Oscillation Theorems for Neutral Difference Equations

B. S. Lalli

Department of Mathematics
University of Saskatchewan
Saskatoon, S7N 0WO, Canada

Abstract—Oscillation criteria for first order difference equations of the form \( \Delta x_n + q_n f(x_{n-\delta}) = 0 \) and \( \Delta (x_n + px_{n-k}) + q_n f(x_{n-\tau}) = F_n \), are established. Here \( \Delta x_n = x_{n+1} - x_n \) is the forward difference operator, \( \delta = \pm 1 \), \( \tau \) and \( k \) are nonnegative integers, \( \{q_n\} \) and \( \{F_n\} \) are sequences of nonnegative real numbers and \( f : \mathbb{R} \rightarrow \mathbb{R} = (-\infty, \infty) \) is continuous. Oscillation criteria for nonlinear difference equations, of sublinear as well as superlinear type, are also established.

Keywords—Oscillation, Difference equations, Delay, Neutral, Sublinear, Superlinear.

1. INTRODUCTION

The problem of oscillation and nonoscillation of solutions of delay difference equations has received a considerable attention during the last few years. Erbe and Zhang [1], Georgiou et al. [2], Gyori and Ladas [3,4], Ladas [5-9], Philos [10-12] have done extensive work on this topic. As described by Ravi Agarwal [13] "difference equations manifest themselves as mathematical models describing real life situations in probability theory, queuing problems, statistical problems, stochastic time series, combinatorial analysis, number theory, geometry, electrical networks, quanta in radiation, genetics in Biology, economics, psychology, sociology, etc. Therefore, difference equations are not the discrete analogues of the differential equations, in fact, they paved the way for the development of the latter. In [14], several examples from the diverse fields have been illustrated which are sufficient to convey the importance of the serious qualitative as well as quantitative study of the difference equations."

This paper is concerned with the oscillation of solutions of neutral difference equations of the form

\( \Delta y_n + p y_{n-k} = 0, \quad n = 0, 1, 2, \ldots, \) \hspace{1cm} (1.1)

and

\( \Delta (x_n + px_{n+k}) - q_n f(x_n) = F_n. \) \hspace{1cm} (1.2: \( \delta \))

where \( \delta = \pm 1 \), \( p \) is a nonnegative real number, \( k \in \mathbb{N} = \{1, 2, \ldots, \} \). The forward difference operator \( \Delta \) is defined as usual, i.e., \( \Delta x_n = x_{n+1} - x_n \), \( n \in \mathbb{N} \) for every sequence of real numbers. The sequence \( \{\tau_n\} \) is a sequence of nonnegative integers with \( \lim_{n \to \infty} \tau_n = \infty \) and \( \{F_n\} \) and \( \{q_n\} \) are also sequences of real numbers. The function \( f \) is a real valued function satisfying \( x f(x) > 0 \) for \( x \neq 0 \).

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Difference equations are also appropriate models for describing situations where population growth is not continuous but seasonal, with overlapping generations. For example, the difference equation

\[ y_{n+1} = y_n \exp \left[ r \left( 1 - \frac{y_n}{K} \right) \right], \]  

(1.3)

has been used to model various animal populations.

By a solution of difference equation, say (1.1), we mean a sequence \( \{x_n\} \) which satisfies the equation for all \( n \in \mathbb{N} \). A solution \( \{x_n\} \) is called oscillatory if the terms \( x_n \) of the sequence are neither eventually positive nor eventually negative. Otherwise, it is called nonoscillatory.

In Section 2, we list some recent results for first order difference equations and establish a result for a nonlinear difference equation (sublinear case) similar to the one by Lalli et al. [15] for a linear difference equation. Some new results for neutral difference equations are put in Section 3.

2. RESULTS FOR FIRST ORDER EQUATIONS

Below, we list some well known recent oscillation criteria for first order difference equations. Consider the difference equations

\[ \Delta y_n + p_n y_{n-m} = 0, \quad \text{and} \]
\[ \Delta y_n + p_n f(y_{n-m}) = 0, \]

(2.1) (2.2)

which are discrete analogues of

\[ y'(t) + p(t) y(t - m) = 0, \quad \text{and} \]
\[ y'(t) + p(t) f(y(t - m)) = 0, \]

(2.3) (2.4)

respectively. The difference equation

\[ y_{n+1} - y_n + \sum_{i=1}^{k} p_i y_{n-m_i} = 0 \]

(2.5)

is considered to be a discrete analogue of the delay equation

\[ y'(t) + \sum_{i=1}^{k} p(t) y(t - m_i) = 0. \]

(2.6)

Following, are some results by Erbe and Zhang [1].

**Theorem 2.1.** [1] Assume that

\[ \liminf_{n \to \infty} p_n \equiv c > 0 \quad \text{and} \quad \limsup_{n \to \infty} p_n > 1 - c. \]

(2.7)

Then

(i) \[ y_{n+1} - y_n + p_n y_{n-m} \leq 0, \]

(2.8)

has no eventually positive solution;

(ii) \[ y_{n+1} - y_n + p_n y_{n-m} \geq 0, \]

(2.9)

has no eventually negative solution.

(iii) every solution of (2.1) oscillates.
Theorem 2.2. [1] Assume that
\[ \liminf_{n \to \infty} p_n \equiv c > \frac{m^m}{(m+1)(m+1)}. \]  
(2.10)

Then the conclusion of Theorem 2.1 holds.

Remark 2.1. If \( m = 0 \) in 2.10, the the right hand side is 1 and (2.1) becomes
\[ y_{n+1} - y_n + p_n y_n = 0, \]
which is oscillatory if \( \liminf_{n \to \infty} p_n > 1 \), whereas its discrete analogue
\[ y'(t) + p(t) y(t) = 0 \]
is nonoscillatory for any continuous function \( p(t) \).

It is known [1] that the condition (2.10) is sharp. In fact, if \( p_n = \frac{m^m}{(m+1)} \) for all \( n \geq 0 \), then (2.1) has a nonoscillatory solution defined by \( y_n = d > 0, y_{k+1} = \frac{m}{(m+1)} y_k, k = N, N+1, \ldots \). It is true (see [1]) that if \( \sup p_n < \frac{m^m}{(m+1)} \), then (2.1) has a nonoscillatory solution.

Next result is also by Erbe and Zhang.

Theorem 2.3. [1] Assume that \( p_{ij} \geq 0 \) and
\[ \sum_{i=1}^{k} \left( \liminf_{n \to \infty} p_{in} \right) \left\{ \frac{(m_i + 1)^{m_i + 1}}{m_i^{m_i}} \right\} > 1. \]  
(2.11)

Then every solution of equation
\[ y_{n+1} - y_n + \sum_{i=1}^{k} p_{in} y_{n-m_i} = 0 \]  
(2.12)
is oscillatory.

Corollary 2.1. [1] Assume that \( p_{ij} \geq 0 \) and
\[ k \left( \prod_{i=1}^{k} \left( \liminf_{n \to \infty} p_{in} \right) \right)^{1/k} > \frac{\bar{m}^n}{(\bar{m}+1)^{\bar{m}+1}}, \]  
(2.13)
where \( \bar{m} = \frac{1}{k} \left( \sum_{i=1}^{k} m_i \right) \). Then every solution of (2.12) oscillates.

Some necessary and sufficient conditions for the oscillation of (2.1) are given below. The following result is due to Ladas.

Theorem 2.4. [5] Every solution of (2.1) oscillates if and only if one of the following conditions hold:
(i) \( m = -1 \) and \( p \leq -1 \);
(ii) \( m = 0 \) and \( p \geq 1 \);
(iii) \( m \in \{-3,-2\} \cup \{1,2,\ldots\} \) and \( p \left\{ \frac{(m+1)^{m+1}}{m^m} \right\} > 1 \).

Remark 2.2. It is well known that every solution of the delay differential equation
\[ y'(t) + p x(t - \tau) = 0 \]
oscillates if and only if
\[ p e^\tau > 1. \]  
(2.14)
As pointed out by Ladas [7], Condition (iii) of Theorem 2.4 can be written as
\[ p(m + 1) \frac{(m + 1)^m}{m^m} > 1, \]  
(2.15)
and that
\[ \frac{(m + 1)^m}{m^m} = \left(1 + \frac{1}{m}\right)^m \rightarrow e, \quad \text{as } m \rightarrow \infty. \]
Therefore, one can think of (2.15) as a discrete analogue of (2.14).

**THEOREM 2.5.** [1] Suppose that either
\[ p_i \in (0, \infty) \quad \text{and} \quad m_i \in (0, 1, \ldots), \quad i = 1, 2, \ldots, k, \quad \text{or} \]
(2.16)
\[ p_i \in (-\infty, 0) \quad \text{and} \quad m_i \in (\ldots, -2, -1), \quad i = 1, 2, \ldots, k. \]
(2.17)

Then, every solution of (2.5) oscillates, provided one of the following two conditions is satisfied:
\[ \sum_{i=1}^{k} p_i \frac{(m_i + 1)^{m_i+1}}{m_i^{m_i}} > 1; \]

Here, also, one can think of (a) and (b) as discrete analogues of (c) and (d), respectively, where

\[ \sum_{i=1}^{k} p_i \tau_i > \frac{1}{e} \quad \text{and} \]
(d)

\[ \left[ \prod_{i=1}^{k} p_i \right]^{1/k} \sum_{i=1}^{k} \tau_i > \frac{1}{e}. \]

One can recall that
\[ x'(t) + \sum_{i=1}^{k} p_i x(t - \tau_i) = 0, \quad p_i \in (0, \infty), \]
oscillates if either (c) or (d) holds.

Here are some sharp oscillation conditions established by Ladas.

**THEOREM 2.6.** [7] Suppose that \( \{p_n\} \) is a nonnegative sequence of real numbers and let \( m \) be a positive integer. Assume that
\[ \liminf_{n \to \infty} \left[ \frac{1}{m} \sum_{i=n-m}^{n-1} p_i \right] > \frac{m^m}{(m + 1)^{m+1}}. \]
(2.18)

Then every solution of (2.5) is oscillatory.

**REMARK 2.3.** Condition (2.18) of Theorem 2.6 is sharp in that the lower bound \( m^m/(m + 1)^{m+1} \) cannot be improved. Moreover, when \( p_n \equiv p \), Condition (2.18) reduces to \( p > m^m/(m + 1)^{m+1} \), which is a necessary and sufficient condition for the oscillation of all solutions of (1.1).

It is well known that every solution of the delay equation
\[ x'(t) + p x(t - \tau) = 0, \quad p \in C[[t_0, \infty), R], \quad \tau \in (0, \infty), \]
(2.19)
oscillates provided that
\[
\liminf_{t \to \infty} \int_{t-\tau}^{t} p(s) \, ds > \frac{1}{e}.
\] (2.20)

As indicated by Ladas [7] (2.18) is a discrete analogue of (2.20), since (2.18) can be written in the form
\[
\liminf_{n \to \infty} \left( \sum_{i=n-m}^{n-1} p_i \right) > \frac{m^m}{(m+1)^{m+1}},
\]
and that
\[
\left[ \frac{m}{(m+1)} \right]^{m+1} = \left\{ \frac{1}{\left( 1 + \frac{1}{m} \right)^{m}} \right\} \frac{m}{m+1} \to \frac{1}{e}, \quad \text{as } m \to \infty.
\]

The following lemma concerns the asymptotic behavior of solutions of delay and advance difference inequalities.

**Lemma 2.1.** [5] Assume that \( p \in (0, \infty) \) and \( k \in \{1, 2, \ldots\} \). Then the following statements hold:

(a) The delay difference inequality
\[
x_{n+1} - x_n + px_{n-k} \leq 0, \quad n = 0, 1, 2, \ldots,
\]
has an eventually positive solution if and only if the delay difference equation
\[
y_{n+1} - y_n + py_{n-k} = 0, \quad n = 0, 1, 2, \ldots,
\]
has an eventually positive solution.

(b) The advanced difference inequality
\[
x_{n+1} - x_n - px_{n+k} \geq 0, \quad n = 0, 1, 2, \ldots,
\]
has an eventually positive solution if and only if the advanced difference equation
\[
y_{n+1} - y_n - py_{n+k} = 0, \quad n = 0, 1, 2, \ldots,
\]
has an eventually positive solution.

The following is a comparison result by Yan and Qian. Consider the delay difference equation
\[
x_{n+1} - x_n + \sum_{i=1}^{m} p_i(n) x_{n-k_i} = 0, \quad n \geq 0,
\] (2.21)
where \( p_i \in (0, \infty), n, k_i \) are nonnegative integers. Put
\[
\bar{p}(n) = \min \left\{ \sum_{i=1}^{m} p_i(n) : 1 \leq i \leq m \right\}, \quad p(n) = \max \left\{ \sum_{i=1}^{m} p_i(n) : 1 \leq i \leq m \right\},
\]
\[
\bar{k} = \min \{k_i : 1 \leq i \leq m\}, \quad k = \max \{k_i : 1 \leq i \leq m\}.
\]

**Theorem 2.8.** [16] Assume that the equation
\[
x_{n+1} - x_n + p(n) x_{n-k} = 0
\]
has a nonoscillatory solution, then (2.21) has also a nonoscillatory solution. Further, if the equation
\[
x_{n+1} - x_n + \bar{p}(n) x_{n-k} = 0
\] (2.22)
oscillates, then so does (2.21).
3. OSCILLATION OF NEUTRAL DIFFERENCE EQUATIONS

In this section, we discuss oscillation criteria for some neutral difference equations. Consider

\[ \Delta(x_n + cx_{n-m}) + p_n x_{n-k} = 0, \quad n = 1, 2, \ldots, \] (3.1)

where \( c \) and \( p \) are real numbers, \( m \) and \( k \) are integers, and \( p_n, m \) and \( k \) are nonnegative. We can think of (3.1) as a discrete analogue of

\[ \frac{d}{dt}(x(t) + px(t-\tau)) + qx(t-\sigma) = 0, \] (3.2)

and for this reason (3.1) is called a neutral difference equation. Let \( M = \max\{m, k\} \). Then by a solution of (3.1) for \( n = 1, 2, \ldots \), we mean a sequence \( \{x_n\} \) which is defined for \( n \geq -M \) and which satisfies (3.1) for all \( n = 0, 1, \ldots \). Clearly, if

\[ x_n = A_n, \quad \text{for } n = -M, \ldots, -1, 0, \] (3.3)

are given, the equation (3.1) has a unique solution satisfying the conditions (3.3). Following, are some useful lemmas due to Erbe et al. [1] and Lalli et al. [15].

**LEMMA 3.1.** [1] Assume that \( c = -1 \) and \( p_n > 0 \) for \( n = 1, 2, \ldots \). Let \( \{x_n\} \) be an eventually positive solution of (3.1), then

\[ z_n = x_n + cx_{n-m} \geq 0, \quad \text{and} \quad \Delta z_n \leq 0, \]

eventually.

**LEMMA 3.2.** [15] Assume that \( -1 < c < 0 \) and that \( \{x_n\} \) is an eventually positive solution of (3.1). Then \( z_n > 0 \) and \( \Delta z_n < 0 \).

Now we list a result by Lalli et al. [15] for linear difference equation (3.1).

**THEOREM 3.1.** [15] Assume that

(i) \( c = -1; \)

(ii) \( p_n \geq 0, n = 1, 2, \ldots, \) and \( \sum_{n=N}^{\infty} p_n = \infty, \) where \( N \) is a positive integer. Then every solution of (3.1) is oscillatory.

We establish a result similar to the above for a nonlinear neutral difference equation (sublinear type)

\[ \Delta(x_n + cx_{n-m}) + p_n f(x_{n-k}) = 0, \quad \text{for } n = 1, 2, \ldots. \] (3.4)

**THEOREM 3.2.** In addition to Conditions (i) and (ii) of Theorem 3.1, suppose that

(a) \( x f(x) > 0 \) for \( x \neq 0, \) \( f \) is nondecreasing;

(b) \( \int_0^{\pm\epsilon} \frac{dx}{f(x)} < \infty, \) for all \( \epsilon > 0. \) Then every solution of (3.4) is oscillatory.

**PROOF.** Let \( \{x_n\} \) be a nonoscillatory solution of (3.4), which we assume to be eventually positive. Set \( z_n = x_n + cx_{n-m}. \) From (3.4), we have

\[ \Delta z_n = -p_n f(x_{n-m}) \leq 0, \]

eventually. Since \( p_n \) is not identically zero, \( z_n \) cannot be eventually identically zero. Thus, \( z_n \) is either eventually negative or it is eventually positive. In case \( z_n < 0, \) eventually, we have

\[ z_n \leq z_N, \quad \text{for } n \geq N. \]

Hence,

\[ x_{N+mn} \leq z_N + x_{N+(n-1)m} \leq \cdots \leq n z_N + x_N. \]
By letting $n \to \infty$, we note that $x_{n+\mu n}$ will become eventually negative, which is a contradiction. Thus, $z_n$ is eventually positive. Since, $f$ is nondecreasing, we have

$$\Delta z_n + p_n f(z_{n-k}) \leq 0,$$

i.e.,

$$\frac{\Delta z_n}{f(z_{n-k})} + p_n \leq 0, \quad n \geq N. \quad (3.5)$$

Note that for $z_{n+1} \leq t \leq z_{n-k}$, $f(t) \leq f(z_{n-k})$. This implies that

$$-\int_{z_{n+1}}^{z_n} \frac{dt}{f(t)} \leq \frac{\Delta z_n}{f(z_{n-k})}.$$ 

We use this estimate in (3.5) to have

$$-\int_{z_{n+1}}^{z_n} \frac{dt}{f(t)} + p_n \leq 0.$$

Whence

$$-\int_{z_{n+1}}^{z_N} \frac{dt}{f(t)} + \sum_{i=N}^{n} p_i \leq 0,$$

which, in view of the hypotheses of the theorem, leads to a contradiction. This completes the proof of the theorem.

For the equation

$$\Delta x_n + p x_{n-k} = f(x_{n-1}), \quad n = 0, 1, 2, \ldots, \quad (3.6)$$

Gyori et al. [4] established the following result.

**Theorem 3.3.** Let $p \in (0, \infty)$ and $k, l$ be integers. Further, assume that

$$0 < p < \frac{k^k}{(k+1)^{k+1}}, \quad (3.7)$$

and

$$f \in C[R, R] \text{ and } u f(u) > 0, \text{ for all } u \in R. \quad (3.8)$$

Then the following statements are equivalent:

(a) every solution of (3.6) tends to zero as $n \to \infty$;

(b) $|f(u)| < p |u|$ for $u \neq 0$.

One can easily deduce the following corollary from the above theorem.

**Corollary 3.1.** Assume that $q \in [0, \infty)$ and that (3.7) holds. Then the trivial solution of

$$\Delta x_n + p x_{n-k} - q x_{n-l} = 0, \quad n = 0, 1, 2, \ldots, \quad (3.9)$$

is globally asymptotically stable if and only if

$$p > q.$$ 

Now we discuss the forced neutral difference equation

$$\Delta (x_n + p x_{n+\delta k}) - q_n f(x_{\tau_n}) = F_n, \quad (3.9: \delta)$$

where $\delta = \pm 1$, $p$ is a nonnegative real number, $k \in N = \{1, 2, \ldots\}$. The sequence $\{\tau_n\}$ is a sequence of nonnegative integers with $\lim_{n \to \infty} \tau_n = \infty$ and $\{F_n\}$ and $\{q_n\}$ are sequences of real numbers with $q_n \geq 0$ eventually. The function $f$ is a real valued function satisfying $xf(x) > 0$ for $x \neq 0$. We need the following two assumptions:

(i) There exists a sequence $\{h_n\}$ of real numbers such that

$$\Delta h_n = F_n, \quad \text{and the sequence } \{h_n\} \text{ is oscillatory;}$$

(ii) The sequence $\{h_n\}$ is periodic of period $k$, i.e., $h_n = h_{n+\delta k}$. 

Our first result is the following theorem.

**Theorem 3.4.** If condition \((H_1)\) holds with \(\{h_n\}\) satisfying

\[
\limsup_{n \to \infty} h_n = \infty, \quad \text{and} \quad \liminf_{n \to \infty} h_n = -\infty. \tag{3.10}
\]

Then every bounded solution of (3.9: 6) is oscillatory.

**Proof.** Assume, for the sake of a contradiction, that (3.9: 6) has a bounded nonoscillatory solution \(\{x_n\}\) which we assume to be eventually positive. There exists a positive integer \(n_0\) such that

\[
x_n > 0, \quad x_{n+k} > 0, \quad \text{and} \quad x_{\tau_n} > 0, \quad \text{for } n \geq n_0. \tag{3.11}
\]

Define

\[
y_n = x_n + x_{n+k}, \quad \text{and} \quad z_n = y_n - h_n. \tag{3.12}
\]

Then

\[
\Delta z_n = q_n f(x_{\tau_n}) \geq 0, \quad \text{for } n \geq n_0. \tag{3.13}
\]

We claim that

\[
z_n > 0, \quad \text{for } n \geq n_0. \tag{3.14}
\]

Otherwise, \(z_n \leq 0\) for \(n \geq n_1\) for some \(n_1 \geq n_0\), and hence,

\[
y_n - h_n \leq 0, \quad \text{i.e., } 0 < y_n \leq h_n, \quad \text{for } n \geq n_1.
\]

This provides a contradiction to the fact that \(\{h_n\}\) is oscillatory. Thus, (3.14) holds and

\[
\Delta(y_n - h_n) > 0, \quad \text{and} \quad y_n - h_n > 0, \quad \text{for } n \geq n_0.
\]

Consequently,

\[
\lim_{n \to \infty} (y_n - h_n) = \beta \in [0, \infty].
\]

From (3.10) it follows that there exists a sequence \(\{n_l\}\) such that \(\lim_{l \to \infty} h_{n_l} = \infty\), and hence,

\[
\lim_{l \to \infty} (y_{n_l} - h_{n_l}) = \beta.
\]

This implies that \(\{y_{n_l}\}\) cannot be bounded, which is a contradiction. This completes the proof of the theorem.

The following theorem deals with the asymptotic behavior of (3.9: 6) in the case when \(q_n \leq 0\) eventually.

**Theorem 3.5.** Assume that \(q_n \leq 0\) eventually. If \((H_1)\) and (3.10) hold, then every solution of (3.9: 6) is oscillatory.

**Proof.** Let \(\{x_n\}\) be a nonoscillatory solution of (3.9: 6), which is eventually positive. Choose a positive integer \(n_0\) such that (3.11) is satisfied. With \(y_n\) and \(z_n\) as defined in (3.12), we have

\[
\Delta z_n = q_n f(x_{\tau_n}) \leq 0, \quad \text{for } n \geq n_0.
\]

As shown in the proof of Theorem 3.4, we have \(z_n > 0\) for \(n \geq n_0\), and hence, it is easy to show that

\[
\Delta(y_n - h_n) \leq 0, \quad \text{and} \quad y_n - h_n > 0, \quad \text{for } n \geq n_0,
\]

which implies that \(\lim_{n \to \infty} (y_n - h_n) = \beta^*\). From (3.10), it follows that there exists a sequence \(\{n_l\}\) such that \(\lim_{l \to \infty} h_{n_l} = -\infty\). Then,

\[
\lim_{l \to \infty} (y_{n_l} - h_{n_l}) = \beta^*.
\]
which implies that \( \{y_n\} \) cannot be positive. With this contradiction, the proof of the theorem is complete.

Now, we consider the case when (3.9: \( \delta \)) is of superlinear type, that is, when \( f \) satisfies the condition

\[
    f \text{ is nondecreasing for } x \neq 0 \quad \text{and} \quad \sum_{i=n^*}^{\infty} \frac{\Delta u_i}{f(u_{i+1})} < \infty,
\]

where \( n^* \) is any positive integer. For convenience, we introduce the following notation:

\[
    A_\alpha = \{n \in N : \tau_n > n + \alpha + 1\}, \quad \alpha \text{ is a positive integer.}
\]

Our oscillation criterion for the superlinear case is the following theorem.

**Theorem 3.6.** Assume that the conditions \((H_1), (H_2), (3.10)\) and \((3.15)\) hold. If

\[
    \sum_{n \in A_\alpha} q_n = \infty,
\]

then

(i) equation (3.9: \(-1\)) is oscillatory provided \(0 \leq p < 1\) and \(\alpha = 0\);

(ii) equation (3.9: \(1\)) is oscillatory provided \(p > 1\) and \(\alpha = k\).

**Proof.** Let \( \{x_n\} \) be a nonoscillatory solution of (3.9: \( \delta \)). As before, we assume that \( \{x_n\} \) is eventually positive. We choose a positive integer \( n_0 \) such that (3.11) is satisfied for \( n \geq n_0 \). With \( y_n \) and \( z_n \) as defined in (3.12), we obtain (3.13) and (3.14). Consider the following two cases.

**Case 1.** \( \delta = -1 \) and \(0 \leq p < 1\). From (3.12), we have

\[
    x_n = z_n + h_n - px_{n-k} = z_n + h_n - px_{n-k}.
\]

In view of \((H_2)\) and (3.13), we can choose a sufficiently large integer \( n_1 \) such that

\[
    x_n \geq (1 - p) (z_n + h_n), \quad \text{for } n \geq n_1.
\]

Now select an integer \( n_2 \geq n_1 \) such that

\[
    x_n \geq (1 - p) (z_n + h_{n_2}) = \xi_n, \quad \text{for } n \geq n_2.
\]

It is easy to check that

\[
    \Delta x_n = \frac{1}{1 - p} \Delta \xi_n, \quad \text{for } n \geq n_2,
\]

and

\[
    \xi_n = (1 - p) (z_n + h_{n_2}), \\
    \geq (1 - p) (z_{n_2} + h_{n_2}) > 0, \quad \text{for } n \geq n_2.
\]

**Case 2.** \( \delta = 1 \) and \( p > 1 \). Once again, from (3.12), we have

\[
    x_n = \frac{1}{p} (z_{n-k} + h_{n-k} - x_{n-k}) = \frac{1}{p} \left( z_{n-k} + h_{n-k} - \frac{1}{p} (z_{n-2k} + h_{n-2k} - x_{n-2k}) \right).
\]

In view of \((H_2)\) and (3.13), there exists an integer \( n_3 \geq n_0 \) such that

\[
    x_n \geq \frac{p - 1}{p^2} (z_{n-k} + h_{n-k}), \quad \text{for } n \geq n_3.
\]
Now choose an integer $n_4 \geq n_3 + k$ such that
\[ x_n \geq \frac{p-1}{p^2} (z_{n-k} + h_{n-k}) = \eta_{n-k}, \quad n \geq n_4. \tag{3.20} \]

It follows that
\[ \Delta z_n = \frac{p^2}{p-1} \Delta \eta_n, \quad \text{and} \quad \eta_n > 0, \quad \text{for } n > n_4. \]

Put $n_5 = \max \{n_2, n_4\}$. From (3.13), (3.18) and (3.20) we get
\[ \Delta \theta_n \geq \gamma q_n f(\theta_{n-\alpha}), \quad \text{for } n \geq n_5, \tag{3.21} \]
where
\[ \gamma = \begin{cases} 1 - p, & \alpha = 0, \\ \frac{p-1}{p^2}, & \alpha = k, \end{cases} \tag{3.22} \]

Divide (3.21) by $f(\theta_{n+1})$ and then sum over $\alpha_n \in [n_5, n] = D$. Since $\theta_{n+1} \geq \theta_{n-\alpha}$ on the set $D$, we have
\[ \sum_{i=5}^{n} \frac{\Delta \theta_i}{f(\theta_{i+1})} \geq \gamma \sum_{i \in D} q_i, \]
and hence,
\[ \sum_{i \in D} q_i < \sum_{i=n_5}^{\infty} \frac{\Delta \theta_i}{f(\theta_{i+1})} < \infty, \]
which contradicts (3.16). This completes the proof of the theorem.

REMARK 3.1. If $f(x) = |x|^\lambda \text{sgn} x, \lambda > 1$, then Condition (3.15) of Theorem 3.6 is automatically satisfied, since
\[ \sum_{i=n^*}^{\infty} \frac{\Delta u_i}{f(u_{i+1})} = \sum_{i=n^*}^{\infty} \frac{\Delta u_i}{(u_{i+1})^\lambda} < \infty, \quad n^* \text{ is any positive integer}, \]
(see [17, Theorem 4.1]).

In the following theorem, we assume that the function $f$ satisfies the condition
\[ \frac{f(x)}{x} > M, \quad \text{for } x \neq 0, \quad M \text{ is a positive constant.} \tag{3.23} \]

THEOREM 3.7. Suppose that there exists an integer $\ell \geq 2$ such that $\tau_n - \alpha \geq n + \ell, n \in N$ ($\alpha$ is defined below). Further, in addition to $(H_1), (H_2)$ and (3.23), assume that
\[ \liminf_{n \to \infty} \sum_{i=n-\ell}^{n-1} q_i > \gamma^* \left( \frac{\ell}{\ell+1} \right)^{\ell+1}, \tag{3.24} \]
then
(i) equation (3.9: -1) is oscillatory provided $0 \leq p < 1, \gamma^* = 1/(M(1-p))$ and $\alpha = 0$;
(ii) equation (3.9: 1) is oscillatory provided $p > 1, \gamma^* = p^2/(M(p-1))$ and $\alpha = k$.

PROOF. Suppose, for the sake of a contradiction, that (3.9: $\delta$) has a nonoscillatory solution $\{x_n\}$ which is eventually positive. There exists a positive integer $n_0$ such that (3.11) is satisfied for $n \geq n_0$. With $y_n$ and $z_n$ as in (3.12), we obtain (3.13) and (3.14). From (3.23) and (3.13), it follows that
\[ \Delta z_n = q_n f(x_{n-\alpha}) \geq M q_n x_{n-\alpha}, \quad n \geq n_0. \tag{3.25} \]

Now we consider two cases:
(1) $\delta = -1$ and $0 \leq p < 1$;
(2) $\delta = 1$ and $p > 1$. 

and proceed as in the proof of Theorem 3.6 to obtain (3.18) and (3.20), respectively. Next, we use (3.18) and (3.20) in (3.25) to get
\[ \Delta \omega_n \geq q_n \omega_{n+\ell-\alpha}, \quad \text{for } n \geq N^* \text{ for some sufficiently large } N^*, \quad (3.26) \]
where
\[ \beta = \begin{cases} M(1-p), & \alpha = 0, \\
M \frac{p-1}{p^2}, & \alpha = k, \end{cases} \quad \text{if } \omega_n = \xi_n; \]
\[ \beta = M \frac{p-1}{p^2}, \quad \text{if } \omega_n = \eta_n. \]
In view of theorem 7.6.1 in [3], inequality (3.26) has no eventually positive solution which is a contradiction. This completes the proof of the Theorem.

**REMARK 3.2.** Theorems 3.6 and 3.7 with \( F_n \equiv 0 \) are the discrete analogues of some special cases of results in [17]. The details are omitted.

For the purpose of illustration we consider the following example.

**EXAMPLE 3.1.** Consider the forced neutral difference equation
\[ \Delta (x_n + x_{n-2}) - (2n - 3)|x_{n+4}|^\lambda \sgn x_{n+4} = (-1)^{n+1}(2n + 1), \quad (3.27) \]
where \( n \in N \) and \( \lambda \geq 1 \). Here
\[ F_n = (-1)^{n+1}(2n + 1), \quad \text{and} \quad h_n = (-1)^n n. \]
All the hypotheses of Theorem 3.4 are satisfied, and hence, every bounded solution of (3.27) is oscillatory. When \( \lambda = 1 \) one such solution is \( x_n = (-1)^n \).

**EXAMPLE 3.2.** Consider the equation
\[ \Delta (x_n + p x_{n+2\delta}) - 3 \left( \frac{2n + 3}{n + 7} \right) |x_{r_n}|^\lambda \sgn x_{r_n} = (-1)^{n+1}, \quad \lambda \geq 1, \quad (3.28: \delta) \]
where \( n \in N, \delta = \pm 1, p \) is a nonnegative real number. Here \( \{r_n\} \) is a sequence of positive integers with \( \lim_{n \to \infty} r_n = \infty \). We take \( h_n = (-1)^n \) with period 2. First, let \( \lambda > 1 \) and suppose that \( r_n \) is of the form: \( n + a \), \( a \) is a positive integer; \( n + (-1)^n \) or \( n^a, a > 1, a \) is a real number. We apply Theorem 3.6 to conclude that

(i) equation (3.28: -1) is oscillatory when \( 0 \leq p < 1 \);

(ii) equation (3.28: 1) is oscillatory when \( p > 1 \).

Now, suppose that \( \lambda = 1 \) and that \( r_n - \alpha \geq n + \ell \), where \( \ell \geq 2 \), \( \alpha = 0 \) when \( \delta = -1 \) and \( \alpha = 2 \) when \( \delta = 1 \). We use Theorem 3.7 to conclude that

(i) equation (3.28: -1) is oscillatory if \( 0 \leq p < 1, \alpha = 0 \) and
\[ \liminf_{n \to \infty} \sum_{i=n-\ell+1}^{n-1} \frac{3}{i+7} \left( \frac{2i + 3}{i + 7} \right) > \frac{1}{1-p} \left( \frac{\ell}{1+\ell} \right)^{\ell+1}; \]

(ii) equation (3.28: 1) is oscillatory if \( p > 1, \alpha = 2 \) and
\[ \liminf_{n \to \infty} \sum_{i=n-\ell+1}^{n-1} \frac{3}{i+7} \left( \frac{2i + 3}{i + 7} \right) > \frac{p^2}{p - 1} \left( \frac{\ell}{1+\ell} \right)^{\ell+1}. \]

It is easy to verify that for \( p = 2, \delta = 1, \lambda = 1 \) and \( r_n = n + 7 \), (3.28: 1) has an oscillatory solution \( x_n = (-1)^n n \).
REFERENCES

18. J. Yan and C. Qian, Oscillation and comparison results for delay difference equations.