Ain Shams Engineering Journal (2014) 5, 271–283

Ain Shams University
Ain Shams Engineering Journal
www.elsevier.com/locate/asej
www.sciencedirect.com

ENGINEERING PHYSICS AND MATHEMATICS

Solving systems of linear Fredholm integro-differential equations with Fibonacci polynomials

Farshid Mirzaee *, Seyede Fatemeh Hoseini

Department of Mathematics, Faculty of Science, Malayer University, Malayer, Iran

Received 19 August 2013; accepted 1 September 2013
Available online 9 October 2013

KEYWORDS
Systems of Fredholm integro-differential equations;
The Fibonacci polynomials;
Collocation method;
Fibonacci polynomials solutions

Abstract In this paper, we introduce a method to solve systems of linear Fredholm integro-differential equations in terms of Fibonacci polynomials. First, we present some properties of these polynomials then a new approach implementing a collocation method in combination with matrices of Fibonacci polynomials is introduced to approximate the solution of high-order linear Fredholm integro-differentail equations systems with variable coefficients under the mixed conditions. Numerical results with comparisons are given to confirm the reliability of the proposed method for solving these systems of equations.

1. Introduction

Systems of integro-differential equations have a major role in the fields of science and engineering, such as nano-hydrodynamics [1], glass-forming process [2], dropwise condensation [3], wind ripple in the desert [4], modelling the competition between tumor cells and the immune system [5] and, examining the noise term phenomenon [6].

The concept of a system of integro-differential equations has motivated a huge amount of research work in recent years. There are several numerical methods for solving system of linear and nonlinear integro-differential equations, for example, the Adomian decomposition methods [7], He's homotopy perturbation method [8,9], variational iteration method [10], the Chebyshev polynomial method [11], the rationalized Haar functions method [12], Galerkin methods with hybrid functions [13], the Tau method [14], the differential transform method [15], Runge–Kutta methods [16], the spline approximation method [17], the block pulse functions method [18], the spectral method [19], the finite difference approximation method [20].

Recently, Mirzaee and Hoseini adapted the matrix method for the Fibonacci polynomials. They have been used the Fibonacci matrix method to find approximate solutions of singularly perturbed differential–difference equations [21].

In the present paper, we introduce a method to solve a system of high-order linear Fredholm integro-differential equations with variable coefficients in the form

$$\sum_{n=0}^{m}\sum_{j=1}^{l} p_{ij}^{n}(x)y_j^{(n)}(x) = g_i(x) + \int_a^b \sum_{j=1}^{l} K_{ij}(x,t)y_j(t)dt,$$
$$i = 1,2,\ldots,l, \quad 0 \leqslant a \leqslant x \leqslant b, \qquad (1)$$

* Corresponding author. Tel./fax: +98 851 2355466.
E-mail addresses: f.mirzaee@malayeru.ac.ir, f.mirzaee@iust.ac.ir (F. Mirzaee).
Peer review under responsibility of Ain Shams University.

ELSEVIER Production and hosting by Elsevier

2090-4479 © 2013 Production and hosting by Elsevier B.V. on behalf of Ain Shams University.
http://dx.doi.org/10.1016/j.asej.2013.09.002

where each equation of this system is under the mixed conditions

$$\sum_{j=0}^{m-1}\left(a_{i,j}^{n}y_{n}^{(j)}(a)+b_{i,j}^{n}y_{n}^{(j)}(b)\right)=\lambda_{n,i},\quad i=0,\ldots,m-1,$$
$$n=1,2,\ldots,l, \tag{2}$$

where $y_j^{(0)}(x)=y_j(x)$ is an unknown function. Also, $p_{ij}^{n}(x)$, $g_i(x)$ and $K_{ij}(x,t)$ are known functions defined on the interval $a\leqslant x,\ t\leqslant b$. Moreover, the functions $K_{ij}(x,t)$ for $i,j=1,2,\ldots,l$ can be expanded Maclaurin series and $a_{i,j}^{n},b_{i,j}^{n}$ and $\lambda_{n,i},\ i,j=0,1,\ldots,m-1,\ n=1,2,\ldots,l$ are appropriate constants.

We want to approximate the solution of (1) as follows:

$$y_i(x)\simeq\sum_{n=1}^{N+1}a_{i,n}F_n(x),\quad i=1,2,\ldots,l,\quad 0\leqslant a\leqslant x\leqslant b, \tag{3}$$

where $a_{i,n},\ n=1,2,\ldots,N+1$ are the unknown Fibonacci coefficients, N is any arbitrary positive integer such that $N\geqslant m$, and $F_n(x),\ n=1,2,\ldots,N+1$ are the Fibonacci polynomials that we introduced them in Section 2.

2. The Fibonacci polynomials and properties

Leonardo of Pisa also known as Leonardus Pisanus, or, most commonly, Fibonacci (from "filius Bonacci"), was an Italian mathematician of the 13th century.

Fibonacci is the best known to the modern world for the spreading of the Hindu–Arabic numerical system in Europe, primarily through the publication in 1202 of his Liber Abaci (Book of Calculation), and for a number sequence named the Fibonacci numbers after him, which he did not discover but used as an example in the Liber Abaci. In Fibonacci's Liber Abaci book, chapter 12, he posed, and solved, a problem involving the growth of a population of rabbits based on idealized assumptions. The solution, generation by generation, was a sequence of numbers later known as Fibonacci numbers. The number sequence was known to Indian mathematicians as early as the 6th century, but it was Fibonacci's Liber Abaci that introduced it to the West. In the Fibonacci sequence of numbers, each number is the sum of the previous two numbers, starting with 0 and 1. This sequence begins 0, 1, 2, 3, 5,... [22].

Definition 1. For any positive real number k, the k-Fibonacci sequence, say $\{F_{k,n}\}_{n\in N}$ is defined recurrently by

$$F_{k,n+1}=kF_{k,n}+F_{k,n-1},\quad n\geqslant 1, \tag{4}$$

with initial conditions

$$F_{k,0}=0,\quad F_{k,1}=1. \tag{5}$$

Particular cases of the k-Fibonacci sequence are constructed from the following relations

if $k=1$, the classical Fibonacci sequence is obtained:

$$F_0=0,\quad F_1=1,\quad F_{n+1}=F_n+F_{n-1},\quad n\geqslant 1,$$

if $k=2$, the Pell sequence appears:

$$P_0=0,\quad P_1=1,\quad P_{n+1}=2P_n+P_{n-1},\quad n\geqslant 1,$$

if $k=3$, the following sequence appears:

$$H_0=0,\quad H_1=1,\quad H_{n+1}=3H_n+H_{n-1},\quad n\geqslant 1.$$

If k be a real variable x then $F_{k,n}=F_{x,n}$ and they correspond to the Fibonacci polynomials defined by

$$F_{n+1}(x)=\begin{cases}1, & n=0\\ x, & n=1,\\ xF_n(x)+F_{n-1}(x), & n>1\end{cases} \tag{6}$$

from where the first six Fibonacci polynomials are

$$F_1(x)=1,$$
$$F_2(x)=x,$$
$$F_3(x)=x^2+1,$$
$$F_4(x)=x^3+2x,$$
$$F_5(x)=x^4+3x^2+1,$$
$$F_6(x)=x^5+4x^3+3x,$$

and from these expressions, as for the k-Fibonacci numbers we can write

$$F_{n+1}(x)=\sum_{i=0}^{\lfloor\frac{n}{2}\rfloor}\binom{n-i}{i}x^{n-2i},\quad n\geqslant 0, \tag{7}$$

where $\lfloor\frac{n}{2}\rfloor$ denotes the greatest integer in $\frac{n}{2}$. Fig. 1 displays the behavior of the first six Fibonacci polynomials.

Note that $F_{2n}(0)=0$ and $x=0$ is the only real root, while $F_{2n+1}(0)=1$ with no real roots. Also for $x=k\in N$ we obtain the elements of the k-Fibonacci sequences [23].

The Fibonacci polynomials have generating function [22]

$$G(x,t)=\frac{t}{1-t^2-tx}=\sum_{n=1}^{\infty}F_n(x)t^n$$
$$=t+xt^2+(x^2+1)t^3+(x^3+2x)t^4+\ldots.$$

The Fibonacci polynomials are normalized so that

$$F_n(1)=F_n,$$

where the F_n is nth Fibonacci number.

Note first, that the equations for the Fibonacci polynomials may be written in matrix form as $F(x)=B\,X(x)$, where $F(x)=[F_1(x),F_2(x),F_3(x),\ldots,F_{N+1}(x)]^T$, $X(x)=[1,x,x^2,x^3,\ldots,x^N]^T$, and B is the lower triangular matrix with entrances the coefficients appearing in the expansion of the Fibonacci polynomials in increasing powers of x. For example, if $N=7$

$$B=\begin{pmatrix}1&0&0&0&0&0&0&0\\0&1&0&0&0&0&0&0\\1&0&1&0&0&0&0&0\\0&2&0&1&0&0&0&0\\1&0&3&0&1&0&0&0\\0&3&0&4&0&1&0&0\\1&0&6&0&5&0&1&0\\0&4&0&10&0&6&0&1\end{pmatrix}.$$

Note that in matrix B the non-zero entrances build precisely the diagonals of the Pascal triangle and the sum of the elements in the same row gives the classical Fibonacci sequence. In addition, matrix B is invertible, and

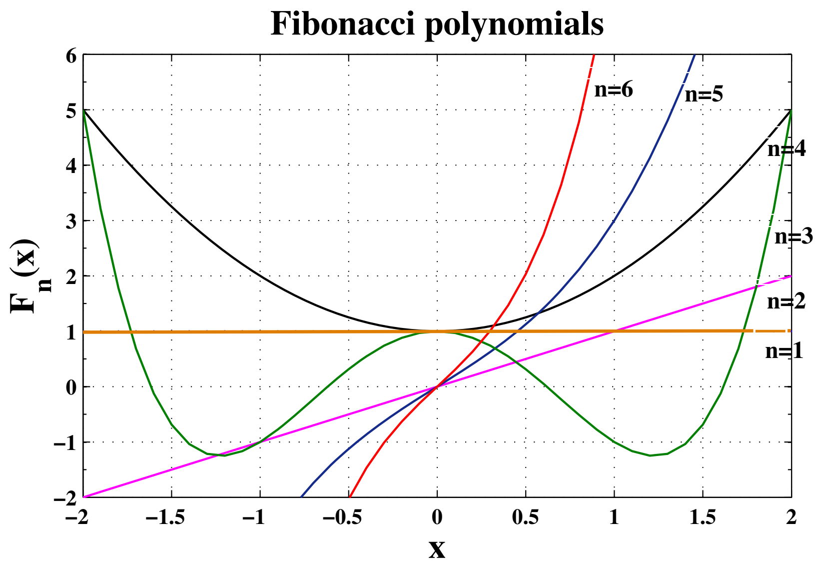

Fig. 1 The behavior of the first six Fibonacci polynomials.

$$B^{-1}=\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0\\ -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0\\ 0 & -2 & 0 & 1 & 0 & 0 & 0 & 0\\ 2 & 0 & -3 & 0 & 1 & 0 & 0 & 0\\ 0 & 5 & 0 & -4 & 0 & 1 & 0 & 0\\ -5 & 0 & 9 & 0 & -5 & 0 & 1 & 0\\ 0 & -14 & 0 & 14 & 0 & -6 & 0 & 1 \end{pmatrix},$$

and, therefore x^n may be written as linear combination of Fibonacci polynomials

$$1 = F_1(x),$$
$$x = F_2(x),$$
$$x^2 = F_3(x) - F_1(x),$$
$$x^3 = F_4(x) - 2F_2(x),$$
$$x^4 = F_5(x) - 3F_3(x) + 2F_1(x),$$
$$x^5 = F_6(x) - 4F_4(x) + 5F_2(x),$$
$$x^6 = F_7(x) - 5F_5(x) + 9F_3(x) - 5F_1(x),$$
$$\vdots$$

These expansions are given in closed form in the following theorem, which is the version of the Zeckendorf's theorem for the Fibonacci polynomials. Zeckendorf's theorem establishes that every integer may be written in a unique way as sum of non-consecutive Fibonacci numbers. For the Fibonacci polynomials we have the following result [23].

Theorem 1 [23]. *For every integer $n \geqslant 1$, x^{n-1} may be written in a unique way as linear combination of the n first Fibonacci polynomials as*

$$x^{n-1} = \sum_{n=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^i \left[\binom{n}{i} - \binom{n}{i-1} \right] F_{n-2i}(x),$$

where $\binom{n}{-1} = 0$.

3. Function approximation

In Section 2, we expressed $F(x)$ in the matrix form as follows:

$$\mathbf{F}(x) = \mathbf{B}\mathbf{X}(x), \tag{8}$$

where

$$\mathbf{F}(x) = [F_1(x), F_2(x), F_3(x), \ldots, F_{N+1}(x)]^T,$$
$$\mathbf{X}(x) = [1, x, x^2, x^3, \ldots, x^N]^T.$$

Suppose the solution of the system of integro-differential Eq. (1) expressed in terms of the Fibonacci polynomials such as (3). Then the function defined in the relation (3) can be written in matrix form

$$y_j(x) \simeq \mathbf{A}_j\mathbf{F}(x), \quad j = 1, 2, \ldots, l, \tag{9}$$

where $\mathbf{A}_j = [a_{j,1}, a_{j,2}, \ldots, a_{j,N+1}]$. Then from Eq. (8)

$$y_j(x) \simeq \mathbf{A}_j\mathbf{B}\mathbf{X}(x), \quad j = 1, 2, \ldots, l. \tag{10}$$

The differentiation of vector $\mathbf{X}(x)$ in Eq. (8), can be expressed as

$$\mathbf{X}^{(1)}(x) = \mathbf{D}\mathbf{X}(x), \tag{11}$$

where

$$\mathbf{D} = \begin{pmatrix} 0 & 0 & 0 & \ldots & 0 & 0\\ 1 & 0 & 0 & \ldots & 0 & 0\\ 0 & 2 & 0 & \ldots & 0 & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & 0 & 0 & \ldots & N & 0 \end{pmatrix}.$$

If we approximate $y_j(x) \simeq \mathbf{B}\mathbf{X}(x)$, then for $i \geqslant 2$ (i is the order derivatives) we get

$$\mathbf{X}^{(i)}(x) = \mathbf{D}\mathbf{X}^{(i-1)}(x) = \mathbf{D}^2\mathbf{X}^{(i-2)}(x) = \ldots = \mathbf{D}^i\mathbf{X}(x), \tag{12}$$

and therefore

$$\mathbf{F}^{(i)}(x) = \mathbf{B}\mathbf{X}^{(i)}(x) = \mathbf{B}\mathbf{D}^i\mathbf{X}(x), \quad i = 1, 2, \ldots, m, \tag{13}$$

then

$$\begin{aligned} y_j^{(i)}(x) &\simeq \mathbf{A}_j\mathbf{F}^{(i)}(x) \\ &= \mathbf{A}_j\mathbf{B}\mathbf{X}^{(i)}(x) \\ &= \mathbf{A}_j\mathbf{B}\mathbf{D}^i\mathbf{X}(x) \\ &= \mathbf{X}^T(x)\mathbf{D}^{Ti}\mathbf{B}^T\mathbf{A}_j^T, \quad i = 1,2,\ldots,m, \quad j = 1,2,\ldots,l. \end{aligned} \tag{14}$$

Therefor, the matrices $\mathbf{y}^{(i)}(x)$, $i = 0, 1, \ldots, m$ can be expressed as follows:

$$\mathbf{y}^{(i)}(x) \simeq \overline{\mathbf{X}}(x)\overline{\mathbf{D}}^i\overline{\mathbf{B}}\mathbf{A}, \tag{15}$$

where

$$\mathbf{y}^{(i)}(x) = \begin{bmatrix} y_1^{(i)}(x) \\ y_2^{(i)}(x) \\ \vdots \\ y_l^{(i)}(x) \end{bmatrix}, \quad \mathbf{A} = \begin{bmatrix} \mathbf{A}_1^T \\ \mathbf{A}_2^T \\ \vdots \\ \mathbf{A}_l^T \end{bmatrix}_{l\times 1}, \quad \overline{\mathbf{X}}(x) = \begin{bmatrix} \mathbf{X}^T(x) & 0 & \ldots & 0 \\ 0 & \mathbf{X}^T(x) & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & \mathbf{X}^T(x) \end{bmatrix}_{l\times l},$$

$$\overline{\mathbf{B}} = \begin{bmatrix} \mathbf{B}^T & 0 & \ldots & 0 \\ 0 & \mathbf{B}^T & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & \mathbf{B}^T \end{bmatrix}_{l\times l}, \quad \overline{\mathbf{D}} = \begin{bmatrix} \mathbf{D}^T & 0 & \ldots & 0 \\ 0 & \mathbf{D}^T & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & \mathbf{D}^T \end{bmatrix}_{l\times l}.$$

4. Method of solution

Consider the system (1) and write it in the matrix form

$$\sum_{i=0}^{m}\mathbf{P}_i(x)\mathbf{y}^{(i)}(x) = \mathbf{g}(x) + \mathbf{I}(x), \tag{16}$$

where

$$\mathbf{P_i}(x) = \begin{bmatrix} p_{1,1}^i(x) & p_{1,2}^i(x) & \ldots & p_{1,l}^i(x) \\ p_{2,1}^i(x) & p_{2,2}^i(x) & \ldots & p_{2,l}^i(x) \\ \vdots & \vdots & \ddots & \vdots \\ p_{l,1}^i(x) & p_{l,2}^i(x) & \ldots & p_{l,l}^i(x) \end{bmatrix}, \quad \mathbf{y}^{(i)}(x) = \begin{bmatrix} y_1^{(i)}(x) \\ y_2^{(i)}(x) \\ \vdots \\ y_l^{(i)}(x) \end{bmatrix}, \quad \mathbf{G}(x) = \begin{bmatrix} g_1(x) \\ g_2(x) \\ \vdots \\ g_l(x) \end{bmatrix},$$

$$\mathbf{I}(x) = \int_a^b \mathbf{K}(x,t)\mathbf{y}(t)dt, \quad \mathbf{K}(x,t) = \begin{bmatrix} K_{1,1}(x,t) & K_{1,2}(x,t) & \ldots & K_{1,l}(x,t) \\ K_{2,1}(x,t) & K_{2,2}(x,t) & \ldots & K_{2,l}(x,t) \\ \vdots & \vdots & \ddots & \vdots \\ K_{l,1}(x,t) & K_{l,2}(x,t) & \ldots & K_{l,l}(x,t) \end{bmatrix},$$

$$\mathbf{I}(x) = \begin{bmatrix} \mathbf{I}_1(x) \\ \mathbf{I}_2(x) \\ \vdots \\ \mathbf{I}_l(x) \end{bmatrix}, \qquad \mathbf{I}_i(x) = \int_a^b \sum_{j=1}^{l} K_{i,j}(x,t)\mathbf{y}_j(t)dt. \tag{17}$$

We can approximate the kernel function $\mathbf{k}_{i,j}(x, t)$ by truncated Taylor series [24] and the truncated Fibonacci series, respectively,

$$K_{i,j}(x,t) \simeq \sum_{m=0}^{N}\sum_{n=0}^{N}\mathbf{k}_{mn}^{ij}x^m t^n, \qquad K_{i,j}(x,t) \simeq \sum_{m=0}^{N}\sum_{n=0}^{N}k_{mn}^{ij}F_m(x)F_n(t), \tag{18}$$

where

$$\mathbf{k}_{mn}^{i,j}(x,t) = \frac{1}{m!n!}\frac{\partial^{m+n}\mathbf{K}(0,0)}{\partial x^m\partial t^n}; \qquad m,n = 0,1,2,\ldots,N,$$
$i,\ j = 1,2,\ldots,l.$

We can write relation (18) in the matrix form as follows:

$$K_{i,j}(x,t) \simeq \mathbf{X}^T(x)\mathbf{k}^{ij}\mathbf{X}(t); \qquad \mathbf{k}^{ij} = [\mathbf{k}_{mn}^{ij}], \tag{19}$$

and

$$K_{i,j}(x,t) \simeq \mathbf{F}^T(x)k^{ij}\mathbf{F}(t); \qquad k^{ij} = [k_{mn}^{ij}]. \tag{20}$$

From Eqs. (19) and (20), we have

$$\begin{aligned} \mathbf{X}^T(x)\mathbf{k}^{ij}\mathbf{X}(t) = \mathbf{F}^T(x)k^{ij}\mathbf{F}(t) \Rightarrow \mathbf{X}^T(x)\mathbf{k}^{ij}\mathbf{X}(t) \\ = \mathbf{X}^T(x)\mathbf{B}^Tk^{ij}\mathbf{B}\mathbf{X}(t). \end{aligned} \tag{21}$$

After substituting the Eqs. (10) and (20) in Eq. (17), we get

$$\begin{aligned} \mathbf{I}_i(x) &= \int_a^b \sum_{j=1}^{l}\mathbf{F}^T(x)k^{ij}\mathbf{F}(t)\mathbf{X}^T(t)\mathbf{B}^T\mathbf{A}_j^T dt \\ &= \sum_{j=1}^{l}\int_a^b \mathbf{F}^T(x)k^{ij}\mathbf{F}(t)\mathbf{X}^T(t)\mathbf{B}^T\mathbf{A}_j^T dt \\ &= \sum_{j=1}^{l}\mathbf{F}^T(x)k^{ij}\mathbf{Q}\mathbf{A}_j^T, \end{aligned} \tag{22}$$

where

$$\mathbf{Q} = \int_a^b \mathbf{F}(t)\mathbf{X}^T(t)\mathbf{B}^T dt = \int_a^b \mathbf{B}\mathbf{X}(t)\mathbf{X}^T(t)\mathbf{B}^T dt = \mathbf{B}\mathbf{H}\mathbf{B}^T,$$

and

$$\mathbf{H} = \int_a^b \mathbf{X}^T(t)\mathbf{X}(t)dt = [h_{r,s}], \qquad h_{r,s} = \frac{b^{r+s+1} - a^{r+s+1}}{r+s+1},$$
$r,s = 0,1,2,\ldots,N,$

then from (8) and (22), we have the matrix

$$\mathbf{I}_i(x) = \sum_{j=1}^{l}\mathbf{X}^T(x)\mathbf{B}^Tk^{ij}\mathbf{Q}\mathbf{A}_j^T. \tag{23}$$

We define the collocation points as follows

$$x_s = a + \frac{b-a}{N}s, \qquad s = 0,1,2,\ldots,N. \tag{24}$$

By placing these points in Eq. (16) we get

$$\sum_{i=0}^{m}\mathbf{P}_i(x_s)\mathbf{y}^{(i)}(x_s) = \mathbf{g}(x_s) + \mathbf{I}(x_s),$$

so we can write

$$\sum_{i=0}^{m}\mathbf{P}_i\mathbf{Y}^i = \mathbf{G} + \mathbf{I}, \tag{25}$$

where

$$\mathbf{P}_i = \begin{bmatrix} \mathbf{P}_i(x_0) & 0 & \ldots & 0 \\ 0 & \mathbf{P}_i(x_1) & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & \mathbf{P}_i(x_N) \end{bmatrix}, \quad \mathbf{Y}^i = \begin{bmatrix} \mathbf{y}^{(i)}(x_0) \\ \mathbf{y}^{(i)}(x_1) \\ \vdots \\ \mathbf{y}^{(i)}(x_N) \end{bmatrix}, \quad \mathbf{G} = \begin{bmatrix} \mathbf{g}(x_0) \\ \mathbf{g}(x_1) \\ \vdots \\ \mathbf{g}(x_N) \end{bmatrix}, \quad \mathbf{I} = \begin{bmatrix} \mathbf{I}(x_0) \\ \mathbf{I}(x_1) \\ \vdots \\ \mathbf{I}(x_N) \end{bmatrix}.$$

By placing the collocation points (24) in relation (15), we obtain

$$\mathbf{y}^{(i)}(x_s) \simeq \overline{\mathbf{X}}(x_s)\overline{\mathbf{D}}^i\overline{\mathbf{B}}\mathbf{A}, \qquad s = 0,1,\ldots,N, \quad i = 0,1,\ldots,l, \tag{26}$$

which can be written as

$$\mathbf{Y}^i \simeq \widetilde{\mathbf{X}}\overline{\mathbf{D}}^i\overline{\mathbf{B}}\mathbf{A}, \tag{27}$$

where

$$\overline{\mathbf{X}}(x_s) = \begin{bmatrix} \mathbf{X}^T(x_s) & 0 & \dots & 0 \\ 0 & \mathbf{X}^T(x_s) & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \mathbf{X}^T(x_s) \end{bmatrix}, \qquad s = 0, 1, \dots, N,$$

and

$$\widetilde{\mathbf{X}} = \begin{bmatrix} \overline{\mathbf{X}}(x_0) \\ \overline{\mathbf{X}}(x_1) \\ \vdots \\ \overline{\mathbf{X}}(x_N) \end{bmatrix}.$$

Then we collocate Eq. (23) at $N+1$ collocation points and hence we have

$$[\mathbf{I}_i(x_s)] = \sum_{j=1}^{l}\mathbf{X}^T(x_s)\mathbf{B}^T k^{ij}\mathbf{Q}\mathbf{A}_j^T, \qquad s = 0, 1, \dots, N,$$

$$i = 1, \dots, l. \tag{28}$$

Similarly, by placing the collocation points into the matrix $\mathbf{I}(x)$ and using relation (28), we have

$$\mathbf{I}(x_s) = \begin{bmatrix} \mathbf{I}_1(x_s) \\ \mathbf{I}_2(x_s) \\ \vdots \\ \mathbf{I}_l(x_s) \end{bmatrix} = \overline{\mathbf{X}}(x_s)\overline{\mathbf{B}}\mathbf{k}_f\overline{\mathbf{Q}}\mathbf{A}, \tag{29}$$

where

$$\overline{\mathbf{B}} = \begin{bmatrix} \mathbf{B}^T & 0 & \dots & 0 \\ 0 & \mathbf{B}^T & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \mathbf{B}^T \end{bmatrix}_{l\times l}, \qquad \mathbf{I}(x_s) = \begin{bmatrix} \mathbf{I}_1(x_s) \\ \mathbf{I}_2(x_s) \\ \vdots \\ \mathbf{I}_l(x_s) \end{bmatrix},$$

$$\mathbf{k}_f = \begin{bmatrix} k^{11} & k^{12} & \dots & k^{1l} \\ k^{21} & k^{22} & \dots & k^{2l} \\ \vdots & \vdots & \ddots & \vdots \\ k^{l1} & k^{l2} & \dots & k^{ll} \end{bmatrix}_{l\times l}, \quad \overline{\mathbf{Q}} = \begin{bmatrix} \mathbf{Q} & 0 & \dots & 0 \\ 0 & \mathbf{Q} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \mathbf{Q} \end{bmatrix}_{l\times l}, \quad \mathbf{A} = \begin{bmatrix} \mathbf{A}_1^T \\ \mathbf{A}_2^T \\ \vdots \\ \mathbf{A}_l^T \end{bmatrix}_{l\times 1}.$$

Therefore, we can express matrix $\mathbf{I}$ in matrix form in terms of the matrix of the Fibonacci coefficients $\mathbf{A}$ by using relation (29) as follows:

$$\mathbf{I} = \begin{bmatrix} \mathbf{I}(x_0) \\ \mathbf{I}(x_1) \\ \vdots \\ \mathbf{I}(x_N) \end{bmatrix} = \widetilde{\mathbf{X}}\overline{\mathbf{B}}\mathbf{k}_f\overline{\mathbf{Q}}\mathbf{A}. \tag{30}$$

After substituting relations (27) and (30) into Eq. (25), we get

$$\left\{\sum_{i=0}^{m}\mathbf{P}_i\widetilde{\mathbf{X}}\overline{\mathbf{D}}^i\overline{\mathbf{B}} - \widetilde{\mathbf{X}}\overline{\mathbf{B}}\mathbf{k}_f\overline{\mathbf{Q}}\right\}\mathbf{A} = \mathbf{G}. \tag{31}$$

Where the dimension of the matrices $\mathbf{P}_i, \widetilde{\mathbf{X}}, \overline{\mathbf{B}}, \overline{\mathbf{D}}, \mathbf{k}_f$, and $\overline{\mathbf{Q}}$ in Eq. (31) is $l(N+1)\times l(N+1)$, and the dimension of the vectors $\mathbf{A}$ and $\mathbf{G}$ is $l(N+1)\times 1$. Hence, the fundamental matrix Eq. (31) corresponding to Eq. (1) can be written in the form

$$\mathbf{WA} = \mathbf{G}. \tag{32}$$

Eq. (32) gives $l(N+1)$ algebraic equations. Since the number of unknowns for each $\mathbf{A}_j$, $j = 1, \dots, l$ is $N+1$, the total number of unknowns is $l(N+1)$.

For the mixed conditions (2), we have

$$\begin{cases} \sum_{j=0}^{m-1}[a^1_{i,j}y_1^{(j)}(a) + b^1_{i,j}y_1^{(j)}(b)] = \lambda_{1,i}, \\ \sum_{j=0}^{m-1}[a^2_{i,j}y_2^{(j)}(a) + b^2_{i,j}y_2^{(j)}(b)] = \lambda_{2,i}, \\ \vdots \\ \sum_{j=0}^{m-1}[a^l_{i,j}y_l^{(j)}(a) + b^l_{i,j}y_l^{(j)}(b)] = \lambda_{l,i}. \end{cases}$$

or

$$\begin{cases} \sum_{j=0}^{m-1}\left[a^1_j y_1^{(j)}(a) + b^1_j y_1^{(j)}(b)\right] = \lambda_1, \\ \sum_{j=0}^{m-1}\left[a^2_j y_2^{(j)}(a) + b^2_j y_2^{(j)}(b)\right] = \lambda_2, \\ \vdots \\ \sum_{j=0}^{m-1}[a^l_j y_l^{(j)}(a) + b^l_j y_l^{(j)}(b)] = \lambda_l. \end{cases}$$

where

Table 1 Numerical results of solutions $y_1(x)$ of Eq. (39).

x	Exact solution	Present method		
	$y_1(x) = \sin(x)$	$y_{1,3}(x)$	$y_{1,7}(x)$	$y_{1,12}(x)$
0.0	0.0	−8.68870193185e−17	−1.47244819419e−16	−2.74526109659e−16
0.1	0.09983341664683	0.09983619519825	0.09983341654799	0.09983341664682
0.2	0.19866933079506	0.19868956158603	0.19866932894328	0.19866933079507
0.3	0.29552020666134	0.29557727035286	0.29552019893790	0.29552020666136
0.4	0.38941834230865	0.38951649268826	0.38941832254098	0.38941834230871
0.5	0.47942553860420	0.47952439978175	0.47942549819685	0.47942553860432
0.6	0.56464247339504	0.56461816282286	0.56464240071734	0.56464247339524
0.7	0.64421768723769	0.64381495300112	0.64421756772347	0.64421768723803
0.8	0.71735609089952	0.71613194150605	0.71735590535183	0.71735609090006
0.9	0.78332690962748	0.78058629952717	0.78332661998099	0.78332690962828
1.0	0.84147098480790	0.83619519825400	0.84147048273325	0.84147098480903

Table 2 Numerical results of solutions $y_2(x)$ of Eq. (39).

x	Exact solution	Present method		
	$y_2(x) = \cos(x)$	$y_{2,3}(x)$	$y_{2,7}(x)$	$y_{2,12}(x)$
0.0	1.0	1.0	1.0	1.0
0.1	0.99500416527803	0.99502673721834	0.99500416563406	0.99500416527803
0.2	0.98006657784124	0.98021389774669	0.98006657748037	0.98006657784125
0.3	0.95533648912561	0.95572190489506	0.95533648366758	0.95533648912564
0.4	0.92106099400289	0.92171118197348	0.92106097725191	0.92106099400297
0.5	0.87758256189037	0.87834215229195	0.87758252524324	0.87758256189053
0.6	0.82533561490968	0.82577523916049	0.82533554636318	0.82533561490996
0.7	0.76484218728449	0.76417086588911	0.76484207238007	0.76484218728495
0.8	0.69670670934717	0.69368945578783	0.69670652786602	0.69670670934788
0.9	0.62160996827066	0.61449143216665	0.62160966322097	0.62160996827171
1.0	0.54030230586814	0.52673721833560	0.54030167580862	0.54030230586965

Table 3 The maximum error $e_{1,N}(x)$ of Eq. (39).

N	3	7	9	10	11	12
Method of [25]	5.0207e−03	5.0207e−07	3.9722e−09	2.6596e−10	2.4875e−11	1.2126e−12
Present method	5.2758e−03	5.0207e−07	3.9721e−09	2.6598e−10	2.4864e−11	1.1364e−12

Table 4 The maximum error $e_{2,N}(x)$ of Eq. (39).

N	3	7	9	10	11	12
Method of [25]	1.3565e−02	6.3006e−07	4.2348e−09	2.9397e−10	2.5629e−11	1.5526e−12
Present method	1.3565e−02	6.3006e−07	4.2348e−09	2.9398e−10	2.5666e−11	1.5137e−12

$$\lambda_i = \begin{bmatrix} \lambda_{i,0} \\ \lambda_{i,1} \\ \vdots \\ \lambda_{i,m-1} \end{bmatrix}_{m\times 1}, \quad a_j^i = \begin{bmatrix} a_{0,j}^i \\ a_{1,j}^i \\ \vdots \\ a_{m-1,j}^i \end{bmatrix}_{m\times 1}, \quad b_j^i = \begin{bmatrix} b_{0,j}^i \\ b_{1,j}^i \\ \vdots \\ b_{m-1,j}^i \end{bmatrix}_{m\times 1}, \quad i = 1,2,\ldots,l,$$

or briefly

$$\sum_{j=0}^{m-1} \left[a_j \mathbf{y}^{(j)}(a) + b_j \mathbf{y}^{(j)}(b)\right] = \lambda, \tag{33}$$

where

$$a_j = \begin{bmatrix} a_j^1 & 0 & \ldots & 0 \\ 0 & a_j^2 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & a_j^l \end{bmatrix}_{l\times l}, \quad b_j = \begin{bmatrix} b_j^1 & 0 & \ldots & 0 \\ 0 & b_j^2 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & b_j^l \end{bmatrix}_{l\times l}, \quad \lambda = \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_l \end{bmatrix}_{l\times 1}.$$

By substituting the Eq. (26) into Eq. (33) at points a and b and simplifying the result we get

$$\sum_{j=0}^{m-1} [a_j \overline{\mathbf{X}}(a) + b_j \overline{\mathbf{X}}(b)] \overline{\mathbf{D}}^j \overline{\mathbf{B}} \mathbf{A} = \lambda. \tag{34}$$

So we can write the conditions in matrix form as follows

$$\mathbf{UA} = \lambda. \tag{35}$$

Finally, to obtain the solution of Eq. (1) under the conditions (2) by replacing the rows of matrix $\mathbf{U}$ and λ by the rows of the matrices $\mathbf{W}$ and $\mathbf{G}$, respectively we obtain

$$\widetilde{\mathbf{W}}\mathbf{A} = \widetilde{\mathbf{G}}. \tag{36}$$

For convenience, if the last mk rows of the matrix $\mathbf{W}$ are replaced, we have the new augmented matrix

$$[\widetilde{\mathbf{W}}; \widetilde{\mathbf{G}}] = \begin{bmatrix} w_{1,1} & w_{1,2} & \ldots & w_{1,l(N+1)} & ; & g_1(x_0) \\ w_{2,1} & w_{2,2} & \ldots & w_{2,l(N+1)} & ; & g_2(x_0) \\ \vdots & \vdots & \vdots & \vdots & ; & \vdots \\ w_{l,1} & w_{l,2} & \ldots & w_{l,l(N+1)} & ; & g_l(x_0) \\ w_{l+1,1} & w_{l+1,2} & \ldots & w_{l+1,l(N+1)} & ; & g_1(x_1) \\ \vdots & \vdots & \vdots & \vdots & ; & \vdots \\ w_{l(N-m+1),1} & w_{l(N-m+1),2} & \ldots & w_{l(N-m+1),l(N+1)} & ; & g_l(x_{N-m}) \\ u_{1,1} & u_{1,2} & \ldots & u_{1,l(N+1)} & ; & \lambda_{1,0} \\ u_{2,1} & u_{2,2} & \ldots & u_{2,l(N+1)} & ; & \lambda_{1,1} \\ \vdots & \vdots & \vdots & \vdots & ; & \vdots \\ u_{l,1} & u_{l,2} & \ldots & u_{l,l(N+1)} & ; & \lambda_{1,m-1} \\ u_{l+1,1} & u_{l+1,2} & \ldots & u_{l+1,l(N+1)} & ; & \lambda_{2,0} \\ \vdots & \vdots & \vdots & \vdots & ; & \vdots \\ u_{ml,1} & u_{ml,2} & \ldots & u_{ml,l(N+1)} & ; & \lambda_{l,m-1} \end{bmatrix}. \tag{37}$$

However, we do not have to replace the last rows. For example, if the matrix $\mathbf{W}$ is singular, then the rows that have the same factor or all zeros are replaced.

If rank $\widetilde{\mathbf{W}} = rank[\widetilde{\mathbf{W}}; \widetilde{\mathbf{G}}] = l(N+1)$, then we can write

$$\mathbf{A} = \widetilde{\mathbf{W}}^{-1}\widetilde{\mathbf{G}}.$$

Thus, the matrix $\mathbf{A}$ (thereby the unknown Fibonacci coefficients) is uniquely determined. Also the system (1) with the conditions (2) has a unique solution. This solution is given by the truncated Fibonacci series (8). However, when $|\widetilde{\mathbf{W}}| = 0$, if $rank\widetilde{\mathbf{W}} = rank[\widetilde{\mathbf{W}}; \widetilde{\mathbf{G}}] < l(N+1)$, then we may find a particular solution. Otherwise if rank $\widetilde{\mathbf{W}} \neq$ rank $[\widetilde{\mathbf{W}}; \widetilde{\mathbf{G}}] < l(N+1)$, then it is not a solution.

5. Accuracy of the solution and error analysis

In this section, accuracy of the solution and error analysis are briefly discussed. In the following lines the main Theorem of this section will be provided.

Let us consider $n + 1$ pairs (x_i, y_i). The problem is to find a polynomial p_k, called the interpolating polynomial, such that

$$p_k(x_i) = c_0 + c_1 x_i + \ldots + c_k x_i^k, \qquad i = 0, 1, \ldots, n.$$

The points x_i are called interpolation nodes. If $n \neq k$, the problem is over or under-determined. Let P_n be the$(n + 1)$-dimensional subspace of $C[a, b]$ spanned by the functions $1, x, \ldots, x^n$. That is, P_n consists of all polynomials of degree at most n.

Given $n + 1$ distinct nodes $x_0, x_1, \ldots, x_n$ and $n + 1$ corresponding values $y_0, y_1, \ldots, y_n$ then there exists a unique polynomial $p_n \in P_n$ such that $p_n(x_i) = y_i$ for $i = 0, 1, \ldots, n$.

If we define

$$l_i \in P_n, \quad l_i = \prod_{\substack{i=0,\\ j\neq i}}^{n} \frac{(x - x_i)}{(x_i - x_j)}, \qquad i = 0, 1, \ldots, n,$$

then $l_i(x_j) = \delta_{ij}$. The polynomials $l_i(x)$ are called Lagrange characteristic polynomials. If $f(x_i) = y_i$ for $i = 0, 1, \ldots, n$, f being a given function, the interpolating polynomial $p_n(x)$ will be denoted by $p_n f(x)$. Let us introduce a lower triangular matrix X of infinite size, called the interpolation matrix on $[a, b]$ whose entries x_{ij} for $i,j = 0, 1, \ldots, n$ represent the points of $[a, b]$, with the assumption that on each row the entries are all distinct. Thus, for any $n \geqslant 0$, the $(n + 1)$-th row of X contains $n + 1$ distinct values that can be identified as nodes, so that, for a given function f, we can uniquely define an interpolating polynomial $p_n f$ of degree n at those nodes.

In the following lines the main Theorems of this section will be provided. In this Theorem, we consider $l = 1$ and $P^n(x) = P^n_{i,j}$ in the Eq. (1) for clarity of presentation. We note that a similar procedure can be applied for the case of $l > 1$.

Theorem 2. *Let the solution of (1) actually computed by the Fibonacci series solution $p_N(x)$ and $y = f(x)$ be the exact solution. Let the coefficient matrix of (36) be $\widetilde{\widetilde{\mathbf{W}}} = \widetilde{\mathbf{W}} + \delta\mathbf{W}$ where $\delta\mathbf{W}$ represents the computational error. Let $\mathbf{X}(x)$ and $\mathbf{B}$ be the matrices which defined in (8). if $\|\widetilde{\widetilde{\mathbf{W}}}^{-1}\|_F \|\delta\mathbf{W}\|_F \leqslant 1$ and $f \in C^{\infty}[a, b]$, then*

$$|p_N(x) - f(x)| \leqslant \frac{r\|\widetilde{\mathbf{A}}\|_F \|\widetilde{\widetilde{\mathbf{W}}}^{-1}\|_F}{1 - r\|\widetilde{\widetilde{\mathbf{W}}}^{-1}\|_F} \|\mathbf{B}\|_F \|\mathbf{X}(b-a)\|_F + \left| \frac{f^{(N+1)}(\xi_x)}{(N+1)!} \prod_{i=0}^{N} (x - x_i) \right|,$$

where r is the highest value of $\|\delta\mathbf{W}\|_F$ and $\widetilde{A}$ is the solution of (36).

Proof. For the proof, see [21].

On the other hand, the error can be estimated by root-mean-square error (RMS). We calculate RMS error by the following formula:

$$\textbf{RMS error} = \left(\frac{\sum_{n=1}^{N+1} (y_i(x_n) - y_{i,N+1}(x_n))}{N+1} \right)^{1/2} \qquad i = 1, 2, \ldots, l,$$

where y_i and $y_{i,N+1}$ are the exact and approximate solutions of the problem. □

6. Numerical examples

In this section, four examples are given to certify the convergence and error bound of the presented method. All results are computed by using a program written in the Matlab. In this regard, we have presented with tables and figures, the values of the exact solution $y_i(x)$, $i = 1, 2,, \ldots, l$, the approximate solutions $y_{i,N}(x)$, $i = 1, 2, \ldots, l$, and the absolute error function $e_{i,N}(x) = |y_i(x) - y_{i,N}|$, $i = 1, 2, \ldots, l$ at the selected points of the given interval. In addition, we define the maximum error for $y_{i,N}(x)$, $i = 1, 2, \ldots, l$ as,

$$e_{i,N}(x) = \|y_{i,N}(x) - y_i(x)\|_\infty = \max\{|y_{i,N}(x) - y_i(x)|, \quad a \leqslant x \leqslant b\}.$$

Example 1 [25]. Consider the system of the linear Fredholm integro-differential equations

$$\begin{cases} y_1^{(2)}(x) - xy_2^{(1)}(x) + 2xy_1(x) = 2x^3 - \frac{37}{12}x^2 + \frac{320}{60}x + 2 + \int_0^1 (x^2 t y_1(t) - xt^2 y_2(t))dt, \\ y_2^{(2)}(x) - 2xy_1^{(1)}(x) + y_2(x) = -5x^2 - \frac{109}{30}x - 1 + \int_0^1 (xty_1(t) + xt^3 y_2(t))dt, \qquad 0 \leqslant x \leqslant 1, \\ y_1(0) = 3, \quad y_1(1) = 2, \quad y_2(0) = 1, \quad y_2(1) = 1, \end{cases} \tag{38}$$

where the exact solutions are $y_1(x) = x^2 - 2x + 3$, $y_2(x) = -x^2 + x + 1$.

The approximate solution $y_i(x)$ by the truncated Fibonacci series

$$y_i(x) = \sum_{n=1}^{3} a_{i,n} F_n(x), \qquad i = 1, 2,$$

where $N = 2, l = 2, m = 2$,

$$g_1(x) = 2x^3 - \frac{37}{12}x^2 + \frac{320}{60}x + 2, \quad g_2(x) = -5x^2 - \frac{109}{30}x - 1,$$
$$p^0_{1,1}(x) = 2x, \quad p^0_{1,2}(x) = 0, \quad p^0_{2,1}(x) = 0, \quad p^0_{2,2}(x) = 1,$$
$$p^1_{1,1}(x) = 0, \quad p^1_{1,2}(x) = -x, \quad p^1_{2,1}(x) = -2x, \quad p^1_{2,2}(x) = 0,$$
$$p^2_{1,1}(x) = 1, \quad p^2_{1,2}(x) = 0, p^2_{2,1}(x) = 0 \text{ and } p^2_{2,2}(x) = 1.$$

The set of collocation points (24) are

$$\left\{ x_0 = 0, \quad x_1 = \frac{1}{2}, \quad x_2 = 1 \right\},$$

and from Eq. (31), we have the fundamental matrix equation as follows

$$\{\mathbf{P}_0 \widetilde{\mathbf{X}} \bar{\mathbf{B}} + \mathbf{P}_1 \widetilde{\mathbf{X}} \bar{\mathbf{D}} \bar{\mathbf{B}} + \mathbf{P}_2 \widetilde{\mathbf{X}} \bar{\mathbf{D}}^2 \bar{\mathbf{B}} - \widetilde{\mathbf{X}} \bar{\mathbf{B}} \mathbf{k}_f \bar{\mathbf{Q}}\} \mathbf{A} = \mathbf{G},$$

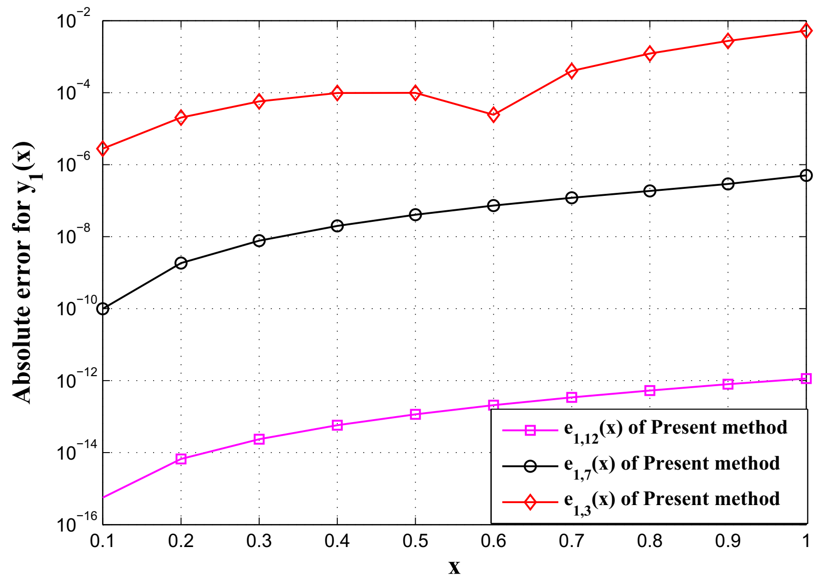

Fig. 2 Comparison of the absolute error functions $e_{1,N}(x)$ of Eq. (39).

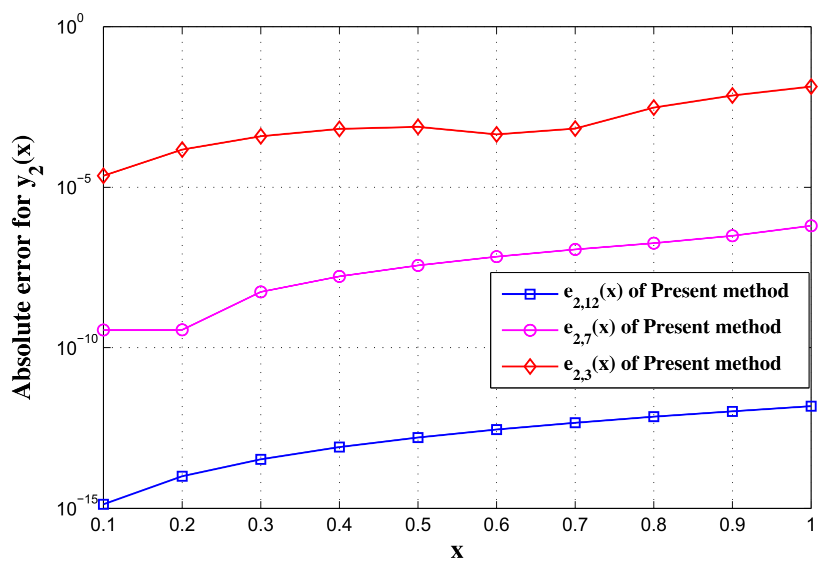

Fig. 3 Comparison of the absolute error functions $e_{2,N}(x)$ of Eq. (39).

Table 5 Numerical results of the absolute error function $e_{1,N}(x)$ in Eq. (40).

x	Present method $e_{1,15}(x)$	BPF method of [26,29] $e_{1,32}(x)$	TF method of [26] $e_{1,32}(x)$	DF method of [27] $e_{1,\ 32}(x)$	RHF method of [28] $e_{1,32}(x)$
0.0	1.4980e−08	1.047e−02	1.151e−03	8.1951e−05	1.548e−02
0.1	1.5485e−08	1.124e−02	1.151e−03	2.1087e−06	1.014e−02
0.2	1.6082e−08	3.560e−03	1.192e−03	4.4883e−05	3.550e−03
0.3	1.6860e−08	4.390e−03	1.285e−03	4.8899e−05	4.480e−03
0.4	1.7984e−08	1.406e−02	1.446e−03	4.6489e−05	1.418e−02
0.5	1.9750e−08	2.572e−02	1.694e−03	1.3319e−04	2.582e−02
0.6	2.2672e−08	1.699e−02	1.699e−03	4.0275e−06	1.693e−02
0.7	2.7623e−08	6.060e−03	1.769e−03	7.3647e−05	6.070e−03
0.8	3.6054e−08	6.540e−03	1.927e−03	8.0238e−05	7.140e−03
0.9	5.0333e−08	2.309e−02	2.199e−03	8.0632e−06	2.492e−02

Table 6 Numerical results of the absolute error function $e_{2,N}(x)$ in Eq. (40).

x	Present method $e_{2,15}(x)$	BPF method of [26,29] $e_{2,32}(x)$	TF method of [26] $e_{2,32}(x)$	DF method of [27] $e_{2,\ 32}(x)$	RHF method of [28] $e_{2,32}(x)$
0.0	1.0244e−08	1.530e−02	6.669e−04	8.0690e−05	1.544e−02
0.1	1.0282e−08	8.260e−03	7.307e−04	4.1264e−06	8.370e−03
0.2	1.0280e−08	2.370e−03	7.430e−04	2.8547e−05	2.480e−03
0.3	1.0232e−08	2.700e−03	7.131e−04	2.6175e−05	2.410e−03
0.4	1.0129e−08	6.500e−03	6.484e−04	2.3356e−06	6.410e−03
0.5	9.9645e−09	9.680e−03	5.549e−04	4.9788e−05	9.660e−03
0.6	9.7271e−09	4.950e−03	5.440e−04	2.4303e−06	4.990e−03
0.7	9.4061e−09	1.390e−03	4.917e−04	1.7341e−05	1.410e−03
0.8	8.9891e−09	7.800e−04	4.023e−04	1.5944e−05	1.590e−03
0.9	8.4619e−09	4.130e−03	2.786e−04	1.3083e−06	4.030e−03

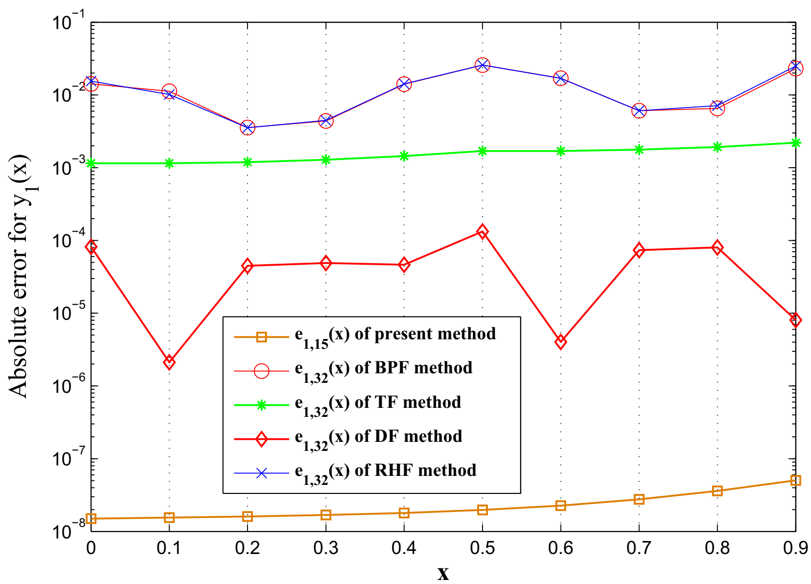

Fig. 4 Comparison of the absolute error functions $e_{1,N}(x)$ of Eq. (40).

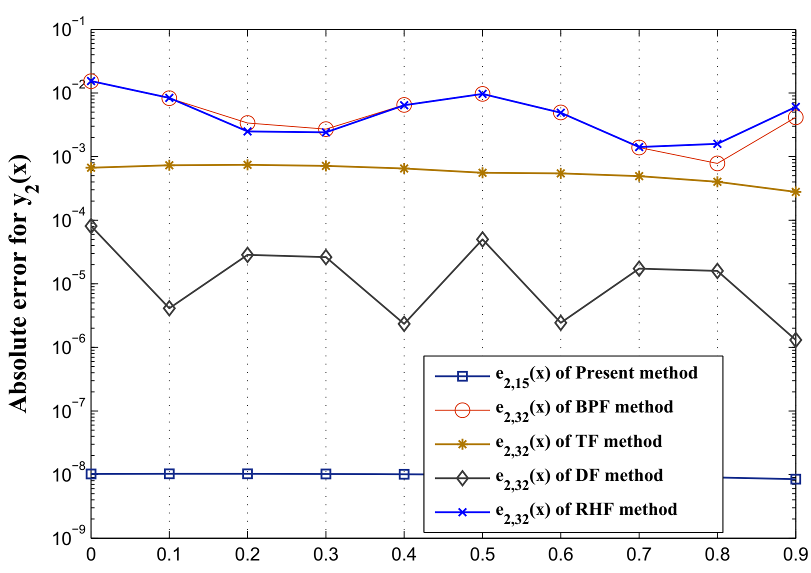

Fig. 5 Comparison of the absolute error functions $e_{2,N}(x)$ of Eq. (40).

where

$$\mathbf{P}_0(x)=\begin{bmatrix}2x & 0\\ 0 & 1\end{bmatrix},\quad \mathbf{P}_1(x)=\begin{bmatrix}0 & -x\\ -2x & 0\end{bmatrix},\quad \mathbf{P}_2(x)=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix},\quad \mathbf{P}_0=\begin{bmatrix}\mathbf{P}_0(0) & 0 & 0\\ 0 & \mathbf{P}_0(\frac{1}{2}) & 0\\ 0 & 0 & \mathbf{P}_0(1)\end{bmatrix},$$

$$\mathbf{P}_1=\begin{bmatrix}\mathbf{P}_1(0) & 0 & 0\\ 0 & \mathbf{P}_1(\frac{1}{2}) & 0\\ 0 & 0 & \mathbf{P}_1(1)\end{bmatrix},\quad \mathbf{P}_2=\begin{bmatrix}\mathbf{P}_2(0) & 0 & 0\\ 0 & \mathbf{P}_2(\frac{1}{2}) & 0\\ 0 & 0 & \mathbf{P}_2(1)\end{bmatrix},\quad \widetilde{\mathbf{X}}=\begin{bmatrix}\overline{\mathbf{X}}(0)\\ \overline{\mathbf{X}}(\frac{1}{2})\\ \overline{\mathbf{X}}(1)\end{bmatrix},$$

$$\overline{\mathbf{X}}(0)=\begin{bmatrix}\mathbf{X}^T(0) & 0\\ 0 & \mathbf{X}^T(0)\end{bmatrix},\quad \overline{\mathbf{X}}\left(\frac{1}{2}\right)=\begin{bmatrix}\mathbf{X}^T(\frac{1}{2}) & 0\\ 0 & \mathbf{X}^T(\frac{1}{2})\end{bmatrix},\quad \overline{\mathbf{X}}(1)=\begin{bmatrix}\mathbf{X}^T(1) & 0\\ 0 & \mathbf{X}^T(1)\end{bmatrix},\quad \mathbf{X}(0)=\begin{bmatrix}1\\ 0\\ 0\end{bmatrix},$$

$$\mathbf{X}\left(\frac{1}{2}\right)=\begin{bmatrix}1\\ \frac{1}{2}\\ \frac{1}{4}\end{bmatrix},\quad \mathbf{X}(1)=\begin{bmatrix}1\\ 1\\ 1\end{bmatrix},\quad \overline{\mathbf{B}}=\begin{bmatrix}\mathbf{B}^T & 0\\ 0 & \mathbf{B}^T\end{bmatrix},\quad \overline{\mathbf{D}}=\begin{bmatrix}\mathbf{D}^T & 0\\ 0 & \mathbf{D}^T\end{bmatrix},$$

$$\mathbf{B}=\begin{bmatrix}0 & 0 & 0\\ 1 & 0 & 0\\ 0 & 2 & 0\end{bmatrix},\quad \mathbf{D}=\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 1 & 0 & 1\end{bmatrix},\quad \mathbf{G}=\begin{bmatrix}\mathbf{g}(0)\\ \mathbf{g}(\frac{1}{2})\\ \mathbf{g}(1)\end{bmatrix},\quad \mathbf{g}(0)=\begin{bmatrix}2\\ -1\end{bmatrix},\quad \mathbf{g}\left(\frac{1}{2}\right)=\begin{bmatrix}\frac{199}{48}\\ \frac{-61}{15}\end{bmatrix},$$

$$\mathbf{g}(1)=\begin{bmatrix}\frac{25}{4}\\ \frac{-289}{30}\end{bmatrix},\quad \mathbf{A}=\begin{bmatrix}\mathbf{A}_1^T\\ \mathbf{A}_2^T\end{bmatrix},\quad \mathbf{A}_1=[a_{1,1}\quad a_{1,2}\quad a_{1,3}],\quad \mathbf{A}_2=[a_{2,1}\quad a_{2,2}\quad a_{2,3}].$$

The augmented matrix for this fundamental matrix equation is

$$[\mathbf{W};\mathbf{G}]=\begin{bmatrix}0 & 0 & 2 & 0 & 0 & 0 & ; & 2\\ 0 & 0 & 0 & 1 & 0 & 3 & ; & -1\\ \frac{7}{8} & \frac{5}{12} & \frac{49}{16} & \frac{1}{6} & \frac{-3}{8} & \frac{-7}{30} & ; & \frac{199}{48}\\ \frac{-1}{4} & \frac{-7}{6} & \frac{-11}{8} & 1 & \frac{1}{2} & \frac{13}{4} & ; & \frac{-61}{15}\\ \frac{3}{2} & \frac{5}{3} & \frac{21}{4} & \frac{1}{3} & \frac{-3}{4} & \frac{-22}{15} & ; & \frac{25}{4}\\ \frac{-1}{2} & \frac{-7}{3} & \frac{-19}{4} & 1 & 1 & 4 & ; & \frac{-289}{30}\end{bmatrix}.$$

From Eq. (35), the matrix form for boundary conditions is

$$[\mathbf{U};\lambda]=\begin{bmatrix}1 & 0 & 1 & 0 & 0 & 0 & ; & 3\\ 1 & 1 & 2 & 0 & 0 & 0 & ; & 2\\ 0 & 0 & 0 & 1 & 0 & 1 & ; & 1\\ 0 & 0 & 0 & 1 & 1 & 2 & ; & 1\end{bmatrix}.$$

Therefore, the new augmented matrix based on conditions from system (37) is

$$[\widetilde{\mathbf{W}};\widetilde{\mathbf{G}}]=\begin{bmatrix}0 & 0 & 2 & 0 & 0 & 0 & ; & 2\\ 0 & 0 & 0 & 1 & 0 & 3 & ; & -1\\ 1 & 0 & 1 & 0 & 0 & 0 & ; & 3\\ 1 & 1 & 2 & 0 & 0 & 0 & ; & 2\\ 0 & 0 & 0 & 1 & 0 & 1 & ; & 1\\ 0 & 0 & 0 & 1 & 1 & 2 & ; & 1\end{bmatrix}.$$

By solving this system, we have

$$\mathbf{A}=[2\quad -2\quad 1\quad 2\quad 1\quad -1].$$

By substituting the elements of this vector into Eq. (10), we have $y_1(x) = x^2 - 2x + 3$, $y_2(x) = -x^2 + x + 1$ which are the exact solutions of the system (38).

Example 2 [25]. Consider the system of the linear Fredholm integro-differential equations

$$\begin{cases} y_1^{(2)}(x)-xy_2^{(1)}(x)-y_1(x)=(x-2)\sin(x)+\int_0^1(x\cos(t)y_1(t)-x\sin(t)y_2(t))dt,\\ y_2^{(2)}(x)-2xy_1^{(1)}(x)+y_2(x)=-2x\cos(x)+\int_0^1(\sin(x)\cos(t)y_1(t)-\sin(x)\sin(t)y_2(t))dt,\quad 0\leqslant x\leqslant 1,\\ y_1(0)=0,\quad y_1^{(1)}(0)=1,\quad y_2(0)=1,\quad y_2^{(1)}(0)=0,\end{cases}\tag{39}$$

where the exact solutions are $y_1(x) = \sin(x)$, $y_2(x) = \cos(x)$.

Here, $l = 2$, $m = 2$,

$$\mathbf{P_0}(x)=\begin{bmatrix}-1 & 0\\ 0 & 1\end{bmatrix},\quad \mathbf{P_1}(x)=\begin{bmatrix}0 & -x\\ -2x & 0\end{bmatrix},\quad \mathbf{P_2}(x)=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix},$$

$$g_1(x) = (x-2)\sin(x),\quad \text{and}\quad g_2(x) = -2x\cos(x).$$

From Eq. (31), the fundamental matrix equation of the problem is

$$\{\mathbf{P_0}\widetilde{\mathbf{X}}\bar{\mathbf{B}}+\mathbf{P_1}\widetilde{\mathbf{X}}\bar{\mathbf{D}}\bar{\mathbf{B}}+\mathbf{P_2}\widetilde{\mathbf{X}}\bar{\mathbf{D}}^2\bar{\mathbf{B}}-\widetilde{\mathbf{X}}\bar{\mathbf{B}}\mathbf{k}_f\bar{\mathbf{Q}}\}\mathbf{A}=\mathbf{G}.$$

Therefore, following the method given in Section 4, we obtain the approximate solution by the Fibonacci polynomials of the problem for $i = 1, 2$ and $N = 3, 7$ and 12, respectively,

$$y_{1,3}(x) = x - 0.163804801746x^3 - 8.68870193185\times 10^{-17},$$

$$y_{2,3}(x) = 0.0267372183356x^3 - 0.5x^2 + 1,$$

$$\begin{aligned}y_{1,7}(x)=&\,0.00834606099211x^5-0.000020155203787x^6\\ &-0.000185009340016x^7-0.00000393305206188x^4\\ &-0.166666480663x^3-7.97769511042\times 10^{-17}x^2\\ &+x-(1.47244819419)\times 10^{-16},\end{aligned}$$

$$\begin{aligned}y_{2,7}(x)=&\,0.0000691369047125x^7-0.00146639307647x^6\\ &+0.0000436530082905x^5+(0.0416540244004)x^4\\ &+0.0000125457168515x^3-0.5x^2+1.0,\end{aligned}$$

$$\begin{aligned}y_{1,12}(x)=&\,0.00000000832179282775x^{12}\\ &-0.0000000269915206247x^{11}\\ &+0.0000000265355217704x^{10}\\ &+0.00000275337650791x^9\\ &+0.0000000014181104426x^8-0.000198413286482x^7\\ &+0.00000000166850643149x^6\\ &+0.00833333330223x^5\\ &+0.00000000000360545427717x^4\\ &-0.166666666666x^3-1.08637630609\times 10^{-16}x^2+x\\ &-2.74526109659\times 10^{-16},\end{aligned}$$

$$\begin{aligned}y_{2,12}(x)=&\,0.0000000190356058937x^{12}\\ &+0.0000000058571030221 8x^{11}\\ &-0.00000027649040581x^{10}\\ &+0.000000000878991687648x^9\\ &+0.0000248010327353x^8\\ &+0.0000000000236146638732x^7\\ &-0.00138888895646x^6\\ &+0.000000000000128633446342x^5\\ &+0.0416666666652x^4\\ &+0.000000000000131119872496x^3-0.5x^2\\ &+2.00591298544\times 10^{-23}x+1.0.\end{aligned}$$

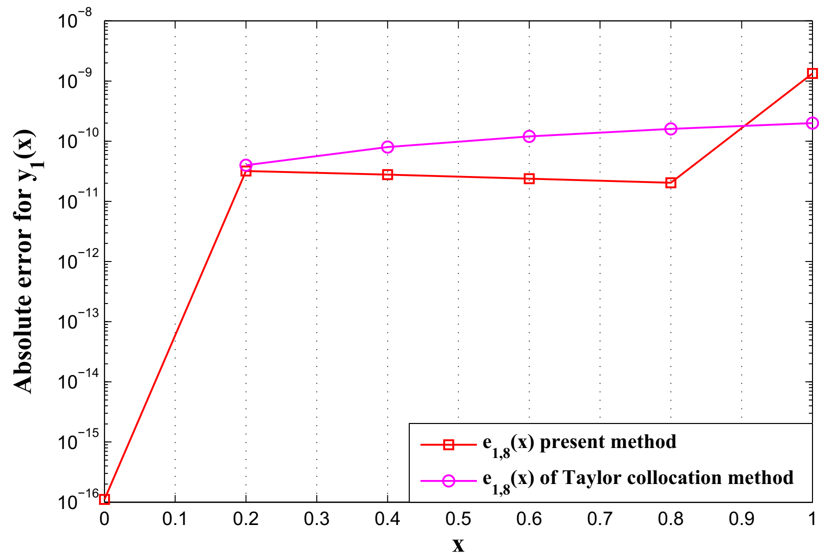

Fig. 6 Comparison of the absolute error functions $e_{1,N}(x)$ of Eq. (41).

Tables 1 and 2 show the numerical solutions and the exact solution of Eq. (39) by the presented method for N = 3, 7, 12. As can be seen from Tables 1 and 2 the results of the solutions obtained by Fibonacci polynomial method for $N = 12$ are almost the same as the results of the exact solutions.

Tables 3 and 4, give the comparison of the result of the maximum error of $e_{i,N}(x)$, $i = 1, 2$ obtained by the present method and the method of [25] for different values of N. From Tables 3 and 4, we see the errors decrease rapidly as N increases.

Figs. 2 and 3 display the absolute error functions obtained by the present method for N = 3, 7, 12.

Example 3 ([26–29]). Consider the system of the linear Fredholm integral equations

$$\begin{cases} y_1(x) = g_1(x) - \int_0^1 e^{x-t} y_1(t)dt - \int_0^1 e^{(x+2)t} y_2(t)dt, \\ \qquad 0 \leqslant x \leqslant 1, \\ y_2(x) = g_2(x) - \int_0^1 e^{xt} y_1(t)dt - \int_0^1 e^{(x+t)} y_2(t)dt, \end{cases} \tag{40}$$

where $g_1(x) = 2e^x + \frac{e^{x+1}-1}{x+1}$ and $g_2(x) = e^x + e^{-x} + \frac{e^{x+1}-1}{x+1}$ with exact solution $y_1(x) = e^x$, $y_2(x) = e^{-x}$. Here,

$$l = 2, m = 2, \quad \mathbf{P_0}(x) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

From Eq. (31), the fundamental matrix equation of the problem is

$$\{\mathbf{P_0}\widetilde{\mathbf{X}}\overline{\mathbf{B}} - \widetilde{\mathbf{X}}\overline{\mathbf{B}}\mathbf{K}_f\overline{\mathbf{Q}}\}\mathbf{A} = \mathbf{G}.$$

Tables 5 and 6 give the comparison of the results of the absolute error functions obtained by the present method for N = 15, the BPF method [26,29], the TF method [26], the

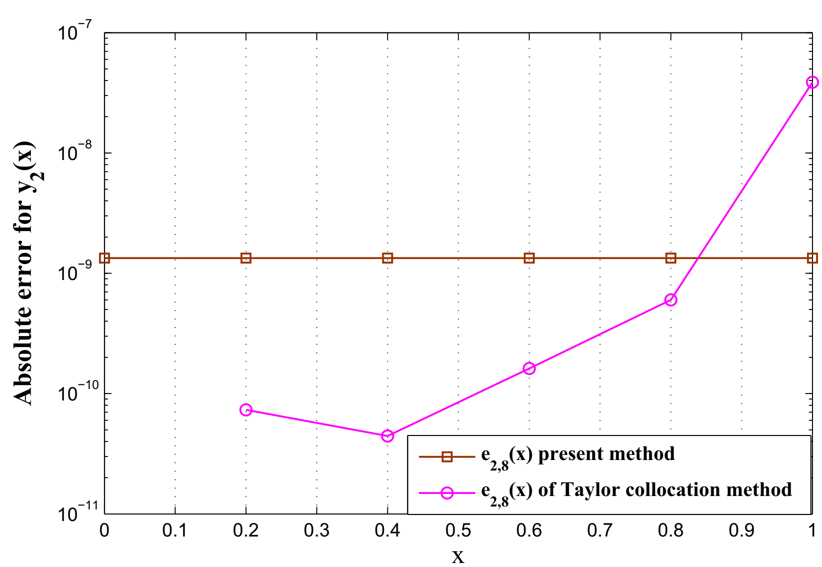

Fig. 7 Comparison of the absolute error functions $e_{2,N}(x)$ of Eq. (41).

Table 7 Numerical results of the absolute error functions $e_{1,N}(x)$, $e_{2,N}(x)$ of Eq. (41)

x	Present method		Taylor collocation method [30]	
	$e_{1,8}(x)$	$e_{2,8}(x)$	$e_{1,8}(x)$	$e_{2,8}(x)$
0.0	1.1102e−16	0.0	0.0	0.0
0.2	3.2057e−11	3.2056e−11	0.40e−10	0.731938e−10
0.4	2.7775e−11	2.7770e−11	0.80e−10	0.444101e−10
0.6	2.3878e−11	2.3858e−11	0.12e−09	0.161688e−09
0.8	2.0362e−11	2.0308e−11	0.16e−09	0.599938e−09
1.0	1.3375e−09	1.3376e−09	0.20e−09	0.387527e−07

DF method [27], and the RHF method [28] for $N = 32$ of Eq. (40). It is seen from Tables 5 and 6 that the results obtained by the present method are better than those obtained by the others. Figs. 4 and 5 display the absolute error functions obtained by the present method for $N = 15$ and the BPF method [26,27], the TF method [26], the DF method [27], and the RHF method [28] for $N = 32$.

Example 4 [30]. Consider the system of the linear differential equations

$$\begin{cases} y_1^{(1)}(x) + y_2^{(1)}(x) + y_1(x) = 1, \\ y_2^{(1)}(x) - 2y_1(x) - y_2(x) = 0, \qquad 0 \leqslant x \leqslant 1, \\ y_1(0) = 0, \quad y_2(0) = 1, \end{cases} \tag{41}$$

where the exact solutions are $y_1(x) = e^{-x} - 1$, $y_2(x) = 2 - e^{-x}$.

So that $l = 2$, $m = 1$, $\mathbf{P_0}(x) = \begin{bmatrix} 1 & 1 \\ -2 & -1 \end{bmatrix}$, $\mathbf{P_1}(x) = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, $g_1(x) = 1$, and $g_2(x) = 0$.

From Eq. (31), the fundamental matrix equation of the problem is

$$\{\mathbf{P_0}\widetilde{\mathbf{X}}\bar{\mathbf{B}} + \mathbf{P_1}\widetilde{\mathbf{X}}\bar{\mathbf{B}}\bar{\mathbf{D}}\}\mathbf{A} = \mathbf{G}.$$

Table 7 shows the numerical results of the absolute error functions obtained by present method and the Taylor collocation method [30] of the system in Eq. (41) for $N = 8$. Also, we compare the absolute error functions obtained by these two methods of the system in Eq. (41) for $N = 8$ in Figs. 6 and 7.

7. Conclusion

In this paper, we have worked out a computational method for approximate solution of high-order system of linear integro-differential equations, based on the expansion of the solution as a series of Fibonacci polynomials. This expansion, beside the collocation method have been used for transforming a system of high-order linear integro-differential to a linear system of algebraic equations that can be solved easily. To obtain the best approximating solution of the system, we take more forms from the Fibonacci expansion of functions, that is, the truncation limit N must be chosen large enough. In addition to these, an interesting feature of this method is to find the analytical solutions if equation has an exact solution that is a polynomial functions. Illustrative examples are given to demonstrate the validity and applicability of proposed method.

References

[1] Xu L, He JH, Liu Y. Electrospun nanoporous spheres with Chinese drug. Int J Nonlinear Sci Numer Simul 2007;8(2):199–202.
[2] Wang H, Fu HM, Zhang HF. A practical thermodynamic method to calculate the best glass-forming composition for bulk metallic glasses. Int J Nonlinear Sci Numer Simul 2007;8(2):171–8.
[3] Sun FZ, Gao M, Lei SH. The fractal dimension of the fractal model of dropwise condensation and its experimental study. Int J Nonlinear Sci Numer Simul 2007;8(2):211–22.
[4] Bo TL, Xie L, Zheng XL. Numerical approach to wind ripple in desert. Int J Nonlinear Sci Numer Simul 2007;8(2):223–8.
[5] Bellomo N, Firmani B, Guerri L. Bifurcation analysis for a nonlinear system of integro-differential equations modelling tumor-immune cells competition. Appl Math Lett 1999;12:39–44.
[6] Wazwaz AM. The existence of noise terms for systems of inhomogeneous differential and integral equations. Appl Math Comput 2003;146:81–92.
[7] El-Sayed SM, Kaya D, Zarea S. The decomposition method applied to solve high order linear Volterra–Fredholm integro-differential equations. Int J Nonlinear Sci Numer Simul 2004;5(2):105–12.
[8] El-Shahed M. Application of He's homotopy perturbation method to Volterra's integro-differential equation. Int J Nonlinear Sci Numer Simul 2005;6(2):163–8.
[9] Ghasemi M, Tavassoli Kajani M, Babolian E. Application of He's homotopy perturbation method to nonlinear integro-differential equation. Appl Math Comput 2007;188:538–48.
[10] Saberi-N J, Tamamgar M. The variational iteration method: a highly promising method for solving the system of the integro-differential equations. Comput Math Appl 2008;56:346–51.
[11] Dascoglua AA, Sezer M. Chebyshev polynomial solutions of systems of higher-order linear Fredholm–Volterra integro-differential equations. J Franklin Inst 2005;342:688–701.
[12] Maleknejad K, Mirzaee F, Abbasbandy S. Solving linear integro-differential equations system by using rationalized Haar functions method. Appl Math Comput 2004;155:317–28.
[13] Maleknejad K, Tavassoli Kajani M. Solving linear integro-differential equation system by Galerkin methods with hybrid functions. Appl Math Comput 2004;159:603–12.
[14] Pour-Mahmoud J, Rahimi-Ardabili MY, Shahmorad S. Numerical solution of the system of Fredholm integro-differential equations by the Tau method. Appl Math Comput 2005;168:465–78.
[15] Arikoglu A, Ozkol I. Solutions of integral and integro-differential equation systems by using differential transform method. Comput Math Appl 2008;56:2411–7.
[16] Baker CTH, Tang A. Stability analysis of continuous implicit Runge–Kutta methods for Volterra integro-differential systems with unbounded delays. Appl Numer Math 1997;24:153–73.
[17] Ganesh M, Sloan IH. Optimal order spline methods for nonlinear differential and integro-differential equations. Appl Numer Math 1999;29:445–78.

[18] Maleknejad K, Safdari H, Nouri M. Numerical solution of an integral equations system of the first kind by using an operational matrix with block pulse functions. Int J Syst Sci 2011;42(1):195–9.
[19] Al-Faour OMA, Saeed RK. Solution of a system of linear Volterra integral and integro-differential equations by spectral method. Al-Nahrain Uni J Sci 2006;6(2):30–46.
[20] Jangveladze T, Kiguradze Z, Neta B. Finite difference approximation of a nonlinear integro-differential system. Appl Math Comput 2009;215:615–28.
[21] Mirzaee F, Hoseini SF. Solving singularly perturbed differential–difference equations arising in science and engineering whit Fibonacci polynomials. Resu Phys 2013;3:134–41.
[22] Grimm RE. The autobiography of leonardo pisano. Fibonacci Quart 1973;11(1):99–104.
[23] Falcon S, Plaza A. On k-Fibonacci sequences and polynomials and their derivatives. Chaos Soliton Fract 2009;39:1005–19.
[24] Kurt N, Sezer M. Polynomial solution of high-order linear Fredholm integro-differential equations with constant coefficients. J Franklin Ins 2008;345:839–50.
[25] Yuzba S, Sahin N, Sezer M. Numerical solutions of systems of linear Fredholm integro-differential equations with Bessel polynomial bases. Comput Math Appl 2011;61:3079–96.
[26] Almasieh H, Roodaki M. Triangular functions method for the solution of Fredholm integral equations system. Ain Shams Eng J 2012;3:411–6.
[27] Almasieh H, Roodaki M. Delta basis functions and their applications to systems of integral equations. Comput Math Appl 2012;63:100–9.
[28] Maleknejad K, Mirzaee F. Numerical solution of linear Fredholm integral equation system by rationalized Haar functions method. Int J Comput Math 2003;80(11):1397–405.
[29] Maleknejad K, Shahrezaee M, Khatami H. Numerical solution of integral equations system of the second kind by block-pulse functions. Appl Math Comput 2005;166:15–24.
[30] Gokmen E, Sezer M. Taylor collocation method for systems of high-order linear differential–difference equations with variable coefficients. Ain Shams Eng J 2013;4:117–25.

Farshid Mirzaee is an Assistant Professor of applied mathematics of Malayer University, Malayer, Iran. His research interests are numerical analysis, Integral equations,Wavelets, Orthogonal functions, Ordinary differential equations, Fuzzy integral equations and Preconditioners

Seyede Fatemeh Hoseini, Department of Mathematics, Faculty of Science, Malayer University, Malayer, Iran. She is currently PH. D. student under the supervision of Assi. Prof. Farshid Mirzaee. Her interests include numerical analysis, Ordinary differential equations, Wavelets and Integral equations.