Real Classification of Complex Quadrics*

William C. Waterhouse
Department of Mathematics
Pennsylvania State University
University Park, Pennsylvania 16802

Submitted by Richard A. Brualdi

ABSTRACT

Canonical forms are given for complex quadric surfaces (and conics) under real changes of variable. The basic idea is to use known results on pairs of real quadratic forms.

INTRODUCTION

In a recent paper [2], P. E. Newstead classified complex conics under real changes of coordinates. His analysis grew out of a study of foci and relies primarily on the intersection points of the conic and its complex conjugate. This has the advantage of keeping the plane geometry clearly in view, but it does not suggest any simple extension to higher dimensions. In this paper I intend to show how a more algebraic approach, one mentioned but not adopted by Newstead, systematically gives us both his results and their extension to quadric surfaces. It will also be clear how anyone who wishes can go on to compute the classifications in higher dimensions.

The basic idea is a very simple one. Our objects of study are complex projective quadrics; that is, they are nontrivial quadratic forms with complex coefficients determined up to a scalar factor. We can of course write any complex quadratic form as $A+iB$, where A and B are real quadratic forms. The changes of variable that we allow are those with real coefficients, and these will take the pair (A, B) to an equivalent pair of real quadratic forms. The other operation that we allow is multiplication by a complex scalar $a+ib$, which changes $A+iB$ to $(aA-bB)+i(bA+aB)$. Now the fact is

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52 Vanderbilt Ave., New York, NY 10017 0024-3795/82/070045-08$2.75

that pairs of real quadratic forms have a rather straightforward classification. (This is true in essence over general fields; a modern treatment can be found in [3] or [4].) All we need to do is to start from that classification and compute which classes are sent to each other by maps $(A, B) \to (aA - bB, bA + aB)$.

1. PAIRS OF REAL FORMS

The classification of pairs of real quadratic forms has a quite simple nature. There are certain basic types, the indecomposable pairs. After change of variables, every pair splits into a direct sum of such indecomposable pairs, and the number of summands of each type is uniquely determined. Thus one can describe a pair up to equivalence precisely by listing the indecomposable pairs of which it is composed. The classification then is completed by giving a catalogue of distinct indecomposable pairs.

The simplest indecomposable pairs are the basic singular pairs, there being just one of these in each odd number of variables. For this paper we need to list explicitly only forms in four variables or less, and hence only two of the singular pairs concern us here: the trivial pair

$$A = 0, \qquad B = 0$$

in one variable, and the three-variable pair

$$A = xz, \qquad B = yz.$$

The nonsingular indecomposables are primarily classified by associating with each a power of an irreducible polynomial; this polynomial is the unique invariant factor of the matrix by which one must multiply to take the symmetric matrix for A to that for B. For each power of an irreducible $(\lambda - r)$ with real root there are two distinct pairs of opposite sign; for each power of a quadratic irreducible there is just one pair. In addition there are pairs (with singular A) that correspond to a place at infinity and are associated with the rational functions λ^{-n}. Explicitly, for $\lambda - r$ and the exponent 1 we have the pair

$$A = x^2, \qquad B = rx^2$$

together with its negative

$$A = -x^2, \qquad B = -rx^2.$$

The indecomposable pairs for $\lambda - r$ and exponents 2, 3, and 4 can be written as

$$A = xy, \qquad B = rxy + y^2,$$

$$A = xy + z^2, \qquad B = r(xy + z^2) + yz,$$

$$A = xy + zw, \qquad B = r(xy + zw) + yz + w^2$$

together with the negatives of these three. (Some customary factors of 2 found in [3], [4] are omitted here to bring our expressions closer to Newstead's. It is easy to check that the transition matrix in each case still does have the correct invariant factor.) The pairs occurring for the infinite place are those we get by setting $r = 0$ and interchanging A and B. We should note that multiplying A and B by the same positive constant does not change the equivalence class of the pair. Finally, for a quadratic irreducible with roots $p \pm iq$ the pairs with exponents 1 and 2 can be written as

$$A = 2xy, \qquad B = 2pxy + y^2 - q^2x^2,$$

$$A = 2(wz + xy), \qquad B = 2p(wz + xy) + 2wy + x^2 - 2q^2xz.$$

These formulas can of course be changed to other equivalent expressions by change of variables, and different authors give different versions of them. In these complex cases the pair can be multiplied by real scalars of either sign without changing its equivalence class.

2. THE EFFECT OF COMPLEX SCALING

Suppose now that we pass from (A, B) to $(a + ib)(A + iB) = A' + iB'$. It is clear that the pair (A', B') will not decompose into a direct sum unless (A, B) does. If we start with the pair (x^2, rx^2), it is trivial to compute that this is changed to a pair (positive or negative) with exponent 1 at the place $r' = (ar + b)/(a - br)$. The same change of place then will also occur for indecomposable pairs of higher exponent associated with $\lambda - r$. Likewise an indecomposable pair associated with $(\lambda - \beta)(\lambda - \bar{\beta})$ will be shifted to one of the same exponent for $\beta' = (a\beta + b)/(a - b\beta)$.

The study of this action seems to be simplified if we write complex scalars in polar form and adjust the standard forms for pairs correspondingly. We can for instance write a real number r uniquely as $r = \tan\theta$ for $-\pi/2 < \theta < \pi/2$, and then (x^2, rx^2) can equivalently be written as $((\cos\theta)x^2, (\sin\theta)x^2)$. Thus

$A + iB$ will be simply $e^{i\theta}x^2$. Extending to $\theta = \pi/2$ will include the case $r = \infty$, and we get the negatives of these pairs by proceeding on around the circle. Hence the indecomposable pairs (A, B) with exponent 1 at a real place give rise precisely to complex forms $e^{i\theta}x^2$ with θ unique modulo 2π. Multiplication by a complex scalar $ue^{i\varphi}$ now just changes θ to $\theta + \varphi$, since the positive factor u does not change the equivalence class. Similarly the forms $A + iB$ for pairs (A, B) of higher exponent at a real place can be expressed (up to equivalence) as the forms at $r = 0$ multiplied by a factor on the unit circle; thus for instance those of exponent 2 are $e^{i\theta}(xy + iy^2)$. A complex scalar again simply rotates the circle.

Consider now a complex place $(\lambda - \beta)(\lambda - \bar{\beta})$, where $\beta = p + iq$ with $q > 0$. Our earlier computation shows that when we multiply by $ue^{i\varphi} = (u\cos\varphi) + i(u\sin\varphi)$, we shift to

$$\beta' = \frac{\beta + \tan\varphi}{1 - (\tan\varphi)\beta}.$$

Now it is a familiar fact about the complex plane [1, p. 25] that $(\beta + s)/(1 - \beta s)$ for real s (including $s = \infty$) traces out a circle. Routine computation shows that its center is on the imaginary axis [at $(\beta\bar{\beta} + 1)/(\bar{\beta} - \beta)$] and that it crosses this axis at two points of the form ti and $t^{-1}i$. (For $\beta = i$, the circle collapses to a point.) Thus there is just one point on each orbit of the form ti with $0 < t \leqslant 1$. This shows us that when (A, B) is an indecomposable pair of exponent 1 at a complex place, we can express $A + iB$ up to equivalence as

$$e^{i\theta}\left[2xy + i\left(y^2 - t^2x^2\right)\right]$$

for a unique t with $0 < t \leqslant 1$. If $t = 1$, the factor $e^{i\theta}$ is irrelevant; otherwise, θ is unique modulo π (not modulo 2π, because in the complex case multiplication by $e^{i\pi} = -1$ does not change the equivalence class). Obviously $A + iB$ for a pair of higher exponent at a complex place can similarly be expressed in terms of the pair for ti.

We can now begin our classification by considering what nontrivial forms $A + iB$ we may have in two variables. First, the pair (A, B) may have a trivial summand together with summand of exponent 1 at a real place. Writing the nontrivial summand as $e^{i\theta}x^2$ and multiplying by a complex scalar, we can reduce to x^2. Next, the pair may be indecomposable of exponent 2 at a real place; scalar multiplication similarly allows us to reduce this to $xy + iy^2$. Third, the pair may be indecomposable of exponent 1 at a complex place; the scaling allows us to move that place to ti with $0 < t \leqslant 1$. Finally, there may be two summands of exponent 1 at real places (which may or may not coincide).

In this case we can write the form as $e^{i\theta}x^2 + e^{i\varphi}y^2$, and our scaling allows us to change θ and φ by an arbitrary constant. There is just one choice of scaling that will make one of the angles zero while the other lies between 0 and π. (This amounts to nothing more than taking two points on a circle and rotating them to a standard position.) Thus we have our first result:

THEOREM 1. *Consider a nonzero quadratic form in two variables with complex coefficients. Then under real changes of variable and complex scalar multiplication it is equivalent to exactly one of the following:*

$$\begin{array}{ll} x^2, & \\ xy + iy^2, & \\ x^2 + e^{i\theta}y^2, & 0 \leqslant \theta \leqslant \pi, \\ 2xy + i(y^2 - t^2x^2), & 0 < t \leqslant 1. \end{array}$$

3. COMPLEX CONICS

We turn now to forms in three variables. We can of course get a pair in three variables by adding a trivial summand to a pair in two variables; no new equivalences are introduced. We also now pick up the basic singular pair in three variables—the complex scaling necessarily preserves its class. Apart from these we have to consider only pairs with no singular summand. First, we may have an indecomposable pair of exponent 3 (at a real place); it can be written as $e^{i\theta}(xy + z^2 + iyz)$, and scaling allows us to set $\theta = 0$. Second, we may have a summand of exponent 2 and another of exponent 1; we can uniquely scale the summand of exponent 2 to make it $xy + iy^2$, while the other piece remains arbitrary. Third, we may have one real and one complex place (of exponent 1); the complex scaling gives us just enough freedom to transform the real-place summand to x^2, while the complex place remains arbitrary. Finally, we may have three summands of exponent 1 at real places. One might normalize here by making one of the angles equal to zero, but this can be done in several ways, and (unless two of the places coincide) there is no natural choice to be made. Hence it is probably simpler to leave this type unnormalized, except for listing the angles in increasing order. In any case, we have now completed our classification of complex conics:

THEOREM 2. *Consider a nonzero quadratic form in three variables with complex coefficients. Then under real changes of variable and complex scalar* *multiplication it is equivalent to one of the forms in Theorem* 1 *or to one of the following*:

$$xz + iyz,$$

$$xy + z^2 + iyz,$$

$$xy + iy^2 + e^{i\theta} z^2, \qquad 0 \leqslant \theta < 2\pi,$$

$$x^2 + 2yz + i(z^2 - y^2),$$

$$x^2 + e^{i\theta}\left[2yz + i(z^2 - t^2 y^2)\right], \qquad 0 < t < 1 \text{ and } 0 \leqslant \theta < \pi,$$

$$e^{i\theta} x^2 + e^{i\varphi} y^2 + e^{i\psi} z^2, \qquad 0 \leqslant \theta \leqslant \varphi \leqslant \psi < 2\pi.$$

TABLE 1
COMPARISON WITH NEWSTEAD'S CLASSIFICATION

Form in Theorem 2	Newstead's type
x^2	L
$xy + iy^2$	H
$x^2 + e^{i\theta} y^2$	
for $\theta = 0, \pi$	$G_\pm$
for $0 < \theta < \pi$	J_r
$2xy + i(y^2 - t^2 x^2)$	
for $t = 1$	M
for $0 < t < 1$	J_c
$xz + iyz$	I
$xy + z^2 + iyz$	C
$xy + iy^2 + e^{i\theta} z^2$	
for $\theta = 0, \pi$	$B_\pm$
for $0 < \theta < \pi$	E_1
for $\pi < \theta < 2\pi$	E_3
$x^2 + 2yz + i(z^2 - y^2)$	K
$x^2 + e^{i\theta}[2yz + i(z^2 - t^2 y^2)]$	F_2
$e^{i\theta} x^2 + e^{i\varphi} y^2 + e^{i\psi} z^2$	
for three coinciding points	A_+
for two coinciding points,	
the third diametrically opposite	A_-
the third not opposite	D_0
for two opposite points, the third elsewhere	D_2
for distinct points, no two opposite,	
all in a semicircle	F_0
not all in a semicircle	F_4

No two of these are equivalent except for the last type, where the three (possibly coincident) points on the unit circle can be changed by an arbitrary rotation of the circle.

Newstead's treatment separates some cases combined here and occasionally chooses different representatives for the equivalence classes. Table 1 lists how the forms in this theorem correspond to the types occurring in Newstead's classification.

4. COMPLEX QUADRICS

We can of course get pairs in four variables by adding a trivial singular summand to the ones we have studied already. There is no basic singular pair in four variables, but we can add the singular pair in three variables to a one-variable pair; normalizing the latter, we get one complex class. Apart from these, we have to consider only pairs made up of nonsingular summands. Suppose first that the pair involves a summand of exponent 3 or 4 at a real place; there can be only one such summand, and the scaling gives us just enough freedom to move that place to the origin. The same is true if there is a unique real summand of exponent 2. Next, we may have two summands of exponent 2 at real places; these we can normalize just as we did in Section 2 for two summands of exponent 1. We can similarly normalize pairs with two complex summands, except that here the θ in $e^{i\theta}$ is determined only modulo π. (Summands with $t = 1$, which are unaffected by scaling, should be listed separately.) If the pair is indecomposable of exponent 2 at a complex place, we just normalize the place as we did in Section 2. If there is a complex summand and two real summands, we normalize the real part. Finally, four summands of exponent 1 at real places may be represented (without normalization) by four points on the unit circle. Thus we have the following classification.

THEOREM 3. *Consider a nonzero quadratic form in four variables with complex coefficients. Then under real changes of variable and complex scalar multiplication it is equivalent to one of the forms in Theorems* 1 *and* 2 *or to one of the following*:

$$x^2 + yw + izw,$$

$$xy + zw + i(yz + w^2),$$

$$
\begin{array}{ll}
xy + z^2 + iyz + e^{i\theta}w^2, & 0 \leqslant \theta < 2\pi, \\
xy + iy^2 + 2zw + i(z^2 - w^2) & \\
xy + iy^2 + e^{i\theta}\left[2zw + i(z^2 - t^2w^2)\right], & 0 < t < 1, \quad 0 \leqslant \theta < \pi, \\
xy + iy^2 + e^{i\theta}z^2 + e^{i\varphi}w^2, & 0 \leqslant \theta \leqslant \varphi < 2\pi, \\
xy + iy^2 + e^{i\theta}(zw + iw^2), & 0 \leqslant \theta \leqslant \pi, \\
2xy + i(y^2 - t^2x^2) + e^{i\theta}\left[2zw + i(z^2 - s^2w^2)\right], & 0 < s, t < 1, \\
 & 0 \leqslant \theta \leqslant \pi/2, \\
2xy + i(y^2 - x^2) + 2zw + i(z^2 - s^2w^2), & 0 < s \leqslant 1, \\
2xy + 2zw + i(x^2 + 2yw - 2t^2xz), & 0 < t \leqslant 1, \\
x^2 + e^{i\theta}y^2 + e^{i\varphi}\left[2zw + i(z^2 - t^2w^2)\right], & 0 \leqslant \theta \leqslant \pi, \quad 0 < t < 1, \\
 & 0 \leqslant \varphi < \pi, \\
x^2 + e^{i\theta}y^2 + 2zw + i(z^2 - w^2), & 0 \leqslant \theta \leqslant \pi, \\
e^{i\theta}x^2 + e^{i\varphi}y^2 + e^{i\psi}z^2 + e^{i\tau}w^2, & 0 \leqslant \theta \leqslant \varphi \leqslant \psi \leqslant \tau < 2\pi.
\end{array}
$$

The only equivalences among these, beyond those listed in Theorem 2, *are in the last case, where the four (possibly coincident) points on the unit circle can be changed by an arbitrary rotation.*

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