Abstract

Characterizations are obtained of those linear operators on the $m \times n$ matrices over an arbitrary semiring that preserve term rank. We also present characterizations of permanent and rook-polynomial preserving operators on matrices over certain types of semirings. Our results apply to many combinatorially interesting algebraic systems, including nonnegative integer matrices, matrices over Boolean algebras, and fuzzy matrices.

1. Introduction and Summary

Suppose $\mathcal{K}$ is a field and $\mathcal{M}$ is the set of all $m \times n$ matrices over $\mathcal{K}$. If $T$ is a linear operator on $\mathcal{M}$ and $f$ is a function defined on $\mathcal{M}$, then $T$ preserves $f$ if $f(T(A)) = f(A)$ for all $A$ in $\mathcal{M}$.

Frobenius (1897 [4]), Marcus and Moyls (1959 [8, 9]), and Marcus and May (1962, [7]) characterized those linear operators on $\mathcal{M}$ that preserve the determinant and characteristic polynomial, the rank, and the permanent, respectively. In 1983, McDonald [10] found that the characterizations obtained for the first three functions were valid for more general rings.

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Typically, the first operations that come to mind for preserving $f$ turned out to be the only ones. For example, $T$ preserves the characteristic polynomial iff $T$ is a similarity transformation, transposition, or composition of such operations [4].

In 1984 and 1985 analogues of Marcus and Moyls's work on rank were obtained by Beasley, Gregory, and Pullman [1–3] for certain types of semirings. These included such combinatorially significant systems as the nonnegative reals, rationals, and integers and the Boolean algebra of two elements. Their results also apply to fuzzy matrices.

In this paper that work is continued. We obtain characterizations of those linear operators on $m \times n$ matrices over arbitrary semirings that preserve term rank (Theorem 1 and Corollary 1.1). We also characterize the linear operators preserving the permanent (Theorem 3 and Corollary 3.1) and those that preserve the rook polynomial (Theorem 4 and Corollary 4.1). However, for these last two functions we assume the matrices have their entries in an "antinegative" commutative semiring without zero divisors (these include the combinatorially significant systems mentioned above).

Section 2 contains most of the definitions and notation. In particular, "semiring" is defined formally at its outset. In brief, a semiring is a ring with unity except that its nonzero elements are not required to have negatives. Matrix operations are defined over semirings as over rings, so are concepts such as invertibility and linearity. Rings, fields, and division rings are all semirings; so are Boolean algebras (with union for addition and intersection for multiplication). The nonnegative members $\mathbb{P}^+$ of any subring $\mathbb{P}$ of the reals $\mathbb{R}$ form a semiring. The real unit interval $\mathbb{F} = [0, 1]$ is a semiring ($\max = +$ and $\min = \times$); it provides the scalars for the fuzzy matrices.

The term rank of a matrix $X$, $t(X)$, is the least number of lines (rows or columns) needed to include all the nonzero entries of $X$.

In Theorem 1 and Corollary 1.1 in Section 3 we show that the following statements about a linear operator $T$ on the $m \times n$ matrices over an arbitrary semiring are equivalent:

(i) $T$ preserves term rank, that is, $t(T(X)) = t(X)$ (for all $X$);
(ii) $t(X) = t(T(X))$ if $t(X) \leq 2$ (for all $X$);
(iii) $t(T(X)) = 1$ if and only if $t(X) = 1$ (for all $X$);
(iv) $T(X)$ is a composition of one or more of the following operations on $X$:
   (a) permute the rows of $X$,
   (b) permute the columns of $X$,
   (c) replace $X$ by $[b_{i,j}r_{i,j}]$ where no $b_{i,j}$ is a zero divisor or zero, and/or
   (d) if $m = n$, transpose $X$. 


Antinegative semirings are those in which only zero has a negative. So no nontrivial ring is antinegative, but $\mathbb{R}^+$, $\mathbb{Z}^+$ (the nonnegative integers), Boolean algebras, and $\mathbb{F}$ are antinegative. In Theorem 3 of Section 5 we show that when $T$ is a linear operator on the $m \times n$ matrices over an antinegative commutative semiring without zero divisors, then:

(a) The following statements are equivalent:

(i) $T$ preserves the permanent, i.e., $\text{per}(X) = \text{per}(T(X))$ for all $X$;

(ii) $T$ preserves term rank, and the permanent of every "diagonal" (see Section 5) of $T(J)$ is 1, where $J$ is the $m \times n$ matrix of 1's.

(b) Also the following are equivalent:

(i) $T$ preserves the rook polynomial;

(ii) $T$ is a composition of row and column permutations and (if $m = n$) transposition;

(iii) $T$ preserves term rank and $T(J) = J$.

Note that Theorem 1 above allows two variations on (a)(ii) and (b)(iii).

The following theorem (see Corollaries 3.1 and 4.1 in Section 5) is valid for nonnegative integer matrices, Boolean matrices, and fuzzy matrices among others (it does not apply to the nonnegative real or rational matrices): When $T$ is a linear operator on the $m \times n$ matrices over an antinegative commutative semiring without zero divisors that has only one unit (multiplicatively invertible element), then the following are equivalent:

(i) $T$ preserves the permanent;

(ii) $T$ preserves the rook polynomial;

(iii) $T$ preserves term rank;

(iv) $T(X)$ is the composition of a row permutation of $X$, a column permutation of $X$, and/or (if $m = n$) transposition of $X$.

2. DEFINITIONS AND NOTATION

A semiring (see e.g., Gregory and Pullman [5] or Kim [6]) is a binary system $(\mathcal{S}, +, \times)$ such that $(\mathcal{S}, +)$ is an abelian monoid (identity 0), $(\mathcal{S}, \times)$ is a monoid (identity 1), $\times$ distributes over $+$, $0 \times s = s \times 0 = 0$ for all $s$ in $S$, and $1 \neq 0$. Usually $\mathcal{S}$ denotes the system and $\times$ is denoted by juxtaposition. (Some authors do not require a semiring to possess a multiplicative identity.)

Algebraic terms such as unit and zero divisor are defined for semirings as they are for rings. Algebraic operations on matrices over a semiring and such
notions as *linearity* and *invertibility* are also defined as if the underlying scalars were in a field.

Here are some examples of semirings which occur in combinatorics. Let $\mathcal{B}$ be any Boolean algebra; then $(\mathcal{B}, \cup, \cap)$ is a semiring. Let $\mathcal{C}$ be any chain with lower bound 0 and upper bound 1; then $(\mathcal{C}, \max, \min)$ is a semiring (a *chain* semiring). In particular, if $\mathcal{F}$ is the real unit interval $[0, 1]$, then $\mathcal{F}$ is a semiring with $\max$ for $+$ and $\min$ for $\times$. If $\mathcal{P}$ is a subring of the reals $\mathbb{R}$ (under real addition and multiplication) and $\mathcal{P}^+$ denotes the nonnegative members of $\mathcal{P}$, then $\mathcal{P}^+$ is a semiring. In particular $\mathbb{Z}^+$, the nonnegative integers, is a semiring.

We let $M_{m,n}(\mathcal{S})$ denote the $m \times n$ matrices over $\mathcal{S}$. The $m \times n$ matrix of 1's is denoted $I_{m,n}$; the $m \times m$ identity matrix is defined as if $\mathcal{S}$ were a field and is denoted $I_m$; the $m \times n$ zero matrix is defined as if $\mathcal{S}$ were a field and is denoted $0_{m,n}$; the $m \times n$ matrix all of whose entries are zero except its $ij$th, which is 1, is denoted $E_{ij}$.

**Conventions.** From now on we will assume that $2 \leq m \leq n$ unless specified otherwise. For the rest of this paper the subscripts $m$ and $n$ may be suppressed and $\mathcal{M}$ will denote $M_{m,n}(\mathcal{S})$, a fixed semiring $\mathcal{S}$ being understood.

The pattern $\bar{A}$ of a matrix $A$ in $\mathcal{M}$ is the $(0,1)$ matrix whose $ij$th entry is 0 if and only if $a_{ij} = 0$. We will also assume that $\bar{A}$ is in $\mathcal{M}(\mathbb{B})$, where $\mathbb{B}$ denotes the Boolean algebra of two elements $\{(0,1), +, \times\}$, where $+$ is $\cup$ and $\times$ is $\cap$.

If $A$ and $B$ are in $\mathcal{M}$, we say $B$ dominates $A$ (written $B \succeq A$ or $A \preceq B$) if $b_{ij} = 0$ implies $a_{ij} = 0$ for all $i, j$. This provides a reflexive, transitive ordering on $\mathcal{M}$. Notice that $A \preceq B$ and $B \preceq A$ implies $A = B$. In fact $\mathcal{M}(\mathbb{B})$ is partially ordered by $\preceq$, its lower bound is 0, and its upper bound is $J$. We'll write $A \equiv B$ if $\bar{A} = \bar{B}$, and $A < B$ if $A \not\equiv B$ but $A \preceq B$.

Note that $A \preceq B$ iff $\bar{A} \preceq \bar{B}$, and that $\bar{A} + \bar{B} \preceq \bar{A} + \bar{B}$ for all $A$ and $B$.

A matrix $M$ in $\mathcal{S}$ is a *monomial* if it has exactly $m$ nonzero entries—one in each row, one in each nonzero column. The pattern of such a matrix is a column permutation of $[I_m | 0_{m,n-m}]$. In particular, $\bar{M}$ is a permutation matrix if $m = n$.

If $L \preceq M$ and $M$ is a monomial, we call $L$ a *submonomial* matrix.

The *term rank* $t(A)$ of a matrix $A$ is the minimum number $k$ such that all the nonzero entries in $A$ are contained in $r$ rows and $k - r$ columns. Evidently the term rank of a matrix is the term rank of its pattern.

**Lemma 1.** *If $X$ and $Y$ are any* $m \times n$ *matrices over* $\mathcal{S}$, *then* $t(X + Y) \leq t(X) + t(Y)$.
Lemma 2. For every $m \times n$ matrix $A$ over $S$, there exists a submonomial matrix $M$ such that $t(A) = t(M)$ and $M \preceq A$.

Proof. See e.g., Ryser [11, p. 55].

Lemma 2, which we'll use frequently in the sequel, is equivalent to the well-known graph-theoretic theorem of König that the vertex covering number of a bipartite graph is the size of a maximum matching.

3. TERM-RANK PRESERVERS

Let $S$ be any semiring and $\mathcal{M} = \mathcal{M}_{m,n}(S)$. Which linear operators over $\mathcal{M}$ preserve term rank? The operations of (1) permuting rows, (2) permuting columns, and (3) (if $m = n$) transposing the matrices in $\mathcal{M}$ are all linear, term-rank preserving operators on $\mathcal{M}$. If we take a fixed $m \times n$ matrix $B$ in $\mathcal{M}$, none of whose entries is a zero divisor in $S$, then its Schur product $B \circ X = [b_{ij} x_{ij}]$ with $X$ has the same term rank as does $X$. The operator $X \to B \circ X$ is linear. Similarly $X \to X \circ B$ is a linear term-rank preserving operator. That these operations and their compositions are the only term-rank preservers is one of the consequences of Theorem 1 below. Such operators are described more formally in the following definition.

If $P$ and $Q$ are $m \times m$ and $n \times n$ permutation matrices and $B$ is an $m \times n$ matrix over $S$ none of whose entries is a zero divisor or zero, then $T$ is a $(P, Q, B)$ operator if

1. $T(X) = P (B \circ X) Q$ for all $X$ in $\mathcal{M}$ or
2. $m = n$ and $T(X) = P (B \circ X^t) Q$ for all $X$ in $\mathcal{M}$.

If $t(T(X)) = k$ whenever $t(X) = k$, we say $T$ preserves term rank $k$. So an operator preserves term rank if it preserves term rank $k$ for every $k \leq m$.

Theorem 1. If $S$ is any semiring, then the following are equivalent for any linear operator $T$ on $\mathcal{M} = \mathcal{M}_{m,n}(S)$:

1. $T$ is a $(P, Q, B)$ operator;
2. $T$ preserves term rank;
3. $T$ preserves term rank 1 and term rank 2.

Proof. That (i) implies (ii) and (ii) implies (iii) is obvious. We now show that (iii) implies (i). Let $\mathcal{E} = \{ E_{ij} : 1 \leq i \leq m, 1 \leq j \leq n \}$. 

We will first show that \( T(E_{i,j}) = c_{i,j}E_{k,l} \) for constants \( c_{i,j} \) and the corresponding mapping on indices \((i,j) \rightarrow (k,l)\) is a bijection.

Suppose \( T(E_{i,j}) \) is not a scalar multiple of an element of \( \mathcal{E} \). Since \( t(E_{i,j}) = 1 \), and \( T \) preserves term rank 1, we have \( t(T(E_{i,j})) = 1 \). Thus \( T(E_{i,j}) \) has either a single nonzero row or a single nonzero column. Without loss of generality, let \( T(E_{i,j}) = a_1E_{k,1} + a_2E_{k,2} + \cdots + a_nE_{k,n} \) for some \( k \), where at least two of the \( a_i \)'s are nonzero. (Note: if the nonzero line is a column, the same argument holds as the following.) Fix \( x \neq j \). Now, since \( t(E_{i,j} + E_{i,x}) = 1 \), it follows that \( T(E_{i,x}) = b_1E_{k,1} + b_2E_{k,2} + \cdots + b_nE_{k,n} \) for some \( b_i \)'s not all zero. Likewise for some fixed \( y \neq i \), \( T(E_{i,j}) = c_1E_{i,1} + c_2E_{i,2} + \cdots + c_nE_{i,n} \) for some \( c_i \)'s not all zero. Now \( t(E_{i,x} + E_{u,j}) = 2 \) while \( T(E_{i,x} + E_{u,j}) = (b_1 + c_1)E_{i,1} + (b_2 + c_2)E_{i,2} + \cdots + (b_n + c_n)E_{i,n} \), contradicting the assumption that \( T \) preserves term rank 2.

We have now shown that \( T(E_{i,j}) = b_{i,j}E_{k,l} \) for constants \( b_{i,j} \), none of which can be zero divisors. Let \( B = [b_{i,j}] \).

If \( T(E_{i,j}) = cT(E_{i,k}) \) for some pairs \((i,j) \neq (k,l)\), choose a pair \((p,q)\) such that \( t(E_{i,j} + E_{p,q}) = 2 \) and either \( p = k \) or \( q = l \), and hence \( t(E_{p,q}) = 1 \). Here, since \( T \) preserves term ranks 1 and 2, \( 2 = t(E_{i,j} + E_{p,q}) = t(T(E_{i,j} + E_{p,q})) = t(T(E_{i,j}) + T(E_{p,q})) = t(T(cE_{i,j} + E_{p,q})) = t(cE_{i,j} + E_{p,q}) = 1 \), a contradiction. Thus the corresponding mapping \( f \) on indices \((i,j) \rightarrow (k,l)\) is a bijection.

Let \( T'(E_{i,j}) = E_{p(i,j)} \) for all \( E_{i,j} \) in \( \mathcal{E} \). Let \( R_i = \{ E_{i,j} : 1 \leq j \leq n \} \), \( C_j = \{ E_{i,j} : 1 \leq i \leq m \} \), \( \mathcal{R} = \{ R_i : 1 \leq i \leq m \} \), \( \mathcal{C} = \{ C_j : 1 \leq j \leq n \} \), and \( \mathcal{L} = \mathcal{R} \cup \mathcal{C} \). Define \( T^*(R_i) = \{ T'(E_{i,j}) : 1 \leq j \leq n \} \) for all \( i \) and \( T^*(C_j) = \{ T'(E_{i,j}) : 1 \leq j \leq n \} \) for all \( j \). Note that \( T^* \) maps \( \mathcal{L} \) onto \( \mathcal{L} \) injectively because \( T \) preserves term ranks 1 and 2 and \( T' \) is bijective. Now suppose \( T^*(R_i) = C_j \) and \( T^*(C_j) = R_i \) for some \( i \neq k \). Then \( T(R_i \cup R_k) \subseteq C_j \cup R_i \) but \( |R_i \cup R_k| = m + n - 1 < 2n - 1 \). This contradicts the fact that \( T' \) is injective on \( \mathcal{E} \).

Therefore there are two cases: (a) \( T^*(\mathcal{R}) = \mathcal{R} \) and \( T^*(\mathcal{C}) = \mathcal{C} \) or (b) \( T^*(\mathcal{R}) = \mathcal{C} \) and \( T^*(\mathcal{C}) = \mathcal{R} \).

Case (a). Let \( \alpha(i) = j \) iff \( T^*(R_i) = R_j \). Then \( \alpha \) permutes \( \{1,2,\ldots,m\} \). Let \( \beta(i) = j \) iff \( T^*(C_j) = C_i \). Then \( \beta \) permutes \( \{1,2,\ldots,n\} \). Let \( P \) be the \( m \times m \) permutation matrix corresponding to \( \alpha \), and \( Q \) be the \( n \times n \) permutation matrix corresponding to \( \beta \). Then for all \( X \), \( T(X) = P(B \circ X)Q \).

Case (b). In this case \( m = n \). Let \( \alpha(i) = j \) if \( T^*(R_i) = C_j \) and \( \beta(i) = j \) if \( T^*(C_j) = R_i \). Then \( \alpha \) and \( \beta \) permute \( \{1,2,\ldots,m\} \). By choosing the permutation matrices \( P \) and \( Q \) corresponding to \( \alpha \) and \( \beta \) appropriately, we have \( T(X) = P(B \circ X^t)Q \) for all \( X \).

We say that an operator \( T \) strongly preserves term rank \( k \) if \( t(T(A)) = k \) if and only if \( t(A) = k \).
Consider the operator on $\mathcal{M}(\mathbb{Z}^+)$ defined by

$$T \begin{bmatrix} x & y \\ z & w \end{bmatrix} = (x + y + z + w) \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}.$$  

This linear operator preserves term rank 1 but does not strongly preserve term rank 1.

**Corollary 1.1.** If $\mathcal{S}$ is any semiring, then a linear operator $T$ on $\mathcal{M}_{m,n}(\mathcal{S})$ preserves term rank if and only if it strongly preserves term rank 1.

**Proof.** Suppose $T$ strongly preserves term rank 1. Let $A$ be any matrix of term rank 2. Then $A = B + C$, where $t(B) = t(C) = 1$. Then $t(T(B)) = t(T(C)) = 1$ by hypothesis. Hence $t(T(A)) \leq 2$ by Lemma 1. But $t(T(A)) \neq 1$ by hypothesis. We can choose $D$ such that $t(D) = 1$ and $t(A + D) = 2$. To see why this is so, note that $a_{i,j} \neq 0$ and $a_{k,l} \neq 0$ for some $i \neq k$ and $j \neq l$; let $D = E_{i,j}$. If $T(A) = 0$, then $t(T(A + D)) = t(T(D))$. But $t(T(D)) = 1$, so $t(A + D) = 1$ by hypothesis, contradicting the fact that $t(A + D) = 2$. Therefore $t(T(A)) = 2$. Then $T$ preserves term rank by Theorem 1.

The converse is immediate. 

Notice that term-rank preserving linear operators need not be invertible. Also the operator

$$T: \begin{bmatrix} a & b \\ c & d \end{bmatrix} \to \begin{bmatrix} a & b \\ d & c \end{bmatrix}$$

is an example of an invertible operator which is not a term-rank preserver. However, if $\mathcal{S}$ is a division ring (or the two-element Boolean algebra), then the $B$-matrix of Theorem 1 has a Schur inverse $B^*$, that is, a matrix $B^*$ such that $B \circ B^* = B^* \circ B = J$. It is either $[b_{i,j}^{-1}]$ or $B$, respectively, and hence the $(P, Q, B)$ operator is invertible. Its inverse is a $(P^t, Q^t, B^*)$ operator.

**Corollary 1.2.** If $\mathcal{S}$ is a division ring or the two-element Boolean algebra, then every term-rank preserving linear operator on $\mathcal{M}(\mathcal{S})$ is invertible.

4. **Antinegative Semirings: Preliminaries**

A semiring $\mathcal{S}$ is antinegative if no nonzero element has a negative in $\mathcal{S}$; see e.g. [5]. That is, if $x + y = 0$, then $x = y = 0$. The examples $\mathcal{B}$, $\mathcal{C}$, $\mathcal{F}$, and
\[ P^+ \text{ of Section 2 are all antinegative semirings. Fields and rings, though they are always semirings, are never antinegative. We noted in Section 2 that if } \mathcal{S} \text{ is an arbitrary semiring, then} \]

\[ A + B \leq \overline{A + B} \quad \text{for all } A, B \text{ in } \mathcal{M}(\mathcal{S}). \quad (\ast) \]

In the proof of the next lemma we use the fact that equality holds in \((\ast)\) if \(\mathcal{S}\) is antinegative.

**Lemma 3.** Suppose \(\mathcal{S}\) is antinegative and \(T\) is a linear operator on \(\mathcal{M}(\mathcal{S})\). For any \(A, B,\) and \(D\) in \(\mathcal{M}(\mathcal{S})\),

\[ \text{if } A + D = B \text{ then } T(A) \leq T(B). \]

**Proof.** \(T(A) \leq T(A) + T(D) = T(B)\) by linearity and antinegativity. \(\blacksquare\)

The following example shows the necessity of the hypothesis that \(\mathcal{S}\) is antinegative. Suppose \(\mathcal{S}\) is the ring of integers,

\[ T\begin{bmatrix} u & v \\ w & x \end{bmatrix} = \begin{bmatrix} u - v & v \\ w - x & x \end{bmatrix}, \]

\[ A = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} \text{ and } B = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}. \]

Then

\[ T(A) = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} \text{ and } T(B) = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}, \]

so \(T(A) \neq T(B)\), although

\[ A + \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} = B. \]

Let \(|X|\) denote the number of nonzero entries in the \(m \times n\) matrix \(X\). Note that \(t(D) = |D|\) for every submonomial matrix \(D\).

**Lemma 4.** Suppose \(\mathcal{S}\) is antinegative and \(D\) is a submonomial matrix, \(A, D\) are in \(\mathcal{M}\), and \(T\) is any linear operator on \(\mathcal{M}\). If \(t(D) < t(A)\) and \(D \leq T(A)\), then for some \(B \leq A, D \leq T(B)\) and \(|B| \leq t(D)\).

**Proof.** Let \(\Omega = \{(i, j): d_{i, j} \neq 0\}\). For each \((i, j) \in \Omega\) there exists an ordered pair \((i', j')\) such that \(E_{i', j'} \leq A\) and \(E_{i, j} \leq T(E_{i', j'})\), because \(D \leq
Let $B = \sum \{ E_{i,j} : (i,j) \in \Omega \}$; then $B \leq A$ and $|B| \leq |\Omega| = t(D)$. But $D = \sum \{ E_{i,j} : (i,j) \in \Omega \} \leq \sum \{ T(E_{i,j}) : (i,j) \in \Omega \} = T(B)$ by antinegativity.

**Lemma 5.** If $T$ is linear operator on $\mathbb{M}(\mathcal{F})$, $\mathcal{F}$ is antinegative, and $T$ strongly preserves term rank $m$, then

$$t(T(X)) \geq t(X) \quad \text{for all } X \in \mathbb{M}.$$  

**Proof.** Suppose $A \in \mathbb{M}$, $k = t(A)$, $r = t(T(A))$, and $r < k$. By Lemma 2 we may choose a submonomial matrix $M \leq A$ having term rank $k$. We may assume $m_{ij} = a_{ij}$ whenever $m_{ij} \neq 0$. Then $T(M) \leq T(A)$ by Lemma 3. Let $r' = t(T(M))$. Then $r' \leq r$ by the definition of $t$. Choose another submonomial matrix $N$ having term rank $m - k$ such that $M + N$ is a monomial. Then $t(M + N) = m$ implies that $t(T(M) + T(N)) = m$ by hypothesis. So $t(T(N)) \geq m - r' \leq m - r$. By Lemma 1. Therefore $t(T(N)) \geq m - r > m - k$. By Lemma 2, $T(M + N) \geq D$ for some monomial $D$. Then $D = D_1 + D_2$, where $D_1$ and $D_2$ are submonomial matrices, $D_1 \leq T(M)$, $D_2 \leq T(N)$. So $t(D_1) \leq r' \leq m \leq t(D_1)$). Here $t(N + B) \leq (m - k) + r' \leq m - k + r < m$. But $T(N) \geq D_2$ and $T(B) \geq D_1$. Therefore $t(T(N + B) \geq D$, and hence $t(T(N + B) = m$, contrary to our hypothesis, as $t(N + B) < m$.

Notice that in this proof we used the fact that $X \geq U$ and $Y \geq V$ imply $X + Y \geq U + V$. This is easy to verify, since $\mathcal{F}$ is antinegative. However, if $\mathcal{F}$ isn’t antinegative, then this need not occur. For example, if

$$X = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \quad U = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix},$$

$$Y = \begin{bmatrix} -1 & -1 \\ 0 & -1 \end{bmatrix}, \quad V = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix},$$

then $X \geq U, Y \geq V$, but

$$X + Y = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \not\geq \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = U + V.$$  

**Corollary.** If $T$ is a linear operator on $\mathbb{M}(\mathcal{F})$, $\mathcal{F}$ is antinegative, and $T$ strongly preserves term rank $m$, then $T$ preserves term rank $m - 1$.

**Lemma 6.** Suppose $T$ is a linear operator on $\mathbb{M} = \mathbb{M}_{m,n}(\mathcal{F})$, $\mathcal{F}$ is antinegative, and $T$ strongly preserves term rank $m$. Suppose $M_1, M_2, \ldots, M_p$
are monomials in $\mathcal{M}$ none of whose entries are zero divisors, $1 \leq p \leq m$, and $M_i \circ M_j = 0$ for all $i \neq j$. If for each $1 \leq i \leq p$, $D_i$ is a monomial dominated by $T(M_i)$, then

$$D_i \circ D_j = 0 \quad \text{for all } i \neq j.$$ 

Proof. Let $\mathcal{E} = \{ E_{i,j} : 1 \leq i, j \leq m \}$. Suppose for some $E$ in $\mathcal{E}$ we have $E \leq D_i$ and $E \leq D_j$ while $i \neq j$. Now for some $F, F'$ in $\mathcal{E}$ we have $F \leq M_i$, $F' \leq M_j$, and $E \leq T(F)$, while $E \leq T(F')$. Let $A \setminus B$ denote the matrix obtained from $A$ by replacing $a_{uv}$ by 0 for all $(u, v)$ such that $b_{uv} \neq 0$. Then put $X = (M_i \setminus F) + F'$. We have $t(X) = m - 1$, because $M_i, M_j$ are $m \times m$ monomial matrices and $M_i \circ M_j = 0$. On the other hand, as we shall prove shortly, $E$ is the only member of $\mathcal{E}$ dominated by $D_i$ that $T(F)$ dominates. By antinegativity, $T(M_i \setminus F) \geq T(M_i) \setminus T(F) \geq D_i \setminus T(F) = D_i \setminus E$; therefore $T(X) \geq D_i$. But $t(T(X)) = m - 1$ by the Corollary to Lemma 5, a contradiction. Therefore $D_i \circ D_j = 0$.

It remains to show that $D_i \circ T(F) \equiv F$. Let $D$ be the submonomial $D_i \setminus T(F)$. Suppose $t(D) \leq m - 2$. Let $A = M_i \setminus F$; then $t(D) < t(A)$ and $T(A) \geq D$. Therefore there is a $B \leq A$ such that $|B| \leq t(D)$ and $T(B) \geq D$ by Lemma 4. Then $B$ is a submonomial and $t(B) < m - 2$ therefore $t(B + F) < m - 1$. But $T(B + F) = T(B) + T(F) \geq D + T(F) \geq D_i$, a monomial, contrary to Lemma 5, which implies $t(T(B + F)) < m - 1$. Therefore $t(D) = m - 1$, that is, $D_i \setminus T(F) \equiv E$. 

Corollary. Suppose $T$ is a linear operator on $\mathcal{M} = \mathcal{M}_{m,n}(\mathcal{S})$, $\mathcal{S}$ is antinegative, and $T$ strongly preserves term rank $m$. Suppose for each $i = 1, 2, \ldots, p$, $M_i$ is a monomial none of whose entries is a zero divisor and whose last $n - m$ columns are 0. If $M_i \circ M_j = 0$ for all $i \neq j$, then every row of $T(\Sigma_{i=1}^p M_i)$ has at least $p$ nonzero entries.

Proof. We could use the same proof as that of Lemma 6 to establish that $D_i \circ D_j = 0$ for all $i \neq j$. The critical point while so doing is to recognize that our assumption about the columns of the $M_i$ allows us to assert that $t(X) = m - 1$. Then $\Sigma_{i=1}^p T(M_i) \geq \Sigma_{i=1}^p D_i$, and the latter sum has at least $p$ nonzero entries per row because $D_i \circ D_j = 0$ for $i \neq j$ and because each $D_i$ contributes 1 to the $k$th row of the sum of the $D_i$'s. 

Theorem 2. If $\mathcal{S}$ is an antinegative semiring, then $T$ strongly preserves term rank $m$ if and only if $T$ preserves term rank.

Proof. Suppose $T$ preserves term rank $m$ strongly. By Lemma 5, $t(T(X)) \geq t(X)$ for all $X$ in $\mathcal{M}$. 


Suppose \( t(T(A)) > t(A) \) for some \( A \). Say \( t(T(A)) = l \). We may suppose that \( l \) is maximal. By permuting rows and columns if necessary we can assume

\[
T(A) \geq \begin{bmatrix} I_l & 0 \\ 0 & 0 \end{bmatrix}.
\]

If there were some \( X \) in \( \mathcal{M} \) with \( (T(X))_{r,s} \neq 0 \) for some \( r, s \) with \( \min(r, s) > l \), then for some \( (i, j) \), \( E_{r,s} \leq T(E_{i,j}) \). Therefore \( t(T(A) + T(E_{i,j})) \geq l + 1 \). So \( t(A + E_{i,j}) \leq t(A) + 1 \) while \( t(T(A + E_{i,j})) \geq l + 1 > t(A) + 1 \), contrary to our choice of \( l \). Therefore

\[
T(X) = \begin{bmatrix} X_1 & X_2 \\ X_3 & 0 \end{bmatrix}, \quad \text{where } X_1 \text{ is } l \times l \text{ and } l \leq m - 1. \quad (\star)
\]

Let \( Q \) be the \( m \times m \) monomial matrix that has \( q_{i,i+1} = 1 \) for \( i = 1, 2, \ldots, m - 1 \) and \( q_{m,1} = 1 \), and \( X = \sum_{j=0}^{m-1} [Q^j]_{0,m-n} \). Then every row of \( T(X) \) has at least \( m \) nonzero entries by the Corollary to Lemma 6. This contradicts (\( \star \)) and establishes the sufficiency of strong term-rank-\( m \) preservation. The converse follows immediately from the definition.

5. ANTONEGATIVE SEMIRINGS: PERMANENT PRESERVERS AND ROOK-POLYNOMIAL PRESERVERS.

In 1962 Marcus and May [7] proved the following theorem.

**Theorem.** If \( \mathcal{S} \) is a field, \( T \) is a linear operator on \( \mathcal{M} = \mathcal{M}_{m,n}(\mathcal{S}) \), and \( m \geq 3 \), then \( T \) preserves the permanent if and only if there exist monomials \( U \) and \( V \) such that \( \text{per}(UV) = 1 \) and either

(a) \( T(X) = UXV \) for all \( X \) in \( \mathcal{M} \) or
(b) \( T(X) = UXV' \) for all \( X \) in \( \mathcal{M} \).

In this section we will obtain some analogues of their theorem valid for antinegative commutative semirings having no zero divisors.

As we want to continue to discuss \( m \times n \) matrices (with \( m \leq n \)) we'll take this as our definition of the **permanent** of an \( m \times n \) matrix \( X \) over any commutative semiring \( \mathcal{S} \);

\[
\text{per}(X) = \sum_{\delta \in \Delta} \prod_{i=1}^{m} x_{i\delta(i)},
\]

where \( \Delta \) is the set of all injections of \( \{1, 2, \ldots, m\} \) into \( \{1, 2, \ldots, n\} \).
For each monomial $D$ there is a unique $\delta$ in $\Delta$ such that for all $(i, j)$, $d_{i,j} \neq 0$ if and only if $j = \delta(i)$. If $B$ is any matrix, a monomial matrix $D$ corresponding to $\delta$ is a diagonal of $B$, written $D = B_\delta$, if $d_{i,\delta(i)} = b_{i,\delta(i)}$ for all $1 \leq i \leq m$.

**Lemma 7.** Suppose $\mathcal{S}$ is any commutative semiring. If $T$ is a $(P, Q, B)$ operator on $\mathcal{M}(\mathcal{S})$ and $\text{per}(D) = 1$ for all diagonals $D$ of $B$, then $T$ preserves the permanent.

**Proof.** Here, $\text{per}(T(X)) = \text{per}(B \circ X)$ or $\text{per}(B \circ X')$. Now $\text{per}(B \circ X) = \sum_{\delta \in \Delta} \prod_{i=1}^{m} b_{i,\delta(i)} x_{i,\delta(i)} - \sum_{\delta \in \Delta} \prod_{i=1}^{m} x_{i,\delta(i)} = \text{per}(X)$.

**Theorem 3.** If $\mathcal{S}$ is antinegative and commutative and has no zero divisors, and $T$ is any linear operator on $\mathcal{M}(\mathcal{S})$, then $T$ preserves the permanent if and only if $T$ is a $(P, Q, B)$ operator and $\text{per}(D) = 1$ for every diagonal $D$ of $B$.

**Proof.** The permanent of $X$ is nonzero if and only if $t(X) = m$ by Lemma 2, because $\mathcal{S}$ is antinegative and has no zero divisors. So if $T$ preserves the permanent, then $T$ strongly preserves term rank $m$. For if $t(X) = m$, then $\text{per}(X) \neq 0$ and hence $\text{per}(T(X)) \neq 0$, so $t(T(X)) = m$. Thus $T$ preserves rank $m$. If $t(X) < m$, then $\text{per}(X) = 0$ and hence $\text{per}(T(X)) = 0$, so $t(T(X)) < m$. Therefore $T$ strongly preserves term rank $m$. By Theorems 1 and 2, $T$ is a $(P, Q, B)$ operator. Let $D = J_\delta$ be any diagonal of $J$, the $m \times n$ matrix of $1$'s. Then $\text{per}(T(D)) = 1$, so $\text{per}(B \circ D) = 1$. But $B_\delta = B \circ D$. Therefore every diagonal of $B$ has permanent 1. The converse is Lemma 7.

An invertible element in a semiring is called a unit. In some semirings 1 is the only unit. For example, Boolean algebras, the nonnegative integers, and the semiring $\mathbb{F}$ underlying the fuzzy matrices (see Section 2) are all antinegative semirings with only one unit.

**Corollary 3.1.** Suppose $\mathcal{S}$ is antinegative and commutative and has no zero divisors and only one unit, and $T$ is a linear operator on $\mathcal{M}(\mathcal{S})$. Then $T$ preserves the permanent if and only if there exist permutation matrices $P$ and $Q$ such that

- (a) $T(X) = PXQ$ for all $X$ in $\mathcal{M}(\mathcal{S})$ or
- (b) $m = n$ and $T(X) = PX'Q$ for all $X$ in $\mathcal{M}(\mathcal{S})$.

**Proof.** The necessity of these conditions follows from Theorem 3. It ensures that $T$ is a $(P, Q, B)$ operator and that $\text{per}(B_\delta) = 1$ for all $\delta \in \Delta$. But then each $b_{i,j} = 1$, so $B = J$. The sufficiency is immediate.
The **rook polynomial** of a matrix $A$ is $R_A(x) = \sum_{j \geq 0} p_j x^j$, where $p_0 = 1$ and $p_j$ is the sum of the permanents of the $j \times j$ submatrices of $A$.

**Theorem 4.** Suppose $\mathcal{S}$ is antinegative and commutative with no zero divisors and $T$ is a linear operator on $\mathcal{M}(\mathcal{S})$. Then $T$ preserves the rook polynomial if and only if there exist permutation matrices $P$ and $Q$ such that

(a) $T(X) = PXQ$ for all $X$ in $\mathcal{M}(\mathcal{S})$ or

(b) $m = n$ and $T(X) = PX^tQ$ for all $X$ in $\mathcal{M}(\mathcal{S})$.

**Proof.** The permanent of $X$ is the leading coefficient of the rook polynomial. If $T$ preserves the rook polynomial, then $T$ preserves the permanent, and hence $T$ is a $(P, Q, B)$ operator and $\text{per}(B_\delta) = 1$ for all $\delta \in \Delta$ by Theorem 3. Choose $(i, j)$. Then $T(E_{i,j}) = P(B \ast E_{i,j})Q = b_{i,j} E_{u,v}$ for some $(u, v)$. But the rook polynomial of $E_{i,j}$ [and hence of $T(E_{i,j})$] is $1 + x$. Therefore $b_{i,j} = 1$. Hence $B = J$. The sufficiency of the conditions is immediate.

**Corollary 4.1.** If $\mathcal{S}$ is antinegative and commutative with no zero divisors and only one unit, and $T$ is any linear operator on $\mathcal{M}(\mathcal{S})$, then $T$ preserves the rook polynomial if and only if $T$ preserves the permanent.

We now give some examples which illustrate the need for the hypotheses in the above theorems. Let $T_i$ be operators on the set of $2 \times 2$ matrices defined by

$$T_1: \begin{bmatrix} a & b \\ c & d \end{bmatrix} \rightarrow \begin{bmatrix} a & b \\ 2c & d \end{bmatrix},$$

$$T_2: \begin{bmatrix} a & b \\ c & d \end{bmatrix} \rightarrow \begin{bmatrix} a/2 & b/2 \\ 2c & 2d \end{bmatrix},$$

$$T_3: \begin{bmatrix} a & b \\ c & d \end{bmatrix} \rightarrow \begin{bmatrix} a - b & b \\ -a + b + c + d & b + d \end{bmatrix}.$$  

Note that $T_1$ preserves term rank but neither the permanent nor the rook polynomial; $T_2$ preserves the permanent and term rank but not the rook polynomial; and $T_3$ preserves the permanent but neither the term rank nor the rook polynomial.

Let $\mathcal{S}$ be the four-element Boolean algebra of subsets of $\{1, 2\}$. Define a linear operator on $\mathcal{M}_{2,2}(\mathcal{S})$ by

$$T_4: \begin{bmatrix} a & b \\ c & d \end{bmatrix} \rightarrow \begin{bmatrix} (1)a + (2)d & (1)c + (2)b \\ (1)b + (2)c & (1)d + (2)a \end{bmatrix}.$$
Then $T_4$ preserves the rook polynomial and hence the permanent, but $T_4$ is not a $(P, Q, B)$ operator and hence is not a term-rank preserver, by Theorem 1. Note that $\mathcal{S}$ is an antinegative semiring, but $\{1\}$ and $\{2\}$ are zero divisors in $\mathcal{S}$.

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