H_0^1, where $\tilde{\phi}$ satisfies weakly $\Delta\tilde{\phi}+\lambda_1\tilde{\phi}=-\Phi_1$ and $\tilde{\phi}|_{\partial D}=0$. Multiplying this equation by Φ_1 and integrating, we deduce $\int\Phi_1^2=0$, a contradiction. Therefore we may assume that $a_i''(0)\,\|u_i\|_\infty\to A$ for some constant A. Again by elliptic regularity we may assume that $(2\phi_i'(0), 2\psi_i'(0)/m_i)\to(\phi, \psi)$, where ϕ satisfies

$$\Delta\phi+\lambda_1\phi=-A\Phi_1+2\Phi_1^2, \qquad \phi|_{\partial D}=0. \qquad (2.28)$$

Multiplying (2.28) by Φ_1 and integrating, we obtain

$$\lim_{i\to\infty} a_i''(0)\,\|u_i\|_\infty=A=2\int\Phi_1^3\bigg/\int\Phi_1^2>0.$$

This completes the proof of Lemma 2.3. ∎

3. THE LIMIT EQUATION (1.3)

3.1. *The Global Solution Curve*

In this subsection we study the solution set to the limit problem (1.3). As we shall be mainly concerned with the case when d is close to λ_1, τ is positive and small, we make the following change of variables:

$$a=\lambda_1+\varepsilon a_1, \qquad d=\lambda_1+d_1\varepsilon, \qquad \tau=\varepsilon\tau_1, \qquad v=\varepsilon z. \qquad (3.1)$$

Then (1.3) can be written as

$$\begin{cases} -\Delta w = \lambda_1 w + \varepsilon w(a_1 - bz/(1+w)), & w|_{\partial D} = 0, \\ -\Delta z = \lambda_1 z + \varepsilon z(d_1 + \tau_1 w/(1+w) - z), & z|_{\partial D} = 0. \end{cases} \tag{3.2}$$

We fix all the parameters in (3.2) except a_1 and ε, and our purpose is to understand the exact solution set $\{(w, z, a_1)\}$ of (3.2) when ε is positive and small. Let $\phi_1 = \Phi_1/\|\Phi_1\|_2$, where Φ_1 is defined as in Sect. 1. Our first result can be stated as follows.

Theorem 3.1. *There exist ε^0 and a_1^0, both small and positive, such that for any $\varepsilon \in (0, \varepsilon^0]$, all the positive solutions (w, z, a_1) of (3.2) form a smooth curve Γ^ε which varies smoothly with ε. Moreover, if $a_1 \leqslant a_1^0$, then there is exactly one positive solution (w, z) to (3.2) and it is non-degenerate and unstable; if $a_1 \geqslant a_1^0$, then the solutions are parameterized by*

$$(w, z, a_1) = (w(s, \varepsilon), z(s, \varepsilon), a_1(s, \varepsilon)), \qquad s_*(\varepsilon) \leqslant s \leqslant s^*(\varepsilon),$$

with $(w(s, 0), z(s, 0), a_1(s, 0)) = (s\phi_1, f(s)\,\phi_1, (s))$, $s_*(0) = 0$, $g(s^*(0)) = a_1^0$, where

$$\begin{aligned} f(s) &= \left(d_1 + \tau_1 s \int_D \frac{\phi_1^3}{1 + s\phi_1}\,dx\right) \Big/ \int_D \phi_1^3\,dx, \\ g(s) &= bf(s) \int_D \frac{\phi_1^3}{1 + s\phi_1}\,dx. \end{aligned} \tag{3.3}$$

Remark 3.2. We shall study the shape of the curve Γ^ε in the next subsection. This is important in understanding the shape of the global bifurcation curve of (1.1). We shall also show in the next subsection how stability of the solutions on Γ^ε can be determined. In particular, we show that Hopf bifurcation sometimes occurs along Γ^ε.

The rest of this subsection is devoted to the proof of Theorem 3.1. Results from the Appendix will be frequently used.

Denote $X = [W^{2,p} \cap H_0^1]^2$, $Y = [L^p(D)]^2$, $p > N$ and define $H: X \to Y$ and $B: X \times R \times R \to Y$ by

$$H(w, z) = (\Delta w + \lambda_1 w, \Delta z + \lambda_1 z),$$

$$B(w, z, a, \varepsilon) = (w(a_1 - bz/(1+w), z(d_1 + \tau_1 w/(1+w) - z)),$$

respectively. Clearly (3.2) is equivalent to

$$H(w, z) + \varepsilon B(w, z, a_1, \varepsilon) = 0.$$

Let X_1 and Y_1 be the L^2 orthogonal complements of span$\{(\phi_1, 0), (0, \phi_1)\}$ in X and Y respectively, and let P and Q denote the orthogonal projections of X and Y onto X_1 and Y_1 respectively. Then any $(w, z) \in X$ can be written as $(w, z) = (s, t)\, \phi_1 + U$, where $U = P(w, z)$, and (3.2) is equivalent to

$$\begin{cases} QH((s, t)\, \phi_1 + U) + \varepsilon QB((s, t)\, \phi_1 + U, a_1, \varepsilon) = 0, \\ (I - Q)\, H((s, t)\, \phi_1 + U) + \varepsilon (I - Q)\, B((s, t)\, \phi_1 + U, a_1, \varepsilon) = 0. \end{cases} \tag{3.4}$$

Since $H((s, t)\phi_1) \equiv 0$ and $(I - Q)\, H(X_1) = \{0\}$, we see immediately that for any $\varepsilon > 0$, (3.4) is equivalent to

$$QH(U) + \varepsilon QB((s, t)\, \phi_1 + U, a_1, \varepsilon) = 0 \tag{3.5}$$

and

$$(I - Q)\, B((s, t)\, \phi_1 + U, a_1, \varepsilon) = 0. \tag{3.6}$$

Since QH is invertible, one easily sees by the Implicit Function Theorem that for any constant $C > 0$, there exists $\varepsilon_0 = \varepsilon_0(C) > 0$ such that for any $(s', t', a_1') \in A \equiv \{(s, t, a_1) : |s|,\ |t|,\ |a_1| \leqslant C\}$, there is a $\delta > 0$ and a small neighborhood N of $((s', t')\, \phi_1, a_1', 0)$ in $X \times R \times R$ such that the solution set of (3.5) in N is given by

$$\{((s, t)\, \phi_1 + U(s, t, a_1, \varepsilon), a_1, \varepsilon) : |s - s'|, |t - t'|, |a_1 - a_1'| < \delta, |\varepsilon| < \varepsilon_0\}.$$

Since A is compact, by a finite covering argument, we see that there exists a $\delta_0 = \delta_0(C) > 0$ and a neighborhood N_0 of $\{((s, t)\, \phi_1, a_1, 0) : |s|, |t|, |a_1| \leqslant C\}$ in $X \times R \times R$ such that the solution set of (3.5) in N_0 is given by

$$\{((s, t)\, \phi_1 + U(s, t, a_1, \varepsilon), a_1, \varepsilon) : |s|, |t|, |a_1| < C + \delta_0, |\varepsilon| < \varepsilon_0\}.$$

To summarize, we have proved the following result.

Lemma 3.3. *(w, z, a_1, ε) is a solution of (3.2) contained in N_0 if and only if $(w, z, a_1, \varepsilon) = ((s, t)\, \phi_1 + U(s, t, a_1, \varepsilon), a_1, \varepsilon)$ for some $|s|, |t|, |a_1| < C + \delta_0$ and some $\varepsilon \in (0, \varepsilon_0)$ which satisfy*

$$(I - Q)\, B[(s, t)\, \phi_1 + U(s, t, a_1, \varepsilon), a_1, \varepsilon] = 0. \tag{3.7}$$

Let

$$u(s, t, a_1, \varepsilon) = (QH)^{-1}\, \{QB((s, t)\, \phi_1 + U(s, t, a_1, \varepsilon), a_1, \varepsilon)\}.$$

One sees immediately that $U = \varepsilon u$ and u is as smooth as U which is as smooth as B. Now define $F\colon M \equiv \{(s, t, a_1, \varepsilon): |s|, |t|, |a_1| < C + \delta_0,\ |\varepsilon| < \varepsilon_0\} \to \operatorname{span}\{(\phi_1, 0), (0, \phi_1)\}$ by

$$F(s, t, a_1, \varepsilon) \equiv (I - Q)\, B[(s, t)\, \phi_1 + \varepsilon u(s, t, a_1, \varepsilon), a_1, \varepsilon].$$

By Lemma 3.3, we know that the solution set of (3.2) in N_0 is determined by that of $F(s, t, a_1, \varepsilon) = 0$ in M. If the conditions of the perturbation results in Appendix are met, then the solution set of $F(s, t, a_1, \varepsilon) = 0$ for small ε is approximated well by that of $F(s, t, a_1, 0) = 0$. To that end we first find the solution set of the latter in the following. We shall use the definition of the operator B and the fact that

$$(I - Q)(w, z) = \left(\int_D \phi_1 w\, dx, \int_D \phi_1 z\, dx \right) \phi_1.$$

After a simple calculation we obtain

$$F(s, t, a_1, 0) = \left(s \left[a_1 - bt \int_D \frac{\phi_1^3}{1 + s\phi_1}\, dx \right], \right.$$

$$\left. t \left[d_1 + \tau_1 s \int_D \frac{\phi_1^3}{1 + s\phi_1}\, dx - t \int_D \phi_1^3\, dx \right] \right) \phi_1.$$

Identifying $(s, t)\, \phi_1$ with (s, t), we find that all the solutions of $F(s, t, a_1, 0) = 0$ are given as

$$\{(0, 0, a_1): a_1 \in R\}, \qquad \{(0, t_0, a_1): a_1 \in R\},$$

$$\{(s, 0, 0): s \in R\}, \qquad \{(s, f(s), g(s)): s \in R\},$$

where $t_0 = d_1 / \int_D \phi_1^3\, dx$, f and g are defined as in (3.3). Let $\sigma_0 < 0$ be chosen such that $f(s) > 0$ for $s \geqslant \sigma_0$. Then the solution set $\Gamma_0 \equiv \{(s, f(s), g(s)): s \geqslant \sigma_0\}$ of $F(s, t, a_1, 0) = 0$ will be of particular importance in understanding the positive solution set of (3.2). We shall see easily that $(0, t_0, bd_1)$ is a simple bifurcation point on $\Gamma_1 \equiv \{(0, t_0, a_1): a_1 \in R\}$ and Γ_0 intersects Γ_1 at this point. First of all, we calculate the relevant partial derivatives of F along Γ_1 and Γ_0. We shall identify F as a map into R^2 by dropping ϕ_1 from its expression. By direct calculations we obtain

$$F_{(s,\,t)}(0,\, t_0,\, a_1,\, 0)\begin{pmatrix} h \\ k \end{pmatrix} = \begin{pmatrix} (a_1 - bd_1)\, h \\ -(t_0 \tau_1 \int_D \phi_1^3\, dx)\, h - d_1 k, \end{pmatrix}$$

$$F_{(s,\,t)\, a_1}(0,\, t_0,\, a_1,\, 0)\begin{pmatrix} h \\ k \end{pmatrix} = \begin{pmatrix} h \\ 0 \end{pmatrix},$$

$$F_{a_1}(s,\, f(s),\, g(s),\, 0)\, \xi = \begin{pmatrix} s\xi \\ 0 \end{pmatrix},$$

$$F_{(s,\,t)}(s,\, f(s),\, g(s),\, 0)\begin{pmatrix} h \\ k \end{pmatrix} = K(s)\begin{pmatrix} h \\ k \end{pmatrix},$$

where

$$K(s) = \begin{pmatrix} bsf(s) \int_D \dfrac{\phi_1^4}{(1+s\phi_1)^2}\, dx & -sb \int_D \dfrac{\phi_1^3}{1+s\phi_1}\, dx \\ \tau_1 f(s) \int_D \dfrac{\phi_1^3}{(1+s\phi_1)^2}\, dx & -f(s) \int_D \phi_1^3\, dx \end{pmatrix}.$$

It is easy to check that $\det K(s) = bsf(s)\, g'(s)$. Now we see that $F_{(s,\,t)}$ is invertible along Γ_1 except at $(0,\, t_0,\, bd_1)$, and it is invertible along Γ_0 except at those points $(s_0,\, f(s_0),\, g(s_0))$ where $g'(s_0) = 0$ or $s_0 = 0$. At the degenerate point $(0,\, t_0,\, bd_1)$,

$$F_{(s,\,t)}(0,\, t_0,\, bd_1,\, 0) = \begin{pmatrix} 0 & 0 \\ -t_0 \tau_1 \int_D \phi_1^3\, dx & -d_1 \end{pmatrix} \equiv K_1.$$

Clearly *Rank* $K_1 = 1$ and *Ker* $K_1 = \operatorname{span}\{\binom{1}{0}\}$, *Range* $K_1 = \operatorname{span}\{\binom{0}{1}\}$. Therefore

$$F_{(s,\,t)\, a_1}(0,\, t_0,\, a_1,\, 0)\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \notin \mathit{Range}\, F_{(s,\,t)}(0,\, t_0,\, bd_1,\, 0).$$

This implies that (H_5) in Appendix is satisfied with $x = 0$ replaced by $(s,\, t) = (0,\, t_0)$. It follows from Crandall–Rabinowitz [8] that $(0,\, t_0,\, bd_1)$ is a simple bifurcation point on Γ_1. If $s_0 \neq 0$ and $g'(s_0) = 0$, then Rank $K(s_0) = 1$. Since $s_0 \neq 0$, it is easy to see that

$$F_{(s,\,t)}(s_0,\, f(s_0),\, g(s_0),\, 0)\begin{pmatrix} h \\ k \end{pmatrix} + F_{a_1}(s_0,\, f(s_0),\, g(s_0),\, 0)\, \xi = K(s_0)\begin{pmatrix} h \\ k \end{pmatrix} + \begin{pmatrix} s_0 \\ 0 \end{pmatrix} \xi$$

is onto. Since any square matrix is a Fredholm operator of index zero, we find that (H_3) in Appendix is satisfied along Γ_0 except at $(0, f(0), g(0)) = (0, t_0, bd_1)$. This last fact will be needed in proving Lemma 3.5 below.

Now define $z(\varepsilon)$ to be the unique positive solution of $-\Delta z = \lambda_1 z + \varepsilon z(d_1 - z)$, $z|_{\partial D} = 0$ for $\varepsilon > 0$. Clearly, $z(\varepsilon) = \theta_{\lambda_1 + \varepsilon d_1}/\varepsilon$. If we define $z(0) = t_0 \phi_1$, then it is easy to use a local bifurcation argument to show that $\varepsilon \to z(\varepsilon)$ is C^∞ (in fact analytic) and can be extended smoothly to $\varepsilon < 0$ but close to 0. By the proof of Lemma 3.3, the solution $(w, z) = (0, z(\varepsilon))$ of (3.2) can be written in the form $(0, z(\varepsilon)) = ((0, t(\varepsilon))\, \phi_1 + \varepsilon u(0, t(\varepsilon), a_1, \varepsilon)$, where $\varepsilon \to t(\varepsilon)$ is C^∞ and $t(0) = t_0$. Hence (H_4) is satisfied with $x(\varepsilon) = (0, t(\varepsilon))$ (recall that we replaced $x = 0$ there by $(s, t) = (0, t_0)$). Clearly (H_1) is satisfied with $p = \infty$. Thus by Proposition A.5, we obtain the following result.

LEMMA 3.4. *There exist a small neighborhood U_0 of $(0, t_0, bd_1, 0)$ in R^4 and a small positive number δ_0 such that*

$$F^{-1}(0) \cap U_0 = \{(s, t_0 + sz(s, \varepsilon), a_1(s, \varepsilon), \varepsilon) : s, \varepsilon \in (-\delta_0, \delta_0)\}$$
$$\cup \{(0, t(\varepsilon), a_1, \varepsilon) \in U_0\},$$

where $(s, \varepsilon) \to z(s, \varepsilon)$ and $(s, \varepsilon) \to a_1(s, \varepsilon)$ are C^∞, and $z(0, 0) = 0$, $a_1(0, 0) = bd_1$.

We see that $(s, t_0 + sz(s, 0), a_1(s, 0)) = (s, f(s), g(s))$, and we assume that $(s, f(s), g(s), \varepsilon) \in U_0$ for $|s| \leqslant s_0$ and $|\varepsilon| < \delta_0$. Now for any $C > 0$ define $T = \{(s, f(s), g(s)) : s_0 \leqslant s \leqslant C\}$. Clearly T is compact and connected. The following result follows directly from Proposition A.3.

LEMMA 3.5. *There is some $\varepsilon_0 = \varepsilon_0(C) > 0$, a neighborhood V of T and a C^∞ map $S: (s_0/2, C+1) \times (-\varepsilon_0, \varepsilon_0) \to R^3$ such that $S(s, 0) = (s, f(s), g(s))$ and all the solutions of $F(s, t, a_1, \varepsilon) = 0$ in $V \times (-\varepsilon_0, \varepsilon_0)$ are given by*

$$\{(S(s, \varepsilon), \varepsilon) : |\varepsilon| < \varepsilon_0, s \in (s_0/2, C+1)\} \cap V \times (-\varepsilon_0, \varepsilon_0).$$

It is easy to see that for any $|\varepsilon| < \varepsilon_1 = \min(\delta_0, \varepsilon_0)$,

$$\Gamma_0^\varepsilon = \{(s, t_0 + sz(s, \varepsilon), a_1(s, \varepsilon), \varepsilon) : s \in (-\delta_0, \delta_0)\}$$
$$\cup \{(S(s, \varepsilon), \varepsilon) : s \in (s_0/2, C+1)\} \cap V$$

is a smooth curve: the two curves in the right side join somewhere in $\{(s, t, a_1) : (s, t, a_1, \varepsilon) \in U_0\} \backslash \{(0, t_0, bd_1)\}$ and coincide in an open set there. By the proof of Proposition A.3, we see that we can extend the function S to obtain a unified parameterization of

$$\Gamma_0^\varepsilon = \{(S(s, \varepsilon), \varepsilon) : s \in (-s_1, C+1)\}, \tag{3.8}$$

where $s_1>0$ and the function S is C^∞ with $S(s,0)=(s, f(s), g(s))$. This last property of S comes from the way the unified parameterization is constructed in the proof of Proposition A.3 and the fact that the parameterizations in Lemmas 3.4 and 3.5 possess this property. To summarize, we have the following result.

PROPOSITION 3.6. *There exist s_1 small and positive, a neighborhood V_C of $T_C=\{(s, f(s), g(s)): -s_1/2\leqslant s\leqslant C\}$ and a small positive number ε_C such that for any fixed $\varepsilon\in(-\varepsilon_C, \varepsilon_C)$, the solutions of $F(s, t, a_1, \varepsilon)=0$ in V_C form two smooth curves: $\Gamma_0^\varepsilon\cap V_C$ and $\Gamma_0^\varepsilon\cap V_C$, where Γ_0^ε is defined as in* (3.8) *and $\Gamma_0^\varepsilon\{(0, t_0+t(\varepsilon), a_1): a_1\in R\}$ is given in Lemma* 3.4. *Moreover, this two curves intersect at some point $S(s(\varepsilon), \varepsilon)$, where $\varepsilon\to S(s(\varepsilon),\varepsilon)$ is smooth and $S(s(0), 0)=(0, t_0, bd_1)$.*

Note that $S(s(\varepsilon),\varepsilon)$ corresponds to the bifurcation point $(w, v, a)=(0, \theta_d, \lambda_1(b\theta_d))$ of (1.3). By the change of variables (3.1), it is easy to see that $S(s(\varepsilon),\varepsilon)=(0, t(\varepsilon), a_1(\varepsilon))$, where

$$t(\varepsilon)=\varepsilon^{-1}\int_D \theta_{\lambda_1+d_1\varepsilon}\phi_1\,dx, \qquad a_1(\varepsilon)=\varepsilon^{-1}[\lambda_1(b\theta_{\lambda_1+d_1\varepsilon})-\lambda_1].$$

Now we go back to problem (3.2). Clearly for any $\varepsilon>0$, Γ_0^ε corresponds to the semi-trivial solution branch $\{(0, z(\varepsilon), a_1): a_1\in R\}$ of (3.2). It is also easy to see that $S((s(\varepsilon), C+1),\varepsilon)\subset\Gamma_0^\varepsilon$ gives a branch of positive solutions of (3.2):

$$\Gamma^\varepsilon(C)\equiv\{(w, z, a_1)=((s, t)\,\phi_1+\varepsilon u(s, t, a_1, \varepsilon), a_1): (s, t, a_1)$$
$$\in S((s(\varepsilon), C+1), \varepsilon)\}.$$

We need some a priori estimates for positive solutions of (3.2) in order to use Proposition 3.6.

LEMMA 3.7. *For any $a_1^0>0$, there exist $C_0>0$ and $\varepsilon_0>0$ such that any positive solution (w, z) of* (3.2) *with $\varepsilon\in(0,\varepsilon_0)$ and $a_1\geqslant a_1^0$ can be written as $(w, z)=(s, t)\,\phi_1+\varepsilon u(s, t, a_1, \varepsilon)$ with $(s, t, a_1)\in V_{C_0}$, where V_C is defined in Proposition* 3.6.

Proof. It suffices to show that there exists $C_0>0$ such that if (w_n, z_n) is a positive solution of (3.2) with $a_1=a_1^n\geqslant a_1^0$ and $\varepsilon=\varepsilon_n\to 0$, then $(w_n, z_n)=(s_n, t_n)\,\phi_1+\varepsilon_n u(s_n, t_n, a_1^n, \varepsilon_n)$, and there is a subsequence of $\{s_n\}$ still denoted by $\{s_n\}$ such that $s_n\to s\in[-s_1/2, C_0]$, $t_n\to f(s)$ and $a_1^n\to g(s)$.

Since $\varepsilon z_n \leqslant \theta_{\lambda_1 + d_1\varepsilon + \tau_1\varepsilon}$ and $\theta_\lambda/(\lambda - \lambda_1) \to \phi_1/\int_D \phi_1^3$ uniformly in D as $\lambda \to \lambda_1+$, we see easily that for all large n,

$$z_n \leqslant 1 + (d_1 + \tau_1)\,\phi_1 \Big/ \int_D \phi_1^3.$$

Then it follows from the equation of w_n that

$$\lambda_1 + \varepsilon_n a_1^n = \lambda_1(\varepsilon_n b z_n/(1 + w_n)) < \lambda_1(\varepsilon_n b \,\|z_n\|_\infty)$$

$$= \lambda_1 + \varepsilon_n b \,\|z_n\|_\infty \leqslant \lambda_1 + \varepsilon_n b \left[1 + (d_1 + \tau_1)\, \|\phi_1\|_\infty \Big/ \int_D \phi_1^3 \right]. \quad (3.9)$$

Hence $\{a_1^n\}$ is bounded, and thus we may assume that $a_1^n \to a_1 \geqslant a_1^0 > 0$. By a simple regularity argument, it is easy to see from the equations in (3.2) that $w_n/\|w_n\|_\infty \to \phi_1/\|\phi_1\|_\infty$ and $z_n/\|z_n\|_\infty \to \phi_1/\|\phi_1\|_\infty$ in C^1 norm. Since $\{\|z_n\|_\infty\}$ is bounded, we may assume that $\|z_n\|_\infty/\|\phi_1\|_\infty \to t \geqslant 0$ and thus $z_n \to t\phi_1$ in C^1.

Next we show that there is some $C_0 > 0$ such that $\|w_n\|_\infty < C_0$ for all n. If not, we may assume that $\|w_n\|_\infty \to \infty$. Multiplying the equation for w_n in (3.2) by ϕ_1 and integrating we obtain

$$\int_D \phi_1 w_n(a_1^n - b z_n/(1 + w_n)) = 0. \quad (3.10)$$

Dividing (3.10) by $\|w_n\|_\infty$ and passing to the limit we deduce that $a_1^n \to 0$, which is impossible. Thus $\{\|w_n\|_\infty\}$ is bounded and we may assume that $\|w_n\|_\infty/\|\phi_1\|_\infty \to s \geqslant 0$, which implies that $w_n \to s\phi_1$. Therefore for all large n, $(w_n, z_n, a_1^n, \varepsilon_n)$ belongs to some N_0 as defined in Lemma 3.3. By Lemma 3.3, we must have $(w_n, z_n) = (s_n, t_n)\,\phi_1 + \varepsilon_n u(s_n, t_n, a_1^n, \varepsilon_n)$. Moreover, we have $s_n \to s$ and $t_n \to t$. Multiplying the equation for z_n in (3.2) by ϕ_1 and integrating, we obtain

$$\int_D \phi_1 z_n(d_1 + \tau_1 w_n/(1 + w_n) - z_n) = 0. \quad (3.11)$$

Passing to the limits in (3.10) and (3.11) we get

$$\begin{aligned} &\int_D s(a_1 - bt\phi_1/(1 + s\phi_1))\,\phi_1^2 = 0, \\ &\int_D t(d_1 + \tau_1 s\phi_1/(1 + s\phi_1) - t\phi_1)\,\phi_1^2 = 0. \end{aligned} \quad (3.12)$$

From (3.9) we also find that $\|z_n\|_\infty \geqslant a_1^n/b \geqslant a_1^0/b$, which implies that $t>0$. It then follows easily from (3.12) that $t=f(s)$ and $a_1=g(s)$ if $s\neq 0$; if $s=0$, by (3.12) we see that $t=d_1/\int_D \phi_1^3=f(0)$. We can again use $\lambda_1+\varepsilon_n a_1^n=\lambda_1(\varepsilon_n b z_n/(1+w_n))$ to obtain

$$a_1=\lim_{n\to\infty}\,[\lambda_1(\varepsilon_n b z_n/(1+w_n))-\lambda_1]/\varepsilon_n=bt\int_D \phi_1^3=bd_1=g(0).$$

Hence we always have $(s_n, t_n, a_1^n)\to(s, f(s), g(s))$ for some $s\in[0, C_0]$. This finishes the proof. ∎

Now we can use Proposition 3.6 to conclude the following.

PROPOSITION 3.8. *For any $a_1^0>0$, there exist $C_0>0$ and $\varepsilon_0>0$ such that any positive solution (w, z, a_1) of (3.2) with $a_1\geqslant a_1^0$ and $\varepsilon\in(0, \varepsilon_0)$ belongs to $\Gamma^\varepsilon(C_0)$ defined before.*

Next we consider the case where a_1 is small.

LEMMA 3.9. *There exists $\varepsilon_1>0$ small such that if $a_1\leqslant\varepsilon_1$ and $\varepsilon\leqslant\varepsilon_1$, then any positive solution (w, z) of (3.2) is non-degenerate. Moreover, the linearization of (3.2) at such a solution has all its eigenvalues with positive real parts and bounded away from zero except two (counting multiplicity) which are close to zero but with nonzero real parts.*

Proof. It suffices to show that if $\varepsilon_n\to 0$, $a_1^n\to 0$ and (w_n, z_n) is a positive solution to (3.2) with $(\varepsilon, a_1)=(\varepsilon_n, a_1^n)$, then for all large n, the eigenvalue problem

$$\begin{cases} -\Delta h=\lambda_1 h+\varepsilon_n\left[a_1^n-\dfrac{bz_n}{(1+w_n)^2}\right]h-\varepsilon_n\dfrac{bw_nk}{1+w_n}+\mu h,\\[2mm] -\Delta k=\lambda_1 k+\varepsilon_n\left[d_1-2z_n+\dfrac{\tau_1 w_n}{1+w_n}\right]k+\varepsilon_n\dfrac{\tau_1 z_n h}{(1+w_n)^2}+\mu k,\\[2mm] h|_{\partial D}=k|_{\partial D}=0. \end{cases} \tag{3.14}$$

has all its eigenvalues with positive real parts except two which are close to zero but with nonzero real parts. For this purpose we need some estimates on w_n and z_n. As in the proof of Lemma 3.7, we easily obtain that $w_n/\|w_n\|_\infty\to\phi_1/\|\phi_1\|_\infty$, $z_n/\|z_n\|_\infty\to\phi_1/\|\phi_1\|_\infty$ and $z_n\to\alpha\phi_1$ for some $\alpha\geqslant 0$. We show next that $\|w_n\|_\infty\to\infty$. If not, we may assume that $\|w_n\|_\infty$ is bounded. By passing to a subsequence, we may assume $w_n\to\beta\phi_1$ for some $\beta\geqslant 0$. If $\beta=0$, then we can use (3.10) to obtain $a_1^n/\|z_n\|_\infty\to b\int_D\phi_1^3/\|\phi_1\|_\infty$. On the other hand, (3.11) implies $\|z_n\|_\infty\to d_1\|\phi_1\|_\infty/\int_D\phi_1^3$. Since $a_1^n\to 0$, this is impossible. Therefore we must have $\beta>0$. Now we pass to the limit in (3.10) to obtain $\alpha = 0$, and then passing to the limit in (3.11) yields

$$d_1 = -\int_D \tau_1 \beta\phi_1^3/(1+\beta\phi_1) < 0.$$

Again we arrive at a contradiction. Therefore we have $\|w_n\|_\infty \to \infty$. Thus $\|z_n\|_\infty/\|w_n\|_\infty \to 0$ and by (3.11) we obtain $\alpha = (d_1+\tau_1)/\int \phi_1^3$.

With these properties of w_n and z_n, we see that (3.14) is a regular perturbation of the problem

$$-\Delta h = \lambda_1 h + \mu h, \qquad -\Delta k = \lambda_1 k + \mu k, \qquad h|_{\partial D} = k|_{\partial D} = 0. \quad (3.15)$$

Since (3.15) has 0 as a double eigenvalue with eigenspace span$\{(0, \phi_1), (\phi_1, 0)\}$, and all the other eigenvalues are bounded away from 0 with positive real parts, it follows from [22] that for all large n, (3.14) has exactly two eigenvalues (counting multiplicity) μ_n^1, μ_n^2 which are close to 0, and all the other eigenvalues are bounded away from 0 and have positive real parts. We are going to show that these two eigenvalues both have non-zero real parts. Let μ_n denote either μ_n^1 or μ_n^2 and (h_n, k_n) be the corresponding eigenvector with $\|h_n\|_2 + \|k_n\|_2 = 1$. Since $\mu_n \to 0$, it is easy to see from (3.14) that subject to choosing a subsequence, $h_n \to \xi\phi_1$ and $k_n \to \eta\phi_1$ in L^2 for some real numbers ξ and η satisfying $|\xi| + |\eta| = 1$. There are two possibilities: (i) $\eta \neq 0$, (ii) $\eta = 0$. In case (i), we can assume that $\eta > 0$ by a simple rescaling of the eigenvectors. Then we multiply the equation for k_n in (3.14) by ϕ_1, integrate it, divide it by ε_n and pass to the limit to obtain

$$d_1\eta - 2\alpha\eta \int_D \phi_1^3 + \tau_1\eta + \eta \lim_{n\to\infty} \mu_n/\varepsilon_n = 0. \quad (3.16)$$

Hence $\mu_n/\varepsilon_n \to d_1 + \tau_1$. Now we do the same thing to the equation of h_n and obtain $\xi\mu_n/\varepsilon_n \to b\eta$. Therefore $\xi = b\eta/(d_1+\tau_1)$ and thus $\eta = (d_1+\tau_1)/(b+d_1+\tau_1)$.

In case (ii), we have $k_n \to 0$ in L^2. We may assume that $h_n \to \phi_1$. By using the equation of h_n we easily see that $\mu_n/\varepsilon_n \to 0$. Now we decompose k_n as $k_n = \eta_n\phi_1 + k_n'$, $\int_D \phi_1 k_n' = 0$. From the equation of k_n we obtain

$$\begin{aligned} \|k_n'\|_2 &\leqslant \varepsilon_n M \,\|(d_1 - 2z_n + \tau_1 w_n/(1+w_n) + \mu_n)\, k_n\|_2 + \varepsilon_n M \left\| \frac{\tau_1 z_n h_n}{(1+w_n)^2} \right\|_2 \\ &\leqslant \varepsilon_n M_1(\|k_n\|_2 + 1/\|w_n\|_\infty^2), \end{aligned} \quad (3.17)$$

where M is the norm of $(-\Delta - \lambda_1)^{-1}$ from the orthogonal complement of span$\{\phi_1\}$ in L^2 to L^2. By passing to a subsequence, we have either $\|k_n'\|_2 = o(1/\|w_n\|_\infty{}^2)$ or $\|k_n'\|_2 = o(\|k_n\|_2)$ and hence $k_n/\eta_n \to \phi_1$ in L^2. If the latter happens, then we multiply the equation for k_n by ϕ_1, integrate it, divide it by $\eta_n \varepsilon_n$ and pass to the limit to find that

$$\int_D (d_1 - 2\alpha\phi_1 + \tau_1)\,\phi_1^2 + \lim_{n\to\infty} (\alpha\tau_1 \,\|\phi_1\|_\infty^2)/(\eta_n \|w_n\|_\infty^2) = 0.$$

That is,

$$\eta_n \,\|w_n\|_\infty^2 \to \alpha\tau_1 \,\|\phi_1\|_\infty^2/(d_1+\tau_1).$$

Then it follows that $\|k_n\|_2 = O(1/\|w_n\|_\infty^2)$ which implies that we always have $\|k_n\|_2 = O(1/\|w_n\|_\infty^2)$. Now, using (3.10) we can easily deduce $a_1^n \,\|w_n\|_\infty \to \alpha\,\|\phi_1\|_\infty$. Finally we multiply the equation of h_n by ϕ_1, integrate it, divide it by $\varepsilon_n/\|w_n\|_\infty$, and obtain by passing to the limit that

$$\lim_{n\to\infty} \|w_n\|_\infty\, \mu_n/\varepsilon_n = -\lim_{n\to\infty} a_1^n \,\|w_n\|_\infty = -\alpha\,\|\phi_1\|_\infty < 0.$$

Hence for large n, either (h_n, k_n) is close to $(b\phi_1/b+d_1+\tau_1,\ (d_1+\tau_1)\,\phi_1/b+d_1+\tau_1)$ and μ_n/ε_n is close to $d_1+\tau_1>0$, or (h_n, k_n) is close to $(\phi_1, 0)$ and $\|w_n\|_\infty\,\mu_n/\varepsilon_n$ is close to $-(d_1+\tau_1)\,\|\phi_1\|_\infty/\int \phi_1^3 < 0$. ∎

PROPOSITION 3.10. *There exists $\varepsilon_2>0$ small such that for any $a_1, \varepsilon \in (0, \varepsilon_2]$,* (3.2) *has a unique positive solution. Moreover, the positive solution is non-degenerate and unstable. By the Implicit Function Theorem, the positive solutions $\{(w, z, a_1): a_1 \in (0, \varepsilon_2]\}$ form a smooth curve $\hat{\Gamma}^\varepsilon$ parameterized by a_1 and it varies smoothly with ε.*

Proof. Let $\varepsilon_1 > 0$ be given by Lemma 3.9. Then we know that every positive solution (w, z) of (3.2) with $\varepsilon, a_1 \in (0, \varepsilon_1]$ is non-degenerate. One easily checks that

$$g(0) = bd_1 \Big/ \int_D \phi_1^3,$$

$$\lim_{s\to\infty} sg(s) = (d_1+\tau_1)\, b \Big/ \int \phi_1^3 > 0, \tag{3.18}$$

$$\lim_{s\to\infty} s^2 g'(s) = -(d_1+\tau_1)\, b \Big/ \int \phi_1^3 < 0.$$

Therefore we can find $S_0 > 0$ large such that $g(S_0) < \varepsilon_1$, $g'(s) < 0$ for $s \geqslant S_0 - 1$ and $g(s) > g(S_0)$ for $0 < s < S_0$. We show next that there exists $\varepsilon_2 \in (0, \varepsilon_1)$ such that for each $a_1 \in [\,g(2S_0), g(S_0)]$ and $\varepsilon \in (0, \varepsilon_2]$, (3.2) has a unique positive solution. This follows easily from Proposition 3.8. In fact, if (w, z) is a positive solution of (3.2) with $a_1 \in [\,g(2S_0), g(S_0)]$ and ε small, then $(w, z, a_1) \in \Gamma^\varepsilon(C_0)$, and if we express $S(\xi, \varepsilon)$ in the form $S(\xi, \varepsilon) = (s(\xi, \varepsilon),\ t(\xi, \varepsilon),\ a_1(\xi, \varepsilon))$, then $(s(\xi, 0),\ t(\xi, 0),\ a_1(\xi, 0)) = (\xi, f(\xi), g(\xi))$. Since $g'(\xi) < 0$ for $\xi \in [S_0 - 1, 2S_0 + 1]$, we can find some $\varepsilon_2 > 0$ so that for $\varepsilon \in (0, \varepsilon_2]$, $\partial a_1(\xi, \varepsilon)/\partial\xi < 0$ for $\xi \in [S_0 - 1, 2S_0 + 1]$ and $[g(2S_0), g(S_0)] \subset a_1([S_0 - 1, 2S_0 + 1], \varepsilon)$. Thus for each $a_1 \in [g(2S_0), g(S_0)]$, there is a unique $\xi = \xi(a_1, \varepsilon)$ such that $a_1 = a_1(\xi(a_1, \varepsilon), \varepsilon)$. This shows that there is exactly one positive solution of (3.2) for $\varepsilon \in (0, \varepsilon_2]$ and $a_1 \in [g(2S_0), g(S_0)]$.

Uniqueness for $a_1 \in (0, g(2S_0)]$ now follows easily by using Lemma 3.9, Proposition 3.8 and a simple continuation argument. Note that the positive solution set $\{(w, z, a_1)\}$ of (3.2) for $a_1 > 0$ bounded away from 0 and any fixed $\varepsilon > 0$ is precompact.

Next we consider the stability of the unique positive solution. By Lemma 3.9, and the Leray–Schauder formula for fixed point index, any such solution (w, z) of (3.2) would have fixed point index 1 unless exactly one of the small eigenvalues of the linearization of (3.2) at (w, z) has negative real part (and hence both small eigenvalues are real). But, as in the last part of the proof of Lemma 2.6, the unique positive solution has fixed point index -1 by the additivity of the degree, since the total degree of the nonnegative solutions is 0 and the degree of the only semi-trivial solution $(0, z)$ is 1. Hence it must be unstable (the linearization at it has exactly one negative eigenvalue, counting multiplicity). This finishes the proof. ∎

Proof of Theorem 3.1. It is clear that $\Gamma^\varepsilon = \hat{\Gamma}^\varepsilon \cup \Gamma^\varepsilon(C_0)$ is a smooth curve which varies smoothly with ε for all small positive ε. Combining Propositions 3.8 and 3.10, we immediately obtain Theorem 3.1. ∎

3.2. *Further Analysis of the Global Solution Curve*

In this subsection, we look more carefully at the solution curve Γ^ε. We first analyze the stability of the solutions $(w, z, a_1) = (w(s, \varepsilon),\ z(s, \varepsilon),\ a_1(s, \varepsilon))$. Note that if $S(\xi, \varepsilon) = (s(\xi, \varepsilon),\ t(\xi, \varepsilon),\ a_1(\xi, \varepsilon))$, then

$$(w(\xi, \varepsilon), z(\xi, \varepsilon)) = ((s(\xi, \varepsilon), t(\xi, \varepsilon))\, \phi_1 + \varepsilon u(s(\xi, \varepsilon), t(\xi, \varepsilon), a_1(\xi, \varepsilon), \varepsilon)). \tag{3.19}$$

The linearization problem of (3.2) at the positive solution $(w, z) = (w(s, \varepsilon), z(s, \varepsilon))$ with $a_1 = a_1(s, \varepsilon)$ can be written as

$$L(s, \varepsilon)(h, k) \equiv -H(h, k) - \varepsilon B_{(w,\, z)}(w(s, \varepsilon), z(s, \varepsilon), a_1(s, \varepsilon))(h, k) = \mu(h, k).$$

As in the proof of Lemma 3.9, $L(s, \varepsilon)$ is a small perturbation of H. Then it follows from [22] that for small $\varepsilon > 0$, all the eigenvalues of $L(s, \varepsilon)$ are bounded away from zero and with positive real parts except two which are close to zero. Denote these two small eigenvalues by $\mu_1(s, \varepsilon)$ and $\mu_2(s, \varepsilon)$.

Then clearly the stability of $(w(s, \varepsilon), z(s, \varepsilon))$ is completely determined by $\mu_1(s, \varepsilon)$ and $\mu_2(s, \varepsilon)$.

PROPOSITION 3.11. *Let $\mu_1(s_0)$ and $\mu_2(s_0)$ be the two eigenvalues of $K(s_0)$, then*

$$\lim_{(s, \varepsilon) \to (s_0, 0)} \mu_i(s, \varepsilon)/\varepsilon = -\mu_i(s_0), \qquad i = 1, 2. \tag{3.20}$$

Proof. We divide the proof into two cases: (i) $\mu_1(s_0) \neq \mu_2(s_0)$, (ii) $\mu_1(s_0) = \mu_2(s_0)$. We consider case (i) first. In this case we give a constructive proof which will be useful later in the discussion of Hopf bifurcations. We want to solve the equation

$$L(s, \varepsilon)(h, k) = \mu(h, k) \tag{3.21}$$

for (h, k, μ) with $(h, k) \neq (0, 0)$. If we can find solutions (h_1, k_1, μ_1) and (h_2, k_2, μ_2) such that (h_1, k_1) and (h_2, k_2) are linearly independent and μ_1, μ_2 are close to 0, then we necessarily have $\mu_1(s, \varepsilon) = \mu_1, \mu_2(s, \varepsilon) = \mu_2$.

In the rest of the proof, we understand that the spaces X, Y, X_1, Y_1 and $\mathrm{span}\{(\phi_1, 0), (0, \phi_1)\}$ are all Banach spaces of complex valued functions over the complex field C. Suppose that P and Q are the projections of the complex spaces X onto X_1 and Y onto Y_1 respectively. Then we look for solutions (h, k, μ) to (3.21) with the following form:

$$\mu = \varepsilon\zeta, \qquad (h, k) = (1, \eta)\, \phi_1 + \varepsilon V, \qquad V \in X_1.$$

Under these change of variables, (3.21) is equivalent to

$$\begin{cases} QH(V) + Q\hat{B}(s, \varepsilon)[(1, \eta)\, \phi_1 + \varepsilon V] - \varepsilon\zeta V = 0, \\ (I - Q)\, \hat{B}(s, \varepsilon)[(1, \eta)\, \phi_1 + \varepsilon V] - \zeta(1, \eta)\, \phi_1 = 0, \end{cases} \tag{3.22}$$

where $\hat{B}(s, \varepsilon) = B_{(w, z)}(w(s, \varepsilon), z(s, \varepsilon), a_1(s, \varepsilon))$. Now let N be a small neighborhood of $(s_0, 0)$ in R^2, and define $G = (G_1, G_2)$, where $G_1 \colon C \times X_1 \times C \times N \to \mathrm{span}\{(\phi_1, 0), (0, \phi_1)\}$ and $G_2 \colon C \times X_1 \times C \times N \to Y_1$ are given by

$$\begin{cases} G_1(\eta, V, \zeta, s, \varepsilon) = (I - Q)\, \hat{B}(s, \varepsilon)[(1, \eta)\, \phi_1 + \varepsilon V] + \zeta(1, \eta)\, \phi_1, \\ G_2(\eta, V, \zeta, s, \varepsilon) = QH(V) + Q\hat{B}(s, \varepsilon)[(1, \eta)\, \phi_1 + \varepsilon V] + \varepsilon\zeta V. \end{cases} \tag{3.23}$$

Then clearly $G(\eta_i, V_i, -\mu_i(s_0), s_0, 0) = 0$, $i = 1, 2$, where $(1, \eta_i)$ is an eigenvector of $K(s_0)$ corresponding to the eigenvalue $\mu_i(s_0)$, and $V_i = (QH)^{-1}\, Q\hat{B}(s_0, 0)[\binom{1}{\eta_i}\, \phi_1]$. Note that since all the entries in $K(s_0)$ are nonzero, any eigenvector of $K(s_0)$ must have both components nonzero. Therefore we can always choose the eigenvector to be of the form $(1, \eta)$. For $i = 1, 2$, let $A_i = G_{(\eta, V, \zeta)}(\eta_i, V_i, -\mu_i(s_0), s_0, 0)$. A direct calculation shows

$$A_i(\eta, V, \zeta) = ((I-Q)\,\hat{B}(s_0, 0)[(0, \eta)\,\phi_1] - \mu_i(s_0)(0, \eta)\,\phi_1 + \zeta(1, \eta_i)\,\phi_1),$$
$$QH(V) + Q\hat{B}(s_0, 0)[(0, \eta)\,\phi_1]).$$

Since $(I-Q)\,\hat{B}(s_0, 0) = K(s_0)$ has two different eigenvalues, it is easy to see that A_i is 1-1 and onto. Therefore it follows from the Implicit Function Theorem that for any (s, ε) near $(s_0, 0)$, $G(\eta, V, \zeta, s, \varepsilon) = 0$ has a unique solution $(\eta, V, \zeta) = (\eta_i(s, \varepsilon),\ V_i(s, \varepsilon),\ \zeta_i(s, \varepsilon))$, where the functions are smooth and $\eta_i(s_0, 0) = \eta_i$, $V_i(s_0, 0) = V_i$, $\zeta_i(s_0, 0) = -\mu_i(s_0)$. Note that the two eigenfunctions of $L(s, \varepsilon)$ obtained in this way must be linearly independent as the corresponding eigenvalues are different. This finishes the proof for case (i).

Next we consider case (ii). We use an argument along the lines of the proof of Lemma 3.9. It suffices to show that if $(w_n, z_n, a_1^n) = (w(s_n, \varepsilon_n), z(w_n, \varepsilon_n), a_1(s_n, \varepsilon_n))$, $s_n \to s_0$, $\varepsilon_n \to 0$, (h_n, k_n, μ_n) is a nontrivial solution to (3.14) with $\|h_n\|_2 + \|k_n\|_2 = 1$ and $\mu_n \to 0$, then, by choosing a subsequence, for all large n, μ_n/ε_n is close to $-\mu_1(s_0) = -\mu_2(s_0)$. Note that, since $L(s_n, \varepsilon_n)$ is real, $\bar{\mu}_n$ is always an eigenvalue of $L(s_n, \varepsilon_n)$ if μ_n is. As in the proof of Lemma 3.9, by passing to a subsequence, we have $h_n \to \alpha\phi_1$, $k_n \to \beta\phi_1$, where $|\alpha| + |\beta| = 1$. Note that now we have $w_n \to s_0\phi_1$, $z_n \to f(s_0)\,\phi_1$, $a_1^n \to g(s_0)$. Hence multiplying the equation for h_n by ϕ_1, integrating over D, dividing by ε_n and then passing to the limit we obtain

$$\alpha\left[g(s_0) - \int_D bf(s_0)\,\phi_1^3/(1+s_0\phi_1)^2 + \lim_{n\to\infty} \mu_n/\varepsilon_n\right]$$
$$-\beta\int_D bs_0\phi_1^3/(1+s_0\phi_1) = 0,$$

that is,

$$\alpha\left[bf(s_0)\,s_0\int_D \phi_1^4/(1+s_0\phi_1)^2 + \lim_{n\to\infty} \mu_n/\varepsilon_n\right]$$
$$-\beta\left[bs_0\int_D \phi_1^3/(1+s_0\phi_1)\right] = 0. \tag{3.24}$$

Doing the same to the equation for k_n we obtain, after some simplifications,

$$\beta\left[-2f(s_0)\int_D \phi_1^3 + \lim_{n\to\infty} \mu_n/\varepsilon_n\right] + \alpha\left[\tau_1 f(s_0)\int_D \phi_1^3/(1+s_0\phi_1)^2\right] = 0. \tag{3.25}$$

From (3.24) and (3.25) it follows that $\mu_n/\varepsilon_n \to \mu_0$ and $-\mu_0$ is an eigenvalue of $K(s_0)$ with eigenvector (α, β). Therefore, for all large n, $-\mu_n/\varepsilon_n$ is close to the double eigenvalue of $K(s_0)$. This finishes the proof. ∎

From the expression of $K(s)$, we see that its eigenvalues $\mu_1(s)$ and $\mu_2(s)$ satisfy

$$\mu_1(s) + \mu_2(s) = f(s)\, h(s),$$

where

$$h(s) = sb \int_D \phi_1^4/(1 + s\phi_1)^2 - \int_D \phi_1^3, \qquad \mu_1(s)\, \mu_2(s) = bsf(s)\, g'(s). \tag{3.26}$$

Thus by Proposition 3.11 we have the following result.

THEOREM 3.12. *If $h(s_0) > 0$, then for (s, ε) close to $(s_0, 0)$, $(w(s, \varepsilon), z(s, \varepsilon), a_1(s, \varepsilon))$ is unstable; If $h(s_0) < 0$, then for (s, ε) close to $(s_0, 0)$, $(w(s, \varepsilon), z(s, \varepsilon), a_1(s, \varepsilon))$ is stable if $g'(s_0) > 0$, and unstable if $g'(s_0) < 0$.*

For a given a_1, the number of positive solutions is determined by that of the solutions s to $a_1 = a_1(s, \varepsilon)$. Since $a_1(s, \varepsilon)$ is close to $g(s)$ for small ε, it reduces to analyzing the curve $a_1 = g(s)$. Since $g(s)$ is analytic and $g'(s) < 0$ for all large s, $g'(s) = 0$ has at most finitely many solutions: s_i, $0 \leqslant i \leqslant k$. For each s_i, there is an integer $p_i > 1$ such that $g^{(p_i)}(s_i) \neq 0$. It follows that if $g'(0) \neq 0$, there exists $\varepsilon_0 > 0$ small such that for any fixed $\varepsilon \in (0, \varepsilon_0]$, $\partial a_1(s, \varepsilon)/\partial s = 0$ has exactly k positive solutions $s_1(\varepsilon) < \cdots < s_k(\varepsilon)$, and $\partial^{p_i} a(s_i(\varepsilon), \varepsilon)/\partial s^{p_i} \neq 0$. Denote $(w_i, z_i, a_1^i) = (w(s_i(\varepsilon), \varepsilon), z(s_i(\varepsilon), \varepsilon), a_1(s_i(\varepsilon), \varepsilon))$, $i = 1, ..., k$. Then we are ready to state and prove the following result.

THEOREM 3.13. *Suppose that $g'(0) \neq 0$. Then there exists $\varepsilon^0 > 0$ small such that for $\varepsilon \in (0, \varepsilon^0]$, (w_i, z_i, a_1^i) defined above are exactly all the degenerate positive solutions of* (3.2). *The positive solution curve Γ^ε given in Theorem* 3.1 *is divided into $k+1$ pieces of smooth curves by these k degenerate points on it*:

$$\Gamma^\varepsilon \backslash \{(w_i, z_i, a_1^i) : 1 \leqslant i \leqslant k\} = \bigcup_{i=1}^{k+1} \Gamma^\varepsilon(i).$$

On each $\Gamma^\varepsilon(i)$, the positive solutions are non-degenerate and can be parameterized by a_1. Moreover, there exists $\underline{b} > 0$ such that if $b < \underline{b}$, then the solutions in each $\Gamma^\varepsilon(i)$ are either all stable, or all unstable, and no Hopf bifurcation can occur along Γ^ε.

To prove Theorem 3.13, we need the following result.

LEMMA 3.14. *For any given $\delta_0 > 0$, there exists $\varepsilon_0 > 0$ such that if $\varepsilon \in (0, \varepsilon_0]$, then the linearization of* (3.2) *at any positive solution (w, z) with $\|w\|_{W^{2,2}} \geqslant \delta_0$ satisfies condition (H_3).*

Proof. It suffices to show that if $\varepsilon_n \to 0$, and if (w_n, z_n, a_1^n) is a degenerate solution of (3.2) with $\varepsilon = \varepsilon_n$, then for all large n, dim $Ker(L_n) =$ codim $Range(L_n) = 1$, and $(w_n, 0) \notin Range(L_n)$, where $L_n = H + \varepsilon_n B_{(w,z)}(w_n, z_n, a_1^n)$. By Propositions 3.8 and 3.10, we see immediately that $(w_n, z_n, a_1^n) = (w(s_n, \varepsilon_n), z(s_n, \varepsilon_n), a_1(s_n, \varepsilon_n))$ for some $s_n \in (s(\varepsilon_n), C_0]$. By passing to a subsequence, we may assume that $s_n \to s_0$. Since $\|w_n\|_{W^{2,2}} \geqslant \delta_0$, we must have $s_0 > 0$. By Proposition 3.11, we know that the two small eigenvalues $\mu_i^n = \mu_i(s_n, \varepsilon_n)$, $i = 1, 2$ of L_n are such that $\mu_1^n/\varepsilon_n \to -\mu_1(s_0)$, $\mu_2^n/\varepsilon_n \to -\mu_2(s_0)$. By our assumption, one of the small eigenvalues must be zero. Let $\mu_1^n = 0$. Then $\mu_1(s_0) = 0$. Since all the entries of $K(s_0)$ are nonzero, $\mu_1(s_0) = 0$ cannot be a double eigenvalue. Hence $\mu_2(s_0)$ must be a nonzero real eigenvalue. Let $(1, \eta_2)$ be an eigenvector corresponding to $\mu_2(s_0)$ (recall that any eigenvector of $K(s_0)$ must have both components nonzero). By the proof of case (i) in Proposition 3.11, we can choose the eigenvector (h_n, k_n) of L_n corresponding to μ_2^n such that it converges to $(1, \eta_2)\,\phi_1$. Since $\mu_2^n \neq 0$ for large n and is real, $(h_n, k_n) = L_n(h_n/\mu_2^n, k_n/\mu_2^n)$ is in the range of L_n. Now we see easily that $(h_n, k_n) \notin X_1$, span$\{(h_n, k_n)\} \oplus X_1 \subset Range(L_n)$. Since $L_n\colon X \to Y$ is a compact perturbation of $H\colon X \to Y$ (note that X embeds compactly into Y), and H is a Fredholm operator of index 0, L_n must be of Fredholm with index 0. But by assumption and the previous discussion, dim $Ker(L_n) = 1$. Therefore we necessarily have $Range(L_n) =$ span$\{(h_n, k_n)\} \oplus X_1$.

It remains to show that $(w_n, 0) \notin Range(L_n)$. Suppose that this is not true. Then we can write

$$(w_n, 0) = \alpha_n(h_n, k_n) + (u_n, v_n), \qquad \int_D u_n\phi_1 = \int_D v_n\phi_1 = 0.$$

It follows then

$$\int_D w_n\phi_1 = \alpha_n \int_D h_n\phi_1, \; 0 = \alpha_n \int_D k_n\phi_1.$$

Passing to the limits, we obtain $\alpha_n \to s_0 > 0$ and $\alpha_n \to 0$ respectively. This contradiction finishes our proof. ∎

Proof of Theorem 3.13. By Propositions A.5 and A.5′ we can find a neighborhood N_0 of $(0, f(0)\,\phi_1, g(0)\,\phi_1)$ such that $\Gamma^\varepsilon \cap N_0$ contains only non-degenerate solutions for small ε. Then by Lemma 3.14 and a result of Crandall–Rabinowitz, we know that if (w_0, z_0, a_1^0) is a degenerate positive solution to (3.2) with $\varepsilon > 0$ small, then there is a neighborhood N of this point such that all the solutions of (3.2) in N forms a smooth curve and this curve can not be parameterized by a_1 near (w_0, z_0, a_1^0). On the other hand, we must have $(w_0, z_0, a_1^0) = (w(s_0, \varepsilon), z(s_0, \varepsilon), a_1(s_0, \varepsilon))$ for some s_0.

Thus we necessarily have $\partial a_1(s_0, \varepsilon)/\partial s = 0$. Hence $s_0 = s_i(\varepsilon)$ and $(w_0, z_0, a_1^0) = (w_i, z_i, a_1^i)$ for some $1 \leqslant i \leqslant k$. Conversely, (w_i, z_i, a_1^i), $i = 1, \dots, k$ are all degenerate solutions of (3.2): otherwise, by the Implicit Function Theorem, all the nearby solutions of (3.2) form a smooth curve which can be parameterized by a_1, i.e., the nearby parts of Γ^ε can be parameterized by a_1. But this contradicts the fact that $\partial a_1(s_i(\varepsilon), \varepsilon)/\partial s = 0$. Let

$$\underline{b} = \int \phi_1^3 \Big/ \left[\max_{s \in [0, \infty)} \int s\phi_1^4/(1+s\phi_1)^2 \right]. \tag{3.27}$$

Clearly, $h(s) < 0$ for $s \in [0, \infty)$ if $b \leqslant \underline{b}$. This implies that $\mu_1(s) + \mu_2(s) < 0$ for all $s \geqslant 0$. By Proposition 3.11 we see that the two small eigenvalues of the linearization of (3.2) along $\Gamma^\varepsilon(C_0)$ can never be a pure imaginary pair. This finishes the proof. ∎

Remark 3.15. If for some fixed *i*, $g'(s) < 0$ for $s \in (s_i, s_{i+1})$, then clearly $\mu_1(s)$ and $\mu_2(s)$ cannot be a conjugate complex pair. Then the same reasoning as in the last part of the above proof shows that Hopf bifurcation does not occur along the corresponding $\Gamma^\varepsilon(i+1)$ when $\varepsilon > 0$ is small.

Next we show that for large b and suitable choices of d_1 and τ_1, Hopf bifurcation along the solution curve Γ^ε of (3.2) can occur for any small $\varepsilon > 0$. More precisely,

THEOREM 3.16. *Let d_1, τ_1 satisfy $\tau_1/d_1 \geqslant 2 \int_D \phi_1^4/(\int_D \phi_1^3)^2$. Then there exists some positive constant $\bar{b} > 0$, depending only on D, such that if $b \geqslant \bar{b}$, for any small $\varepsilon > 0$, Hopf bifurcation occurs along the positive solution curve Γ^ε of* (3.2).

Proof. It is easy to check that

$$\begin{aligned} \frac{g'(s)}{b} \int_D \phi_1^3 &= -d_1 \int_D \frac{\phi_1^4}{(1+s\phi_1)^2} + \tau_1 \int_D \frac{\phi_1^3}{1+s\phi_1} \\ &\quad \times \left[\int_D \frac{\phi_1^3}{1+s\phi_1} - 2s \int_D \frac{\phi_1^4}{(1+s\phi_1)^2} \right], \\ h'(s)/b &= \int_D \frac{\phi_1^4}{(1+s\phi_1)^2} - 2s \int_D \frac{\phi_1^5}{(1+s\phi_1)^2}. \end{aligned}$$

Let $\tau_1/d_1 \geqslant 2 \int_D \phi_1^4/(\int_D \phi_1^3)^2$. It is easy to see that there exists $\delta_0 > 0$ small, independent of b, d_1 and τ_1, such that if $s \in [0, \delta_0]$, then $g'(s) > 0$ and $h'(s) > 0$. Set

$$\bar{b} = \min_{s \in (0, \delta_0]} \int_D \phi_1^3 \Big/ \left(s \int_D \phi_1^4/(1+s\phi_1)^2 \right). \tag{3.28}$$

Hence for any $b \geqslant \bar{b}$, there exists some $s_0 \in (0, \delta_0]$ such that

$$\begin{aligned} bs_0 \int_D \phi_1^4/(1+s_0\phi_1)^2 &= \int_D \phi_1^3, \\ bs \int_D \phi_1^4/(1+s\phi_1)^2 &< \int_D \phi_1^3, \qquad s \in (0, s_0). \end{aligned} \tag{3.29}$$

Summarizing the above discussions we have

$$g'(s_0) > 0, \qquad h(s_0) = 0, \quad h'(s_0) > 0, \quad h(s) < 0, \quad s \in (0, s_0). \tag{3.30}$$

It follows from (3.26) that the two eigenvalues of $K(s_0)$ are given by

$$\mu_1(s_0) = i\beta_0, \qquad \mu_2(s_0) = -i\beta_0, \qquad \beta_0 = \sqrt{bs_0 f(s_0)\, g'(s_0)} > 0.$$

It follows then from bifurcation theory for linear operators (see, e.g., [22]) that for s near s_0, the two eigenvalues of $K(s)$ are conjugate complex pairs:

$$\mu_1(s) = \alpha(s) + i\beta(s), \qquad \mu_2(s) = \alpha(s) - i\beta(s),$$

where α and β are smooth functions and $\alpha(s_0) = 0$, $\beta(s_0) = \beta_0$. We also have $\alpha'(s_0) = f(s_0)\, h'(s_0)/2 > 0$. Now we use Proposition 3.11 to find that for (s, ε) close to $(s_0, 0)$, the two small eigenvalues $\mu_1(s, \varepsilon)$ and $\mu_2(s, \varepsilon)$ of the linearization of (3.2) about (w, z) at $(w(s, \varepsilon), z(s, \varepsilon), a_1(s, \varepsilon))$ are conjugate complex pairs, and both are simple eigenvalues. It then follows from results on simple eigenvalues that

$$\mu_1(s, \varepsilon) = \alpha(s, \varepsilon) + i\beta(s, \varepsilon), \qquad \mu_2(s, \varepsilon) = \alpha(s, \varepsilon) - i\beta(s, \varepsilon),$$

where α and β are smooth functions. By the proof of case 1 of Proposition 3.11, we must have $\mu_i(s, \varepsilon) = \varepsilon\zeta_i(s, \varepsilon)$ for all s close to s_0 and positive small ε, where $\zeta_i(s, \varepsilon)$ is smooth for all s close to s_0 and all (not necessarily positive) ε close to 0, and $\zeta_i(s, 0) = \mu_i(s)$. Thus we can find $\varepsilon_0 > 0$ small such that for any $\varepsilon \in (0, \varepsilon_0]$, there is a unique $s_\varepsilon \in (s_0 - \varepsilon_0, s_0 + \varepsilon_0)$ such that $\alpha(s_\varepsilon, \varepsilon) = 0$, $\alpha'_s(s_\varepsilon, \varepsilon) > 0$, and $\beta(s, \varepsilon) \neq 0$ for $s \in (s_0 - \varepsilon_0, s_0 + \varepsilon_0)$. Therefore, Hopf bifurcation occurs at $(w, z, a_1) = (w(s_\varepsilon, \varepsilon), z(s_\varepsilon, \varepsilon), a_1(s_\varepsilon, \varepsilon))$ (see, e.g., [7, 20]). ∎

Remark 3.16′. In view of Theorem 3.12 and (3.30), it is easy to see that the positive solution $(w(s, \varepsilon), z(s, \varepsilon))$ loses stability as s passes through s_ε. Note that $g'(s) > 0$ for $s < s_\varepsilon$. Therefore for $\varepsilon > 0$ small, $s \to a_1(s, \varepsilon)$ is increasing for $s < s_\varepsilon$. This fact will be needed in proving part (iii) of Theorem 4.1. It seems possible to use the method presented in [20] to determine the stability of the periodic solutions obtained by the Hopf bifurcation. As the calculation seems rather tedious and long, we do not pursue it here. We suspect that the periodic solutions are stable.

Finally, we discuss the shape of the bifurcation curve Γ^{ε}. For small ε, the problem is reduced to analyzing the function $g(s)$. Clearly,

$$g(0)=d_1 b\Big/\int \phi_1^3, \qquad g(\infty)\equiv \lim_{s\to\infty} g(s)=0.$$

From the expression of $g'(s)$, one easily sees that there exists some small positive number δ_1 such that $g'(s)<0$ for all $s\geqslant 0$ if $\tau_1/d_1<\delta_1$. On the other hand, if $\tau_1/d_1>\int \phi_1^4/(\int \phi_1^3)^2$, then $a_1^*=\sup_{s\in[0,\infty)} g(s)$ is achieved at some $s>0$ and $a_1^*>g(0)$. Set

$$a_1^*(\varepsilon)=\max_{s\in[0,\infty)} a_1(s,\varepsilon)$$

and recall that the positive solution curve Γ^{ε} intersects the semi-trivial solution curve $\{(0, z, a_1)\}$ at the point

$$(w(s(\varepsilon),\varepsilon), z(s(\varepsilon)\,\varepsilon), a_1(s(\varepsilon),\varepsilon))=(0, \varepsilon^{-1}\theta_{\lambda_1+d_1\varepsilon}, \varepsilon^{-1}[\lambda_1(b\theta_{\lambda_1+d\varepsilon}-\lambda_1]).$$

That is, the positive solution curve Γ^{ε} bifurcates from the semi-trivial solution branch $\{(0, z, a_1)\}$ at $a_1=a_1(\varepsilon)\equiv a_1(s(\varepsilon),\varepsilon)$. Now by the above discussions we have the following result.

THEOREM 3.17. *There exists $\varepsilon_0>0$ small such that, for any $0<\varepsilon<\varepsilon_0$, the a_1 range which the positive solution curve Γ^{ε} covers is $(0, a_1^*(\varepsilon))$ or $(0, a_1^*(\varepsilon)]$. Moreover, the a_1 range is the former if $\tau_1/d_1<\delta_1$, and in this case, (3.2) has a unique positive solution for any $a_1\in(0, a_1^*(\varepsilon))$; on the other hand, if $\tau_1/d_1>\int \phi_1^4/(\int \phi_1^3)^2$, then $a_1^*(\varepsilon)>a_1(\varepsilon)$, the desired a_1 range is $(0, a_1^*(\varepsilon)]$, and (3.2) has at least two positive solutions for any $a_1\in(a_1(\varepsilon), a_1^*(\varepsilon))$.*

From the above we see that if $\tau_1/d_1>\int \phi_1^4/(\int \phi_1^3)^2$, then the curve Γ^{ε} has at least one turning point at $a_1=a_1^*(\varepsilon)$ and Γ^{ε} lies to the left of this point. It would be interesting to know whether there is exactly one turning point. We show in the following that this is indeed the case if τ_1/d_1 is large for some special type domain D.

THEOREM 3.18. *Suppose that the domain D is a ball in R^N with $N\leqslant 3$, or $D=[a_1, b_1]\times[a_2, b_2]\times\cdots\times[a_N, b_N]$ with $N\leqslant 6$. Then there exists $\delta>0$ small, depending only on D, such that if $d_1/\tau_1\leqslant\delta$, then $g(s)$ has a unique critical point in $(0,\infty)$ denoted by s_0. Moreover, $g''(s_0)<0$. It follows that, in these cases, Γ^{ε} is exactly "$\supset$" shaped.*

To establish Theorem 3.18, we first prove the following lemma.

LEMMA 3.19. *Suppose that $\Phi \not\equiv 0$ is a continuous, non-negative function in $\bar{D}$. If $\frac{1}{|D|}\int_D \Phi \geqslant \frac{4}{27}\max_{\bar{D}}\Phi$, then the function*

$$g_1(s) \equiv s^{1/2}\int_D \Phi^3/(1+s\Phi)\,dx, \qquad s\in(0,\infty)$$

has a unique critical point.

Proof. Without loss of generality we may assume that $\max_{\bar{D}}\Phi = 1$. It is easy to check that

$$\begin{aligned} 2s^{1/2}g_1'(s) &= \int_D \Phi^3/(1+s\Phi) - 2s\int_D \Phi^4/(1+s\Phi)^2, \\ 2sg_1''(s) + g_1'(s) &= 4s^{3/2}\int_D \Phi^5/(1+s\Phi)^3 - 3s^{1/2}\int_D \Phi^4/(1+s\Phi)^2. \end{aligned} \tag{3.31}$$

Claim. If $g_1'(s)=0$, then $g_1''(s)<0$.

To establish this assertion, we consider two cases: (i) $s<3$. For this case, our assertion follows readily from (3.31) as

$$2sg_1''(s) = s^{1/2}\int_D \Phi^4(s\Phi-3)/(1+s\Phi)^3 < 0. \tag{3.32}$$

(ii) $s \geqslant 3$. For this case we argue by contradiction. Suppose that there exists $s\geqslant 3$ such that $g_1'(s)=0$ and $g_1''(s)\geqslant 0$. It is easy to check that $g_1'(s)=0$ is equivalent to

$$\int_D \Phi^3/(1+s\Phi) = 2\int_D \Phi^3/(1+s\Phi)^2. \tag{3.33}$$

The assumption $g_1'(s)=0$ and $g_1''(s)\geqslant 0$ implies that

$$4s\int_D \Phi^5/(1+s\Phi)^3 \geqslant 3\int_D \Phi^4/(1+s\Phi)^2, \tag{3.34}$$

which is the same as

$$\int_D \Phi^4/(1+s\Phi)^2 \geqslant 4\int_D \Phi^4/(1+s\Phi)^3. \tag{3.35}$$

Multiplying (3.35) by s and simplifying, by (3.33) we find

$$4\int_D \Phi^3/(1+s\Phi)^3 \geqslant 3\int_D \Phi^3/(1+s\Phi)^2. \tag{3.36}$$

We can further simplify (3.36) to obtain

$$7\int_D \Phi^2/(1+s\Phi)^2 \geqslant 4\int_D \Phi^2/(1+s\Phi)^3 + 3\int_D \Phi^2/(1+s\Phi). \tag{3.37}$$

By Holder and Cauchy inequalities we have

$$\begin{aligned}\int_D \Phi^2/(1+s\Phi)^2 &\leqslant \left(\int_D \Phi^2/(1+s\Phi)^3\right)^{1/2} \left(\int_D \Phi^2/(1+s\Phi)\right)^{1/2}\\ &\leqslant 2/3\int_D \Phi^2/(1+s\Phi)^3 + 3/8\int_D \Phi^2/(1+s\Phi),\end{aligned} \tag{3.38}$$

which can be rewritten as

$$\int_D \Phi^2/(1+s\Phi)^3 > 3/2\int_D \Phi^2/(1+s\Phi)^2 - 9/16\int_D \Phi^2/(1+s\Phi). \tag{3.39}$$

Substituting (3.39) into (3.37) we obtain

$$4\int_D \Phi^2/(1+s\Phi)^2 > 3\int_D \Phi^2/(1+s\Phi). \tag{3.40}$$

Repeating those arguments from (3.36) to (3.40), we can similarly show that

$$4\int_D \Phi/(1+s\Phi) > 3\int_D \Phi. \tag{3.41}$$

By the elementary inequality $(1+s\Phi) \geqslant 2(s\Phi)^{1/2}$ and $s>3$, from (3.41) it follows that

$$\int_D \Phi < 2/(3\sqrt{3})\int_D \Phi^{1/2} \leqslant 2/(3\sqrt{3})\left(\int_D \Phi\right)^{1/2} |D|^{1/2}.$$

That is, $\int_D \Phi/|D| < 4/27$, which contradicts the assumption. This establishes our assertion.

As $g_1(0)=0$ and $\lim_{s\to\infty} g_1(s)=0$, we can conclude that g_1 has at least a critical point in $(0, \infty)$. By the conclusion of the *Claim* and some elementary argument we see that $g_1(s)$ has exactly one critical point denoted by

s_0. Furthermore, from the proof we see that $g_1''(s_0) < 0$. It is easy to see that g_1 attains local (and thus global) maximum at $s = s_0$. ▮

Proof of Theorem 3.18. Let $\Phi = \Phi_1$ in Lemma 3.19. When $D = \Sigma_N$, then Φ_1 is given by

$$\Phi_1(x) = \sin[\pi(x_1 - a_1)/(b_1 - a_1)] \cdots \sin[\pi(x_N - a_N)/(b_N - a_N)].$$

Therefore $\int_D \Phi/|D| = (2/\pi)^N > 4/27$ provided that $N \leqslant 6$. When D is a ball in R^3, e.g., the unit ball, then $\Phi_1(x) = sin(\pi\,|x|)/(\pi\,|x|)$. Hence for this case, $\int_D \Phi/|D| = 4/\pi^2 > 4/27$. It remains to consider the case when D is a ball in R^2. Without loss of generality we may assume that D is the unit ball in R^2.

Claim 1. $\int_D \Phi_1 \geqslant 4\pi/(e\lambda_1(D))$.

To establish this assertion, we consider

$$u_t = \Delta u \quad \text{in} \quad D \times (0, \infty), \qquad u|_{\partial D} = 0, \qquad u(x, 0) = \Phi_1(x). \tag{3.42}$$

Let $G(t, x) = (4\pi t)^{-1}\, e^{-|x|^2/(4t)}$ be the fundamental solution of the heat equation in R^2. Then we have

$$u(x, t) \leqslant \int_D G(t, x - y)\, \Phi_1(y)\, dy \leqslant \frac{1}{4\pi t} \int_D \Phi_1(y)\, dy, \qquad \forall x \in D, \qquad t > 0.$$

Since $e^{-\lambda_1(D)\,t}\Phi_1(x)$ is the solution of (3.42), hence

$$4\pi t e^{-\lambda_1(D)\,t} \leqslant \int_D \Phi_1, \qquad \forall t > 0. \tag{3.43}$$

The assertion now follows from (3.43) by letting $t = 1/\lambda_1(D)$.

Claim 2. If D is the unit ball in R^2, then $\lambda_1(D) < 3 + 2\sqrt{2}$.
Set $\phi_0(x) = 1 - |x|^{\sqrt{2}}$. Then

$$\lambda_1(D) = \inf_{\phi \neq 0} \frac{\int_D |\nabla \phi|^2}{\int_D \phi^2} < \frac{\int_D |\nabla \phi_0|^2}{\int_D \phi_0^2} = \frac{\int_0^1 r(\phi_0')^2\, dr}{\int_0^1 r\phi_0^2\, dr} = 3 + 2\sqrt{2}.$$

Now it follows from our two assertions that

$$\frac{\int_D \Phi_1}{|D|} = \frac{\int_D \Phi_1}{\pi} \geqslant \frac{4}{e\lambda_1(D)} \geqslant \frac{4}{e(3 + 2\sqrt{2})} > \frac{4}{27}.$$

Therefore by Lemma 3.19 we see that $g_1(s)$ with $\Phi = \phi_1 = \Phi_1/\|\Phi_1\|_2$ has a unique critical point, and it is a strict local (global) maximum. Clearly, the function $[g_1(s)]^2$ also has these properties. Since

$$g(s) = \frac{b\tau_1}{\int \phi_1^3} \left[\frac{d_1}{\tau_1} \int \frac{\phi_1^3}{1 + s\phi_1} + g_1^2(s) \right],$$

by a simple perturbation argument we see that for d_1/τ_1 small, the same conclusion holds true for g. This completes the proof of Theorem 3.18. ∎

4. THE MAIN RESULTS

In this section, we use results from Sections 2, 3 and the Appendix to prove our main result

THEOREM 4.1. *For any fixed $b > 0$, there exists an open set $O = O(b) \subset (0, \infty) \times (0, \infty)$, such that for any $(\tau, d) \in O$, we can find $M = M(b, d, \tau)$ large so that for each $m \geq M$:*

(i) *All positive solutions (u, v, a) of (1.1) lie on an unbounded smooth curve Γ which bifurcates from the semi-trivial solution curve $\{(0, \theta_d, a): a > \lambda_1\}$ at the point $(0, \theta_d, \lambda_1(b\theta_d))$. Moreover, Γ is roughly S-shaped: there exist $a_* = \lambda_1 + O(1/\sqrt{m}) \in (\lambda_1, \lambda_1(b\theta_d))$ and $a^* > \lambda_1(b\theta_d)$, both depending on b, d, τ and m, such that (1.1) has a positive solution if and only if $a \geq a_*$; it has exactly one positive solution for $a = a_*$ and $a > a^*$, at least two positive solutions for $a \in (a_*, \lambda_1(b\theta_d)) \cup \{a^*\}$, and at least three positive solutions for $a \in (\lambda_1(b\theta_d), a^*)$.*

(ii) *There are at most a finite number k (≥ 2) of degenerate positive solutions of (1.1) which we denote by $\{(u_i, v_i, a_i): 1 \leq i \leq k\}$. The positive solution curve Γ given in part (i) is divided into $k+1$ pieces of smooth curves by these k degenerate points on Γ:*

$$\Gamma \backslash \{(u_i, v_i, a_i): 1 \leq i \leq k\} = \bigcup_{i=1}^{k+1} \Gamma(i).$$

On each $\Gamma(i)$, the positive solutions are non-degenerate and can be parameterized by a;

(iii) *There exist $\bar{b} > \underline{b} > 0$, depending only on D, such that if $b \leq \underline{b}$, then the solutions in each $\Gamma(i)$ are either all stable, or unstable, and Hopf bifurcation does not occur along Γ; if $b \geq \bar{b}$, then Hopf bifurcation occurs along $\Gamma(1)$ at some $a_0 \in (\lambda_1(b\theta_d), a^*)$, where $\Gamma(1)$ has $(0, \theta_d, \lambda_1(b\theta_d))$ in its closure. Moreover, there are two asymptotically stable solutions if $a \in (\lambda_1(b\theta_d), a_0)$.*

(iv) *There exists a subset* O_1 *of* O *such that if* $(\tau, d) \in O_1$, *and if the domain* D *is a ball in* R^N *with* $N \leqslant 3$, *or* $D = [a_1, b_1] \times \cdots \times [a_N, b_N]$ *with* $N \leqslant 6$, *then the curve* Γ *is exactly S-shaped: there are exactly two degenerate points on* Γ, *and* (1.1) *has no positive solution for* $a < a_*$, *exactly one positive solution for* $a = a_*$ *(neutrally stable), exactly two positive solutions for* $a \in (a_*, \lambda_1(b\theta_d)]$ *(one stable and one unstable), exactly three positive solutions for* $a \in (\lambda_1(b\theta_d), a^*)$ *(two stable, one unstable if* $b \leqslant \underline{b}$; *if* $b \geqslant \bar{b}$, *then Hopf bifurcation occurs as in part* (iii) *above), exactly two positive solutions for* $a = a^*$ *(one stable, one neutrally stable), and exactly one positive solution for* $a \in (a^*, \infty)$ *(stable).*

Remark 4.2. (i) The open sets O and O_1 are defined as follows.

$$O = \{(\varepsilon\tau_1, \lambda_1 + \varepsilon d_1): d_1/\tau_1 \in (0, \Delta_0), \varepsilon \in (0, \varepsilon_0(b, d_1, \tau_1))\},$$

where $\Delta_0 = (\int_D \phi_1^3\,dx)^2/(2\int_D \phi_1^4\,dx)$, $\phi_1 = \Phi_1/\|\Phi_1\|_2$, and $\varepsilon_0(b, d_1, \tau_1)$ is a function of (b, d_1, τ_1) taking small positive values only;

$$O_1 = \{(\varepsilon\tau_1, \lambda_1 + \varepsilon d_1): d_1/\tau_1 \in (0, \delta_0), \varepsilon \in (0, \varepsilon_0(b, d_1, \tau_1))\},$$

where $\delta_0 < \Delta_0$ is a small positive number depending only on D.

(ii) The numbers $\underline{b}$ and $\bar{b}$ are given by (3.27) and (3.28) respectively.

Proof. Let $r = 1/m$ and $(w, v) = (u/r, v)$, then (1.1) becomes

$$\begin{cases} \Delta w + w(a - rw - bv/(1+w)) = 0 & \text{in } D, \quad w|_{\partial D} = 0, \\ \Delta v + v(d + \tau w/(1+w) - v) = 0 & \text{in } D, \quad v|_{\partial D} = 0. \end{cases} \tag{4.1}$$

Define the operator F: $[W^{2,p} \cap H_0^1]^2 \to [L^p]^2$, $p > N$, by

$$F(w, v, a, r) = \begin{pmatrix} \Delta w + w(a - bv/(1+w)) \\ \Delta v + v(d + \tau w/(1+w) - v) \end{pmatrix} - r\begin{pmatrix} w^2 \\ 0 \end{pmatrix}. \tag{4.2}$$

It is obvious that $F(w, v, a, r) = 0$ if and only if (w, v, a, r) solves (4.1), and that $F(w, v, a, 0) = 0$ if and only if (w, v, a) solves (1.3).

Firstly, we suppose that $(\tau, d) \in O$ with $\varepsilon_0(b, d_1, \tau_1)$ small enough such that all the results on (1.3) obtained in Section 3 hold. Thus all the positive solutions of $F(w, v, a, 0) = 0$ form a nice curve T_0 which is given by some Γ^ε in Section 3 under a simple change of variables. By Proposition 3.10, there exists $\varepsilon_0 > 0$ such that if $a \in (\lambda_1, \lambda_1 + \varepsilon_0)$, then (1.3) has a unique positive solution (w, v), and it is non-degenerate and unstable. Let $T_1 = \{(w, v, a) \in T_0: a \geqslant \lambda_1 + \varepsilon_0/2\}$. Then T_1 is bounded and contains all the degenerate points of T_0.

As in the proof of Lemma 3.4, by a standard local bifurcation analysis, there exists $r_1 > 0$ small, independent of a, such that for any fixed $r \in (0, r_1)$, all the positive solutions of $F(w, v, a, r) = 0$ which are close to $(0, \theta_d, \lambda_1(b\theta_d))$ lie on a smooth curve Γ_1^r starting from this point. Furthermore, by our choice of Δ_0 in the definition of O, we know that Γ_1^r bends to the right of $(0, \theta_d, \lambda_1(b\theta_d))$.

Next we study bounded positive solutions of $F(w, v, a, r) = 0$ for $r \in (0, r_1)$ and (w, v, a) bounded away from $(0, \theta_d, \lambda_1(b\theta_d))$. Let T_2 denote T_1 with a small neighborhood of $(0, \theta_d, \lambda_1(b\theta_d))$ taken out from it. Lemma 3.14 implies that condition (H3) is satisfied for F at any positive solution of (1.3) on T_2. Therefore, by Proposition A.3, for any fixed small positive r, near T_2, the positive solutions to $F(w, v, a, r) = 0$ are well approximated by solutions to $F(w, v, a, 0) = 0$. More precisely, there exists $r_2 > 0$ smaller than r_1 and independent of a, a neighborhood V of T_2 such that, provided $r \in (0, r_2)$, all the positive solutions (w, v, a) of $F(w, v, a, r) = 0$ in V form a smooth curve Γ_2^r, which is C^∞ close to T_2. As Γ_1^r and Γ_2^r overlap in some open interval of a, thus $\Gamma_1^r \cup \Gamma_2^r$ is a smooth curve.

Now we apply Theorem 2.1 to find some $r_3 > 0$ smaller than r_2, and some $\varepsilon_1 > 0$ such that for each $r \in (0, r_3)$, there is a unique $a_* = \lambda_1 + O(\sqrt{r}) < \lambda_1(b\theta_d)$, such that (4.1) has no positive solution for $a < a_*$, exactly one positive solution for $a = a_*$ and exactly two positive solutions for $a \in (a_*, a_* + \varepsilon_1)$, and these solutions form a smooth curve Γ_3^r. Since (1.3) has a unique non-degenerate positive solution for $a \in (\lambda_1, \lambda_1 + \varepsilon_0)$, by the proof of Lemma 2.2, we can choose $\varepsilon_1 > \varepsilon_0/2$. Thus Γ_3^r overlaps with Γ_2^r, and then $\Gamma_1^r \cup \Gamma_2^r \cup \Gamma_3^r$ is a smooth curve.

Choose $A > 0$ such that any solution $(w, v, a) \in \Gamma_1^r \cup \Gamma_2^r \cup \Gamma_3^r$ is such that $a < A/4$. Then by Lemma 2.5 one sees that there is some $r_4 > 0$ smaller than r_3 such that if $r \in (0, r_4)$, then the positive solutions of (4.1) which are not on $\Gamma_1^r \cup \Gamma_2^r \cup \Gamma_3^r$ but with $a < A$ must be close to $(\theta_a, \theta_{d+\tau}, a)$-here we mean that (u, v, a) is close to this point. It then follows from the implicit function theorem that these solutions form a smooth curve Γ_4^r which joins $\Gamma_1^r \cup \Gamma_2^r \cup \Gamma_3^r$.

Finally, we can easily show that for all $r > 0$ small, and any $a > A/2$, any positive solution (w, v) of (4.1) is non-degenerate and linearly stable. To see this, first observe that, by a simple super and sub-solution argument, $u > \theta_{a-b(d+\tau)}$. Substituting this into the equation of u and use super-sub solution argument once more, one easily obtains $u \geqslant \theta_{a-\varepsilon}$ for any given $\varepsilon > 0$ if r is small enough. The non-degeneracy and stability assertions now follow easily from this estimate. It then follows from a degree argument in the same spirit as in the proof of part 2), Theorem 3.1 in [19] that (1.1) and hence (4.1) has a unique positive solution (w, v) for any given $a > A/2$. These solutions (w, v, a) form a smooth curve Γ_5^r which joins Γ_4^r.

Clearly $\Gamma^r = \bigcup_{i=1}^5 \Gamma_i^r$ contains all the positive solutions of (4.1). The conclusions of Theorem 4.1 now follow from results in Section 3, the C^∞

closeness of Γ_2^r to T_2, and the fact that the linearization of F with respect to (w, v) varies continuously with (w, v, a, r). ∎

5. APPENDIX: SOME ABSTRACT RESULTS

Suppose that

(H_1): X and Y are real Banach spaces and $F: X \times R \times R \to Y$ a C^p-map $(p \geqslant 2)$ which sends $(x, \lambda, \varepsilon) \in X \times R \times R$ to $F(x, \lambda, \varepsilon) \in Y$;

(H_2): D is a component of the set of solutions (x, λ) of $F(x, \lambda, 0) = 0$ and T a compact connected subset of D;

(H_3): for any $(x, \lambda) \in T$, $F_x(x, \lambda, 0): X \to Y$ is a Fredholm operator of index 0, and the mapping $B(h, \mu) = F_x(x, \lambda, 0)\, h + F_\lambda(x, \lambda, 0)\, \mu: X \times R \to Y$ is onto. In other words, for any $(x, \lambda) \in T$, either (i) $F_x(x, \lambda, 0): X \to Y$ has a continuous inverse, or (ii) dim $Ker(F_x(x, \lambda, 0)) =$ codim $Range(F_x(x, \lambda, 0)) = 1$ and $F_\lambda(x, \lambda, 0) \notin Range(F_x(x, \lambda, 0))$.

We want to know how T is perturbed to give a solution set T_ε of $F(x, \lambda, \varepsilon) = 0$ when ε is small. Note that, under the above assumptions, as in Dancer [10], the linear operator $(h, \mu) \to F_x(x, \lambda, \varepsilon)\, h + F_\lambda(x, \lambda, \varepsilon)\, \mu$ is onto for (x, λ) in a small neighborhood U of T and all small ε, as T is compact, and the operator depends continuously on $(x, \lambda, \varepsilon)$.

Now for any fixed $(x_0, \lambda_0) \in T$, either case (i) or case (ii) in (H_3) happens. If case (i) happens, then it follows from the implicit function theorem that there exist a neighborhood V of x in X and a small number $\delta > 0$ such that

$$\begin{aligned} &F^{-1}(0) \cap (V \times (-\delta, \delta) \times (-\delta, \delta)) \\ &\qquad = \{(x(\lambda, \varepsilon), \lambda, \varepsilon): \lambda \in (\lambda_0 - \delta, \lambda_0 + \delta), \varepsilon \in (-\delta, \delta)\}, \end{aligned}$$

where $(\lambda, \varepsilon) \to x(\lambda, \varepsilon)$ is C^p, and $x(\lambda_0, 0) = x_0$.

For $\varepsilon \in (-\delta, \delta)$, define $A(\varepsilon): X \times (-\delta, \delta) \to X \times R$ by $A(\varepsilon)(x, \lambda) = (x(\lambda, \varepsilon) + x, \lambda)$. Then clearly $A^{-1}(\varepsilon)(y, \mu) = (y - x(\mu, \varepsilon), \mu)$, and thus $A(\varepsilon)$ is a C^p diffeomorphism between a small neighborhood N of the point $(0, \lambda_0)$ in $X \times R$ and the small neighborhood $W = V \times (-\delta, \delta)$ of (x_0, λ_0) in $X \times R$. Moreover, it maps the line segment $I = \{(0, \lambda): \lambda \in (\lambda_0 - \delta, \lambda_0 + \delta)\}$ onto the curve $J_\varepsilon = \{(x(\lambda, \varepsilon), \lambda): \lambda \in (\lambda_0 - \delta, \lambda_0 + \delta)\}$ for every $\varepsilon \in (-\delta, \delta)$. If case (ii) happens at (x_0, λ_0), then by a well-known results in Crandall–Rabinowitz [9], there exist a small neighborhood W of (x_0, λ_0) and a small $\delta > 0$ such that

$$F^{-1}(0) \cap (W \times (-\delta, \delta)) = \{(x(s, \varepsilon), \lambda(s, \varepsilon), \varepsilon): s, \varepsilon \in (-\delta, \delta)\},$$

where $x(s, \varepsilon)$ and $\lambda(s, \varepsilon)$ are C^p with $x(0, 0) = x_0$, $\lambda(0, 0) = \lambda_0$ and $x(s, \varepsilon) = x_0 + su_0 + z(s, \varepsilon)$. Here u_0 spans $Ker(F_x(x_0, \lambda_0, 0))$ and $z(s, \varepsilon)$ belongs to a complement Z of span$\{u_0\}$ in X.

Now for any $\varepsilon \in (-\delta, \delta)$, define $A(\varepsilon)\colon R \times Z \times (-\delta, \delta) \to X \times R$ by

$$A(\varepsilon)(t, z, s) = (su_0 + z(s, \varepsilon) + z, \lambda(s, \varepsilon) + t).$$

Then it is easily seen that

$$A^{-1}(\varepsilon)(ru_0 + z, \mu) = (\mu - \lambda(r, \varepsilon), z - z(r, \varepsilon), r).$$

Thus $A(\varepsilon) + (x_0, 0)$ is a C^p diffeomorphism between a small neighborhood N of the origin in $R \times Z \times R$ and the neighborhood W of (x_0, λ_0) given above. Moreover, it maps the line segment $I = \{(0, 0, s)\colon s \in (-\delta, \delta)\}$ onto the curve $J_\varepsilon = \{(x(s, \varepsilon), \lambda(s, \varepsilon)\colon s \in (-\delta, \delta)\}$ for every $\varepsilon \in (-\delta, \delta)$.

Summarizing our above discussions in a unified fashion, we obtain the following result.

Proposition A.1. *Under the assumptions $(H_1)-(H_3)$, for any $(x, \lambda) \in T$, there exists a small neighborhood $W = W(x, \lambda)$ of (x, λ) and a small positive number $\delta = \delta(x, \lambda)$ such that*

(a) *for any $\varepsilon \in (-\delta, \delta)$, $F_\varepsilon^{-1}(0) \cap W = \{(x(s, \varepsilon), \lambda(s, \varepsilon))\colon s \in (-\delta, \delta)\} \equiv J_\varepsilon = J_\varepsilon(x, \lambda)$, where $F_\varepsilon(x, \lambda) = F(x, \lambda, \varepsilon)$, the functions $(s, \varepsilon) \to x(s, \varepsilon)$ and $(s, \varepsilon) \to \lambda(s, \varepsilon)$ are C^p;*

(b) *there exists a C^p function $\phi(x, \lambda, \varepsilon)$ such that for any fixed $\varepsilon \in (-\delta, \delta)$, the map ϕ^ε given by $(x, \lambda) \to \phi(x, \lambda, \varepsilon)$, is a C^p diffeomorphism between W and a neighborhood of the origin in $X \times R$. Moreover, ϕ^ε maps J_ε onto the line segment $\{(0, s)\colon s \in (-\delta, \delta)\}$.*

The above result implies that T is a compact, connected 1-submanifold in $X \times R$ and thus it is homeomorphic to the circle S^1 or to the closed interval $[-1, 1]$ (see, e.g., [1]). Moreover, it follows from the compactness of T that we can find finitely many W which cover T. Denote them by $W_1, ..., W_k$ with

$$F_\varepsilon^{-1}(0) \cap W_i = \{(x_i(s, \varepsilon), \lambda_i(s, \varepsilon))\colon s \in (-\delta_i, \delta_i)\} \equiv J_i^\varepsilon, \qquad i = 1, ..., k. \quad (5.1)$$

Let $W = W_1 \cup ... \cup W_k$. We see that $F_\varepsilon^{-1}(0) \cap W = J_1^\varepsilon \cup \cdots \cup J_k^\varepsilon \equiv J^\varepsilon$ is also a 1-submanifold in $X \times R$. Clearly J^0 is a connected 1-submanifold containing T. It is easy to see that by deleting unnecessary W_i's and the corresponding J_i^0's and then reorder them, we may assume that J_i^0 intersects only J_{i-1}^0 and J_{i+1}^0 for $i = 2, ..., k-1$. If T is homeomorphic to the circle, we suppose also that J_1^0 intersects only J_2^0 and J_k^0, and J_k^0 intersects only J_{k-1}^0 and J_1^0.

By shrinking δ_i slightly if needed, we may assume that the functions $x_i(s, \varepsilon)$ and $\lambda_i(s, \varepsilon)$ are defined for s in an open interval containing $[-\delta_i, \delta_i]$. Hence $\bar{J}_i^\varepsilon$ is part of a well parameterized curve. From now on, we will only consider the closed curve pieces $\bar{J}_i^\varepsilon$ and $\bar{J}^\varepsilon$. To save notations, we will omit the bars. We suppose that the intersection properties are retained for the closed curve pieces J_i^ε. The following result of Dancer [10, Theorem 2] gives the relationship between J^0 and J^ε.

PROPOSITION A.2. *There exist an* $\varepsilon_0 > 0$, *a small neighborhood* V *of* T *in* $X \times R$ *and a continuous function* $S: J^0 \times (-\varepsilon_0, \varepsilon_0) \to X \times R$ *such that*

(i) $F(S(u, \varepsilon), \varepsilon) = 0$ *if* $u \in J^0$ *and* $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$,

(ii) $S(u, 0) = u$ *if* $u \in J^0$ *and*

(iii) *if* $F(v, \varepsilon) = 0$ *with* $(v, \varepsilon) \in V \times (-\varepsilon_0, \varepsilon_0)$ *then* $v = S(u, \varepsilon)$ *for some* $u \in J^0$. *Moreover, for each fixed* ε, $u \to S(u, \varepsilon)$ *is* $1-1$.

It follows from Proposition A.2 that if T is homeomorphic to the circle then $J^0 = T$ and for all small ε, J^ε is homeomorphic to the circle; and if T is homeomorphic to $[-1, 1]$, then each J^ε is homeomorphic to $[-1, 1]$. Moreover, in either case, J^ε is a continuous deformation of J^0 when ε is small. It was remarked in [10] that the map S in Proposition A.2 can be shown to be C^{p-1}. In the following, we improve this by showing that J^ε is in fact a C^p deformation of J^0. Note that the proof below is used in Section 3.

Now let us consider (5.1) with the conventions made before Proposition A.1. Denoting $\psi_1^\varepsilon(s) = \psi_i(s, \varepsilon) = (x_i(s, \varepsilon), \lambda_i(s, \varepsilon))$ for convenience of notation, and replacing s by $-s$ when needed, we may assume that $\psi_i^\varepsilon(-\delta_i) \in J_{i-1}^\varepsilon$ and $\psi_i^\varepsilon(\delta) \in J_{i+1}^\varepsilon$ for $i = 2, ..., k-1$ if J^ε is homeomorphic to $[-1, 1]$, and $i = 1, ..., k$ with 0 identified with k and 1 identified with $k+1$ if J^ε is homeomorphic to S^1. We will take this convention and not distinguish these two cases from now on.

Claim. We can parameterize $J_1^\varepsilon \cup J_2^\varepsilon$ by some $\psi^\varepsilon: [-\delta_1, \delta_2 - c] \to X \times R$ (where c is a constant) with the following properties:

(i) $\psi^\varepsilon[-\delta_1, \delta_2 - c] = J_1^\varepsilon \cup J_2^\varepsilon$;

(ii) $\psi^\varepsilon(t) = \psi_1^\varepsilon(t)$ for t near $-\delta_1$, $\psi^\varepsilon(t) = \psi_2^\varepsilon(t + c)$ for t near $\delta_2 - c$;

(iii) $(t, \varepsilon) \to \psi^\varepsilon(t)$ is C^p;

(iv) there exist an open set $N \subset X \times R$ which contains $J_1^\varepsilon \cup J_2^\varepsilon$ for all small ε, and a map $\phi^\varepsilon: N \to X \times R$ such that $(u, \varepsilon) \to \phi^\varepsilon(u)$ is C^p, $\phi^\varepsilon(\psi^\varepsilon(t)) = (0, t)$.

We will call ϕ^ε in (iv) an extended inverse of ψ^ε, and denote it by $(\psi^\varepsilon)^{-I}$. Note that, by Proposition A.1, each ψ_i^ε has an extended inverse $(\psi_i^\varepsilon)^{-I}$.

If the parameterization in the above Claim can be done, then by repeating this procedure finitely many times, we obtain a global parameterization of J^ε as follows.

PROPOSITION A.3. *Under the assumptions of Proposition* A.2, *for each small* ε, *there is a mapping* $\Psi^\varepsilon\colon [-1, 1] \to X \times R$ (*with* -1 *and* 1 *identified if* J^0 *is homeomorphic to* S^1) *such that the range of* Ψ^ε *is* J^ε, $(t, \varepsilon) \to \Psi^\varepsilon(t)$ *is* C^p, *and* Ψ^ε *has an extended inverse* $(\Psi^\varepsilon)^{-I}$ *with* $(u, \varepsilon) \to (\Psi^\varepsilon)^{-I}(u)$ *a* C^p *map. In particular, the map* S *in Proposition* A.2 *can be chosen to be* C^p: $S(u, \varepsilon) = \Psi^\varepsilon[\beta(u)]$, *where* $\beta(u) \in R$ *is the second component of* $(\Psi^0)^{-I}(u)$.

Note that the map S given in Proposition A.3 may be different from that obtained in [10]. Note also that the map S in Proposition A.3 actually maps J^0 onto J^ε for any fixed small ε.

Proof of the Claim. We shall parameterize $J_1^\varepsilon \cup J_2^\varepsilon$ as described above by making use of the local parameterizations of J_i^ε. Let $\eta(\varepsilon)$ and $\xi(\varepsilon)$ be defined by $\psi_2^\varepsilon(-\delta_2) = \psi_1^\varepsilon(\eta(\varepsilon))$ and $\psi_1^\varepsilon(\delta_1) = \psi_2^2(\xi(\varepsilon))$. Then clearly

$$J_1^\varepsilon \cap J_2^\varepsilon = \{\psi_1^\varepsilon(s) : s \in [\eta(\varepsilon), \delta_1]\} = \{\psi_2^\varepsilon(s) : s \in [-\delta_2, \xi(\varepsilon)]\}.$$

For t in a small neighborhood of $[\eta(\varepsilon), \delta_1]$ in R and s in a small neighborhood of $[-\delta_2, \xi(\varepsilon)]$ in R, we set $\psi_1^\varepsilon(t) = \psi_2^\varepsilon(s)$ and obtain $(0, s) = (\psi_2^\varepsilon)^{-I}(\psi_1^\varepsilon(t)) \equiv (0, h^\varepsilon(t))$. It follows from the properties of ψ_1^ε and ψ_2^ε given in Proposition A.1 that h^ε is a C^p diffeomorphism, $(t, \varepsilon) \to h^\varepsilon(t)$ and $(s, \varepsilon) \to (h^\varepsilon)^{-1}(s)$ are both C^p. In particular, $\eta(\varepsilon) = (h^\varepsilon)^{-1}(-\delta_2)$ and $\xi(\varepsilon) = h^\varepsilon(\delta_1)$ are C^p. Moreover, we must have $(h^\varepsilon)'(t) > 0$ for $t \in [\eta(\varepsilon), \delta_1]$. This last assertion follows from the fact that h^ε is an increasing function and both $(h^\varepsilon)'(t)$ and $((h^\varepsilon)^{-1})'(s)$ are continuous.

Choose a constant $c > 0$ such that $\xi(\varepsilon) < c + \delta_1 < \delta_2$ for all small ε. Then define $g(t) = c + t$. Clearly $g(\delta_1) > h^\varepsilon(\delta_1)$. Hence we can find σ_0 such that $\eta(\varepsilon) < \sigma_0 < \delta_1$ for all small ε, and $g(t) > h^\varepsilon(t)$ for all $t \in [\sigma_0, \delta_1]$ and all small ε. Choose any $\sigma^0 \in (\sigma_0, \delta_1)$ and let $\rho \in C^\infty(R)$ be such that ρ is non-decreasing, $\rho(t) \equiv 0$ for $t \leqslant \sigma_0$, $\rho(t) \equiv 1$ for $t \geqslant \sigma^0$. Then define

$$H^\varepsilon(t) = [1 - \rho(t)]\, h^\varepsilon(t) + \rho(t)\, g(t).$$

Evidently, $H^\varepsilon(t) = h^\varepsilon(t)$ for $t \leqslant \sigma_0$, $H^\varepsilon(t) = g(t)$ for $t \geqslant \sigma^0$ and $(t, \varepsilon) \to H^\varepsilon(t)$ is C^p. Moreover, for $t \in [\sigma_0, \sigma^0]$,

$$\begin{aligned}(H^\varepsilon)'(t) &= [1 - \rho(t)](h^\varepsilon)'(t) + \rho(t) + \rho'(t)[g(t) - h^\varepsilon(t)]\\ &\geqslant \min\{\min_{[\sigma_0, \sigma^0]} (h^\varepsilon)'(t), 1\} > 0.\end{aligned}$$

Hence a simple application of the implicit function theorem shows that $t \to H^\varepsilon(t)$ is a C^p diffeomorphism, and $(t, \varepsilon) \to H^\varepsilon(t)$ and $(s, \varepsilon) \to (H^\varepsilon)^{-1}(s)$ are also C^p. Note that H^ε maps $[\eta(\varepsilon), \delta_2 - c]$ onto $[-\delta_2, \delta_2]$. Now we choose two constants σ_* and σ^* such that $\eta(\varepsilon) < \sigma_* < \sigma^* < \sigma_0$ for all small ε. Then choose $\tau \in C^\infty(R)$ such that $\tau(t) \equiv 0$ for $t \leqslant \sigma_*$, $\tau(t) \equiv 1$ for $t \geqslant \sigma^*$. Finally we define

$$\psi^\varepsilon(t) = [1 - \tau(t)]\, \psi_1^\varepsilon(t) + \tau(t)\, \psi_2^\varepsilon(H^\varepsilon(t)).$$

We want to show that ψ^ε is the required parameterization for $J_1^\varepsilon \cup J_2^\varepsilon$. Clearly, $\psi^\varepsilon(t) = \psi_1^\varepsilon(t)$ for $t \in [-\delta_1, \sigma_0]$ (recall that $\psi_1^\varepsilon(t) = \psi_2^\varepsilon(h^\varepsilon(t)) = \psi_2^\varepsilon(H^\varepsilon(t))$ for $t \in [\eta(\varepsilon), \sigma_0]$). Moreover, $\psi^\varepsilon(t) = \psi_2^\varepsilon(H^\varepsilon(t))$ for $t \in [\sigma^*, \delta_2 - c]$ and $\psi^\varepsilon(t) = \psi_2^\varepsilon(c + t)\ 0$ for $t \in [\sigma^0, \delta_2 - c]$. It then follows from the properties of ψ_1^ε, ψ_2^ε and H^ε that $(t, \varepsilon) \to \psi^\varepsilon(t)$ is C^p. It is also easy to see that $\psi^\varepsilon[-\delta_1, \delta_2 - c] = J_1^\varepsilon \cup J_2^\varepsilon$. It remains to show that ψ^ε has an extended inverse $(\psi^\varepsilon)^{-I}$ and $(u, \varepsilon) \to (\psi^\varepsilon)^{-I}(u)$ is C^p. By Proposition A.1 and the way J_i^ε are chosen, for each i, we can find an open set N_i which contains J_i^ε for all small ε, and a map ϕ_i^ε: $N_i \to X \times R$ such that $(u, \varepsilon) \to \phi_i^\varepsilon(u)$ is C^p and $\phi_i^\varepsilon = (\psi_i^\varepsilon)^{-I}$. Now choose an interior point u_0 in $J_1^0 \cap J_2^0$. By Proposition A.1, we can find a small neighborhood N_0 of u_0 such that

$$N_0 \cap (J_1^0 \cap J_2^0) = \{u_0 + sw_0 + w(s) \colon -\delta < s < \delta\}, \tag{5.2}$$

where δ is a small positive number, $\|w_0\| = 1$, $w(s)$ belongs to a complement W of $\operatorname{span}\{w_0\}$ in $X \times R$, $s \to w(s)$ is C^p with $w'(0) = 0$. For each small positive number r, $J_1^0 \cup J_2^0$ can be covered by finitely many balls with centers on the curve and the same radius r. Let $\{B_i \colon i = 1, ..., l\}$ be such a cover with the property that B_i intersects only B_{i-1} and B_{i+1} (a similar convention to the one used before applies if $J_1^0 \cup J_2^0$ is homeomorphic to S^1) and the union of the balls are contained in $N_1 \cup N_2$. Note that it is possible to choose the balls with this intersection property because by Proposition A.1, the curve can be expressed locally like (5.2) above near each point on it. We may assume that B_i is centered at $\psi^0(t_i)$ and $-\delta_1 < t_1 < \cdots < t_l < \delta_2 - c$. By choosing r small enough, we can find $j_0 (\geqslant 3)$ balls $B_{i_0+1}, ..., B_{i_0+j_0}$ with their union contained in the small neighborhood N_0 of u_0, and a small positive number s_0 such that $u = u_0 + sw_0 + w \in B_{i_0+1}$, $w \in W$ implies that $s \leqslant -s_0$ and $u = u_0 + sw_0 + w \in B_{i_0+j_0}$, $w \in W$ implies that $s \geqslant s_0$ (we replace w_0 by $-w_0$ if needed). Note that the union of these j_0 balls covers part of the curve which contains u_0.

Now we define a linear functional l on $X \times R$ by $l(u) = s$, where $u = sw_0 + w$, $w \in W$, and choose $r \in C^\infty(R)$ satisfying $r(t) \equiv 0$ for $t \leqslant -s_0$ and $r(t) \equiv 1$ for $t \geqslant s_0$. We decompose ϕ_2^ε as $\phi_2^\varepsilon(u) = (\alpha^\varepsilon(u), \beta^\varepsilon(u))$ where

$\alpha^\varepsilon(u) \in E$ and $\beta^\varepsilon(u) \in R$ and define $\tilde{\phi}_2^\varepsilon(u) = (\alpha^\varepsilon(u), (H^\varepsilon)^{-1}(\beta^\varepsilon(u)))$. Finally define $\phi^\varepsilon \colon \bigcup_{i=1}^{l} B_i \to X \times R$ by

$$\phi^\varepsilon(u) = (1 - r(l(u-u_0))\, \phi_1^\varepsilon(u) + r(l(u-u_0))\, \tilde{\phi}_2^\varepsilon(u) \qquad \text{for} \quad u \in \bigcup_{j=1}^{j_0} B_{i_0+j};$$

$$\phi^\varepsilon(u) = \phi_1^\varepsilon(u) \qquad \text{for} \quad u \in \bigcup_{i=1}^{i_0} B_i;$$

$$\phi^\varepsilon(u) = \tilde{\phi}_2^\varepsilon(u) \qquad \text{for} \quad u \in \bigcup_{i=i_0+j_0+1}^{l} B_i.$$

By the continuity of ψ^ε we know that for all small ε, $J_1^\varepsilon \cup J_\varepsilon^2$ are contained in the union of the l balls. To check that $(u, \varepsilon) \to \phi^\varepsilon(u)$ is C^p, by the intersection property of the balls, we need only check that on $B_{i_0} \cup B_{i_0+1}$ and on $B_{i_0+j_0} \cup B_{i_0+j_0+1}$. But it turns out that on the first two balls, ϕ^ε agrees with ϕ_1^ε and on the other two balls, it agrees with $\tilde{\phi}_2^\varepsilon$. It is straightforward to check that $\phi^\varepsilon(\psi^\varepsilon(t)) = (0, t)$ for t in a small neighborhood of $[-\delta_1, \delta_2 - c]$. This completes our proof of the Claim. ▮

Remark A.4. 1. In Proposition A.1, if F is analytic, then the local parameterizations are also analytic. However, we have only a C^∞ global parameterization in Proposition A.3 even if F is analytic.

2. The perturbation parameter ε need not be a real number. It can be an element in a Banach space.

Next we consider the case where (H_1) and the following two conditions are assumed.

(H_4): $F(x(\varepsilon), \lambda, \varepsilon) = 0$ for all (λ, ε) near $(\lambda_0, 0)$, where $\varepsilon \to x(\varepsilon)$ is C^p and $x(0) = 0$.

(H_5): $\dim \mathit{Ker}(F_x(0, \lambda_0, 0)) = \operatorname{codim} \mathit{Range}(F_x(0, \lambda_0, 0)) = 1$, $\mathit{Ker}(F_x(0, \lambda_0, 0)) = \{x_0\}$ and $F_{x\lambda}(0, \lambda_0, 0)\, x_0 \notin \mathit{Range}(F_x(0, \lambda_0, 0))$.

Then a simple variant of the proof of Theorem 1.7 in Crandall–Rabinowitz [8] shows the following result is valid.

Proposition A.5. *Under the assumptions* (H_1), (H_4) *and* (H_5), *there is a neighborhood* U *of* $(0, \lambda_0, 0)$ *and a positive number* δ *such that*

$$F^{-1}(0) \cap U = \{(sx_0 + sz(s, \varepsilon), \lambda(s, \varepsilon), \varepsilon) \colon s, \varepsilon \in (-\delta, \delta)\}$$
$$\cup \{(x(\varepsilon), \lambda, \varepsilon) \colon (x(\varepsilon), \lambda, \varepsilon) \in U\},$$

where z *and* λ *are* C^{p-1} *with* $z(0,0) = 0$, $\lambda(0,0) = \lambda_0$, $z(s, \varepsilon)$ *belongs to a complement* Z *of* $\mathit{span}\{x_0\}$ *in* X.

Remark A.6. A similar result (but in a less general form) to Proposition A.5 can be found in Chow and Hale [5], where they suppose that $u(\varepsilon) \equiv 0$ and $F_x(0, \lambda, \varepsilon)$ does not depend on ε.

There is also an easy generalization of Theorem 1.16 in Crandall–Rabinowitz [9] to the above case. Following the proof there, we have the following result.

PROPOSITION A.5′. *Under the conditions and notations of Proposition* A.5, *there exist open intervals* I, J *with* $\lambda_0 \in I$, $0 \in J$ *and* C^{p-1} *functions* $\gamma: I \times J \to R$, $\mu: J \times J \to R$, $u: I \times J \to X$ *and* $w: J \times J \to X$ *such that*

$$F_x(x(\varepsilon), \lambda, \varepsilon)\, u(\lambda, \varepsilon) = \gamma(\lambda, \varepsilon)\, u(\lambda, \varepsilon) \qquad \textit{for} \quad \lambda \in I, \quad \varepsilon \in J,$$

$$F_x(sx_0 + sz(s, \varepsilon), \lambda(s, \varepsilon), \varepsilon)\, w(s, \varepsilon) = \mu(s, \varepsilon)\, w(s, \varepsilon) \qquad \textit{for} \quad s, \varepsilon \in J.$$

Moreover,

$$\gamma(\lambda_0, 0) = \mu(0, 0) = 0, \qquad u(\lambda_0, 0) = w(0, 0) = x_0,$$

$$u(\lambda, \varepsilon) - x_0 \in Z, \qquad w(s, \varepsilon) - x_0 \in Z,$$

and $\gamma_\lambda(\lambda_0, 0) \neq 0$

$$\lim_{s \to 0,\, \mu(s, 0) \neq 0} \frac{-s\lambda_s(s, 0)\, \gamma_\lambda(\lambda_0, 0)}{\mu(s, 0)} = 1.$$

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