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Mathematical Representations of Sociolinguistic Restraints on Three-Person Conversations

VERN S. POYTHRESS

Clare College, University of Cambridge, Cambridge, England

This paper applies group theory and a statistical analysis by questions to the examination of sociolinguistic restraints on three-person conversation. It is shown in four situations of increasing generality that conversational invariance under permutation of participant roles implies restriction of conversational changes to description by a small subgroup of all possible transformations. The last of the four situations is n-person conversation; hence, the mathematical techniques here used are applicable to situations of greater complexity than the three-person conversations on which the present article focuses. A final section discusses possible applications to situations in descriptive phonology and grammar.

1. INTRODUCTION

Imagine an overheard conversation: "As long as Bill keeps interrupting with sarcastic remarks, we'll never be able to conduct a serious discussion."

Even a casual inspection of cocktail-party conversation or the procedure of a business meeting easily uncovers regularities in the interchange of speakers and addressees (cf. Pike, 1973; Poythress, 1973). Such regularities are in part statistical, since deviations in the form of interruptions or social surprises occur from time to time. But a mere statistical inventory of who is likely to speak to whom can be unilluminating. From the standpoint of sociolinguistics, we should like to know *why* statistical patterns fall out as they do.

This paper applies several mathematical tools to the *same* conversational situation. By doing so, it shows a way in which naïve insights into linguistic regularities can be transformed into more exact statements about statistical patterns.

2. TRIAL AND ERROR APPROACH TOWARD A GROUP-THEORETIC MODEL

First, let us summarize the approach that has been taken to conversational interchange in previous literature (see particularly Wise and Lowe, 1972;

Pike, 1973; Poythress, 1973). For the sake of simplicity suppose that we are dealing with a conversation among three persons, A(be), B(ill), and C(harlie) (see Section 6 for a relaxation of this assumption). Let a *snapshot* of the system be a specification of who is speaker and who is addressee. Suppose further that only one person can speak at a given time, and must address exactly one other person. Then there are exactly six possible snapshots,

$$1: \begin{pmatrix} S & A & X \\ A & B & C \end{pmatrix}, \quad 2: \begin{pmatrix} S & A & X \\ A & C & B \end{pmatrix}, \quad 3: \begin{pmatrix} S & A & X \\ B & A & C \end{pmatrix}, \quad 4: \begin{pmatrix} S & A & X \\ B & C & A \end{pmatrix},$$

$$5: \begin{pmatrix} S & A & X \\ C & A & B \end{pmatrix}, \quad 6: \begin{pmatrix} S & A & X \\ C & B & A \end{pmatrix}.$$

Here S, A, and X denote the roles of *s*peaker, *a*ddressee, and third party, respectively. $\left(\begin{smallmatrix} S & A & X \\ A & B & C \end{smallmatrix}\right)$ denotes the snapshot where A is speaker and B is addressee: A is speaking to B while C listens. Similarly, $\left(\begin{smallmatrix} S & A & X \\ C & A & B \end{smallmatrix}\right)$ denotes the snapshot where C speaks to A. For short, we may label the six snapshots 1, 2, 3, 4, 5, and 6 in the order above. Let $N = \{1, 2, 3, 4, 5, 6\}$ be the set of all snapshots.

Next, let a *transformation* g of the system be a permutation of the six snapshots; that is, a 1–1 map $g \in N^N$ of the snapshots N onto one another. Thus $g(i) = g(j)$ implies $i = j$. The transformations describe how the conversation moves from one stage to another. The set of all transformations forms the group S_6 under the operation of composition $((g \cdot h)(i) = g(h(i)))$. Moreover, the structure of this group is then a model for the structure of the progress of the conversation.

So far so good. However, not *all* transformations $g \in S_6$ are sociolinguistically relevant. Some describe highly unnatural sequences of snapshots. Hence, Pike's article (1973) spends considerable effort in examining what subgroups of S_6 and what generating elements in these subgroups might best represent *natural* conversation types. According to Pike's evaluation,[1] the most relevant elements of S_6 are the following:

(a) $I = \left(\begin{smallmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 2 & 3 & 4 & 5 & 6 \end{smallmatrix}\right) = (1)$, the identity permutation. The speaker remains speaker and the addressee remains addressee.

(b) $r = \left(\begin{smallmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 5 & 1 & 6 & 2 & 4 \end{smallmatrix}\right) = (13)(25)(46)$, the "reversal" transformation. The speaker becomes addressee and the addressee becomes speaker. The appropriateness of r means that the addressee has the social right to reply to the speaker.

[1] Using ideas and notation first developed in Pike and Lowe (1969) and Lowe (1969).

(c) $s = (12)(34)(56)$, the "shift" transformation. Here speaker continues as speaker, the addressee becomes listener, and the listener becomes addressee. The appropriateness of s means that the speaker can shift his attention from one person to the other. Somewhat less likely is

(d) $t = (16)(24)(35)$, where speaker and listener interchange. Pike points out that repeated use of t can occur in the social situation of "badgering." A and B (say) take turns scolding C (Pike, 1973, p. 135).

Any two of r, s, and t generate a subgroup H of order 6 of S_6, canonically isomorphic to the set of permutations on the roles S, A, and X. (The elements of H are I, r, s, $t = rsr$, $rs = (154)(236)$, $sr = (145)(263)$. Multiplication here proceeds from right to left.) We can also consider elements outside H, such as $w = (145)(236)$ and $g_0 = (165342)$. But repeated use of transformations w or g_0 results in quite unnatural conversational sequences.

Thus a relatively uncomplicated analysis of three-person conversation leads to a study of the subgroup H of S_6. However, the *reasons* for choosing H rather than the whole of S_6 have not yet been clarified. What sociolinguistic properties of three-person conversation have led us to the subgroup H? We now focus on formal *motivation* for choosing H out of S_6.

3. Group-Theoretic Analysis of Three-Person Conversation

To demonstrate the unique relevance of the subgroup H, we need only one postulate. Assume that the structure of conversation among A, B, and C is independent of the social roles of A, B, and C. That is, assume that conversational rules are invariant under an interchange or relabeling of the participants A, B, and C. This is equivalent to saying that permutations of A, B, and C are symmetries of the conversational system.

Let G be the set of all those elements of S_6 which are obtained by permuting the labels A, B, C in snapshots 1, 2, 3, 4, 5, 6. G is generated by

$$(AB) = (13)(24)(56) \quad \text{and} \quad (AC) = (16)(25)(34). \tag{1}$$

Let a be an arbitrary element of S_6. Then, if a is a sociolinguistically relevant transformation, it should be the case that a produces the same type of conversational change even when A, B, and C are relabeled. How can we express this mathematically? Let i be a snapshot, and $g \in G$ an arbitrary element of G. $g(i)$ represents a relabeling of the snapshot i. $a(g(i))$ is the transformation a of this relabeled snapshot. Similarly, $g(a(i))$ is a relabeling of the transformation a of i. If a is sociolinguistically relevant, it should make no difference when the relabeling is performed. Hence, $ag(i) = ga(i)$. Since i was arbitrary, $ag = ga$.

In sum, $a \in S_6$ is sociolinguistically relevant only if $ag = ga$ for all $g \in G$. By definition, this is so if and only if a is in the centralizer of G in S_6. And the subgroup H of S_6 is precisely the centralizer of G.

4. Transition Probabilities for Three-Person Conversations

Now we apply the insights of Section 3 to the more complex situation where probabilities of various transformations of snapshots are introduced. Suppose that the conversation has stages 1, 2,..., n, and that a description of the conversation consists in the specification of a snapshot $s_p \in N$ for each stage p, $p = 1, 2,..., n$. Given that the system is in snapshot j at stage p, let $c^p(i, j)$ be the probability that the system will make a transition to snapshot i at stage $p + 1$. Let C^p denote the 6×6 matrix $\| c^p(i, j)\|_{i,j}$.

Suppose further that the probabilities $c^p(i, j)$ and $c^q(k, l)$ associated with two distinct stages $p \neq q$ are independent. Then the product of the matrices $C^m ... C^3C^2C^1$ gives the probabilities of transition from stage 1 to stage $m + 1$.

In particular, suppose that x_i $(i = 1, 2,..., 6)$ is the probability that the system will be found in snapshot i at stage 1. Let $X = [x_1x_2 \cdots x_6]$ be the column matrix of x_i's. Then the probability y_i that the system will be in snapshot i at stage $m + 1$ is given by $Y = C^m \cdots C^2C^1X$, where $Y = [y_1y_2 \cdots y_6]$ (column matrix).

Now let us suppose that the course of conversations is invariant under relabeling of the participants A, B, and C. What restrictions does this place on the matrices C^p? First we must specify how a permutation of participants A, B, and C affects a probability matrix $X = [x_1 \cdots x_6]$.

Let $\delta(i, j)$ be the Kronecker delta: $\delta(i, j) = 0$ for $i \neq j$, and $\delta(i, j) = 1$ for $i = j$. For $a \in S_6$, let $f^a(i, j) = \delta(i, a(j))$, where $a(j)$ is the image of j under the permutation a of N. Let $U_a = \| f^a(i, j)\|$. Then for any column matrix X, $U_aX = Y$ implies $y_i = \sum_j f^a(i, j)x_j = \sum_j \delta(i, a(j))x_j$. Let $k = a^{-1}(i)$. Then $y_i = \sum_j \delta(a(k), a(j))x_j = \sum_j \delta(k, j)x_j = x_k$. Thus the new column vector Y is obtained from X by permuting the probabilities x_i, or, equivalently, by permuting the indices i. The operation U_a is equivalent to relabeling each snapshot k as $a(k)$.

Now any permutation of A, B, and C induces a permutation $g \in G \subsetneq S_6$ of snapshots. It should make no difference when the relabeling takes place, whether before $(C^q \cdots C^pU_g)$ or after $(U_gC^q \cdots C^p)$ the conversational transformations $(C^q \cdots C^p)$. Hence the condition that the conversation probabilities be independent of the labeling of A, B, and C is equivalent to the following: for all $g \in G$, and all positive integers p, q such that $1 \leqslant p \leqslant q < n$,

$$U_g C^q \cdots C^{p+2} C^{p+1} C^p = C^q \cdots C^{p+1} C^p U_g . \tag{2}$$

In particular, it follows from (2) with $p = q$ that $U_g C^p = C^p U_g$. In terms of elements, this means that for all i and k,

$$\sum_j f^g(i, j)\, c^p(j, k) = \sum_j c^p(i, j)\, f^g(j, k).$$

Substituting the definition of $f^a(i, j)$,

$$\sum_j \delta(i, g(j))\, c^p(j, k) = \sum_j c^p(i, j)\, \delta(j, g(k)).$$

Letting $g^{-1}(i) = l$,

$$\sum_j \delta(g(l), g(j))\, c^p(j, k) = \sum_j c^p(i, j)\, \delta(j, g(k))$$

$$\sum_j \delta(l, j)\, c^p(j, k) = \sum_j c^p(g(l), j)\, \delta(j, g(k))$$

$$c^p(l, k) = c^p(g(l), g(k)).$$

These relations hold for all $g \in G$. Hence the matrix C^p may look something like the following:

$$C_1 = (1/21) \begin{pmatrix} 6 & 4 & 5 & 1 & 2 & 3 \\ 4 & 6 & 2 & 3 & 5 & 1 \\ 5 & 1 & 6 & 4 & 3 & 2 \\ 2 & 3 & 4 & 6 & 1 & 5 \\ 1 & 5 & 3 & 2 & 6 & 4 \\ 3 & 2 & 1 & 5 & 4 & 6 \end{pmatrix}. \tag{3}$$

To make the patterning of this matrix visible, it is convenient to rearrange its entries, according to the method developed by Pike and others (1963, 1964, 1968) to deal with morphological data. Instead of indexing the entries $c(i, j)$ by initial snapshot j and final snapshot i, we can choose to index them by initial snapshot j plus the relation between initial and final snapshot; or by final snapshot i plus such a relation. Such a rearrangement is one way of focusing on the relation between initial and final snapshots.

For this purpose, define $b(v, j) = c^p(v(j), j)$ for all $j \in N$ and all $v \in S_6$.

Whereas the c's are indexed by i and j, b's are indexed by final snapshot j and an element v which measures the relation of snapshots. Now the conditions $c^p(i,j) = c^p(g(i), g(j))$ can be rewritten in terms of b's. $b(v,j) = c^p(v(j), j) = c^p(g(v(j)), g(j)) = c^p(gv(j), g(j))$, when we have taken $i = v(j)$. Moreover, $c^p(gv(j), g(j)) = b(v', g(j))$ for v' such that $v'g(j) = gv(j)$. This must be true for all j, so $v'g = gv$, $v' = gvg^{-1}$.

The pattern of equal probabilities will become visible if $b(v,j)$ and $b(v', g(j))$ belong to the same row or to the same column. Since in general we cannot require $j = g(j)$, an arrangement in the same column is not possible. But if $v = v'$, $b(v,j)$ and $b(v', g(j))$ will be in the same row. But $v' = gvg^{-1}$, so $v = v'$ is equivalent to $v = gvg^{-1}$. *All* of the pattern will be visible if this holds for all $g \in G$. Hence once again v should be in the centralizer H of G in S_6.

If C_1 is as in (3), the corresponding matrix $B = \| b(v,j)\|_{v\in H, j\in N}$ is

$$
\begin{array}{c c}
 & \begin{array}{cccccc} j=1 & j=2 & j=3 & j=4 & j=5 & j=6 \end{array} \\
\begin{array}{c} v=I \\ v=r \\ v=s \\ v=rsr \\ v=sr \\ v=rs \end{array} &
\begin{pmatrix}
6 & 6 & 6 & 6 & 6 & 6 \\
5 & 5 & 5 & 5 & 5 & 5 \\
4 & 4 & 4 & 4 & 4 & 4 \\
3 & 3 & 3 & 3 & 3 & 3 \\
2 & 2 & 2 & 2 & 2 & 2 \\
1 & 1 & 1 & 1 & 1 & 1
\end{pmatrix}
\end{array}
$$

Thus we have again explained the sociolinguistic relevance of the subgroup H. One further refinement of method will now give us even greater generality.

5. Question Analysis of Three-Person Conversation

The argument of Section 4 depended on the assumption that the probabilities of transition $c^p(i,j)$ were independent of the previous history of the conversation (cf. the "nonhistorical" conversation types of Poythress (1973)). In fact, this can be at best only an approximation to the truth. But the argument of Section 4 can be generalized to include more general cases, once we have altered our notational devices.

We wish to speak in general terms about states α of a sociolinguistic system. In intuitive terms, a *state* α is a collection of items of information which enable one to make at least probabilistic predictions and postdictions about the system. For example, α might be a classical or quantum mechanical density function which specifies a statistical ensemble of systems each of which is in some one atomic state. More precisely, a state α is considered to be completely described by a function $F_\alpha(Q)$ which, given any appropriate yes–no question Q about the system, gives the probability $F_\alpha(Q)$ that the analyst's answer to Q is "yes" when the system is in state α. For all α and Q, $0 \leqslant F_\alpha(Q) \leqslant 1$. Two states α and α' are considered identical iff $F_\alpha(Q) = F_{\alpha'}(Q)$ for all questions Q in the relevant class of questions.

In the case of three-person conversations, we may suppose that the state α is fixed for any given type of conversation and any given participants A, B, and C. The relevant questions are of the type $Q_i{}^p$: "At stage p, is the conversation in snapshot i?" Note that the probabilities $F_\alpha(Q_i{}^p)$ are not independent of one another. Let $Q_i{}^pQ_j{}^q$ denote the question, "Is it true both that at p the conversation is in snapshot i and that at stage q the conversation is in snapshot j?" $Q_i{}^pQ_j{}^q$ is to be answered "yes" if and only if both $Q_i{}^p$ and $Q_j{}^q$ are answered "yes."

Now suppose that the type of conversation in question is invariant under relabeling of A, B, and C. What condition should this impose on the probabilities $F_\alpha(Q_i{}^pQ_j{}^q)$? In a manner analogous to Sections 3 and 4, invariance has to do with a relation between the original probability $F_\alpha(Q_i{}^pQ_j{}^q)$ and a probability associated with the situation altered by a permutation $g \in G$. As before, $G \subseteq S_6$ is the subgroup of S_6 generated by (AB) and (AC) of (1). The questions $Q^p_{g(i)}$ $(i \in N)$ are the result of relabeling A, B, and C, according to the permutation g, in stage p. $Q^p_{g(i)}Q^q_{g(j)}$ represents a question where the labels have been altered *both* in stage p *and* in stage q. We might expect that $F_\alpha(Q_i{}^pQ_j{}^q) = F_\alpha(Q^p_{g(i)}Q^q_{g(j)})$ for all i and j, if the probability of going from snapshot j in stage q to snapshot i in stage p is really independent of the labeling A, B, C. However, such a condition would be too strong, since for various reasons the initial snapshots j in stage q may not be equally probable.

Actually, the best that we can expect is that the conditional probabilities will be equal:

$$F_\alpha(Q_i{}^p \mid Q_j{}^q) = F_\alpha(Q^p_{g(i)} \mid Q^q_{g(j)})$$

where $F_\alpha(Q_i{}^p \mid Q_j{}^q) = F_\alpha(Q_i{}^pQ_j{}^q)/F_\alpha(Q_j{}^q)$ is the probability that $Q_i{}^p$ is yes if $Q_j{}^q$ is, and $F_\alpha(Q^p_{g(i)} \mid Q^q_{g(j)}) = F_\alpha(Q^p_{g(i)}Q^q_{g(j)})/F_\alpha(Q^q_{g(j)})$ is the probability that $Q^p_{g(i)}$ is yes if $Q^q_{g(j)}$ is. Thus we have

$$F_\alpha(Q_j{}^q)/F_\alpha(Q^q_{g(j)}) = F_\alpha(Q_i{}^pQ_j{}^q)/F_\alpha(Q^p_{g(i)}Q^q_{g(j)}). \tag{4}$$

Since some of the probabilities $F_\alpha(Q_i{}^p)$ might be zero, we prefer to write (4) as

$$F_\alpha(Q_j{}^q)\, F_\alpha(Q^p_{g(i)}Q^q_{g(j)}) = F_\alpha(Q^q_{g(j)})\, F_\alpha(Q_i^p\, Q_j{}^q). \tag{5}$$

Now let us express the identity (5) as a relation purely among the 36 quantities $F_\alpha(Q_i{}^pQ_j{}^q)$ $(i, j = 1, 2, \ldots, 6)$ rather than as a relation between these quantities and the 6 quantities $F_\alpha(Q_j{}^q)$. Substituting k for i in (5), we have

$$F_\alpha(Q_j^q)\, F_\alpha(Q^p_{g(k)}Q^q_{g(j)}) = F_\alpha(Q^q_{g(j)})\, F_\alpha(Q_k^pQ_j^q). \tag{6}$$

Multiplying the right-hand side of (5) with the left-hand side of (6), and the left-hand side of (5) with the right-hand side of (6), we obtain

$$\begin{aligned} &F_\alpha(Q^q_{g(j)})\, F_\alpha(Q_j^q)\, F_\alpha(Q_i^p Q_j^q)\, F_\alpha(Q^p_{g(k)}Q^q_{g(j)}) \\ &\quad = F_\alpha(Q_j^q)\, F_\alpha(Q^q_{g(j)})\, F_\alpha(Q_k^p Q_j^q)\, F_\alpha(Q^p_{g(i)}Q^q_{g(j)}). \end{aligned}$$

If $F_\alpha(Q_j{}^q) \neq 0 \neq F_\alpha(Q^q_{g(j)})$,

$$F_\alpha(Q_i^pQ_j^q)\, F_\alpha(Q^p_{g(k)}Q^q_{g(j)}) = F_\alpha(Q_k^pQ_j^q)\, F_\alpha(Q^p_{g(i)}Q^q_{g(j)}). \tag{7}$$

On the other hand, if $F_\alpha(Q_j{}^q) = 0$, $F_\alpha(Q_i{}^pQ_j{}^q) = F_\alpha(Q_k{}^pQ_j{}^q) = 0$. If $F_\alpha(Q^q_{g(j)}) = 0$, $F_\alpha(Q^p_{g(k)}Q^q_{g(j)}) = F_\alpha(Q^p_{g(i)}Q^q_{g(j)}) = 0$. Hence in any case (7) holds.

Contrariwise, assume that (7) holds. Since the questions $Q_1{}^p, Q_2{}^p, Q_3{}^p, \ldots, Q_6{}^p$ are disjoint, and exactly one must have a yes answer, $F_\alpha(Q_j{}^q) = \sum_i F_\alpha(Q_i{}^pQ_j{}^q)$. Now

$$\begin{aligned} &F_\alpha(Q_j^q)\, F_\alpha(Q^p_{g(k)}Q^q_{g(j)}) \\ &\quad = \sum_i F_\alpha(Q_i^pQ_j^q)\, F_\alpha(Q^p_{g(k)}Q^q_{g(j)}) \\ &\quad = \sum_i F_\alpha(Q_k^pQ_j^q)\, F_\alpha(Q^p_{g(i)}Q^q_{g(j)}) = F_\alpha(Q_k^pQ_j^q) \sum_i F_\alpha(Q^p_{g(i)}Q^q_{g(j)}) \\ &\quad = F_\alpha(Q_k^pQ_j^q)\, F_\alpha(Q^q_{g(j)}). \end{aligned}$$

The last equality holds since $g(i)$ runs through the integers N as i does. Hence (7) implies (5). Hence (5) and (7) are equivalent. Thus we can confine ourselves to a study of (7), which deals exclusively with the 36 quantities $F_\alpha(Q_i{}^pQ_j{}^q)$.

Treating the quantities $a(i, j) = F_\alpha(Q_i{}^pQ_j{}^q)$ as matrix elements of $A = \| a(i, j)\|_{i,j}$, we can once again attempt to reorder the matrix until parallel elements appear in the same row or column. However, (7) is not a strong enough condition to guarantee that any two $a(i, j)$'s are equal. So collecting "parallel" elements does not mean collecting *equal* elements but merely proportional elements. The best we can hope for is a condition such as

$$b(i, j)\, b(k, l) = b(i, l)\, b(k, j) \tag{8}$$

when a's are properly rearranged.

The condition (8) is important because it is equivalent to the independence of the probabilities of rows and columns, in the following sense.

LEMMA. *Let* $B = \| b(i,j) \|$ *be an* $m \times n$ *matrix with entries* $b(i,j)$ *in a field* K. *Consider the following*:

(i) B *is of rank* 1;

(ii) $b(i,j)\, b(k,l) = b(i,l)\, b(k,j)$ *for all* $1 \leqslant i,\ k \leqslant m,\ 1 \leqslant j,\ l \leqslant n$;

(iii) *there exist* a_i $(1 \leqslant i \leqslant m)$ *and* c_j $(1 \leqslant j \leqslant n)$ *such that* $b(i,j) = a_i c_j$;

(iv) *there exists a column matrix* A *and a row matrix* C *such that* $B = AC$;

(v) *there exist a set of mutually exclusive events* E_i $(1 \leqslant i \leqslant m)$, *a set of mutually exclusive events* F_j $(1 \leqslant j \leqslant n)$, *and a constant* $c > 0$, *such that each event* E_i *is independent of each* F_j, *and such that* $cb(i,j)$ *is the probability of joint occurrence of* E_i *and* F_j.

Then (i), (ii), (iii), *and* (iv) *are all equivalent. Moreover, if the* $b(i,j)$ *are all nonnegative real numbers*, (v) *is equivalent to any one of* (i)–(iv).

Proof. (i), (iii), and (iv) are trivially equivalent. (v) follows easily from (iii) by picking E_i and F_j so that $p(E_i) = a_i(c)^{1/2}$, $p(F_j) = c_j(c)^{1/2}$. c can always be chosen small enough so that $\sum_i p(E_i)$ and $\sum_j p(F_j)$ are both less than 1. On the other hand, assuming that (v) holds, set $a_i = p(E_i)$, $c_j = (1/c)\, p(F_j)$, and (iii) follows.

Now it remains to show that (ii) is equivalent to (iii). Assume (iii). $b(i,j)\, b(k,l) = a_i c_j a_k c_l = a_i c_l a_k c_j = b(i,l)\, b(k,j)$. Hence (ii) holds. Now assume (ii). If all the $b(i,j)$'s are zero, the result is trivial. So suppose $b(u,v) \neq 0$. Let $c_j = b(u,j)$ and $a_i = b(i,v)/b(u,v)$. We claim $b(i,j) = a_i c_j$. For $a_i c_j = (b(i,v)/b(u,v))\, b(u,j) = (b(i,v)\, b(u,j))/b(u,v) = (b(i,j)\, b(u,v))/b(u,v) = b(i,j)$. Q.E.D.

Now let $u \in \{(i,j) \mid i,j \in \{1,2,\ldots,6\}\} = N \times N$ range over double indices i and j. Let S_{36} be the set of permutations of $N \times N$. Let $w \in S_{36}$ be one such permutation. Then $b(u) = a(w^{-1}(u))$ defines a matrix $B = \| b(i,j) \|$ which is a reordering of the entries of the matrix $A = \| a(i,j) \|$. If $w(i,j) = (i',j')$, w is uniquely determined by the two functions $w_1, w_2 : N \times N \to N$ such that $i' = w_1(i,j)$, $j' = w_2(i,j)$. That is, $w(i,j) = (w_1(i,j), w_2(i,j))$.

(7) Can be rewritten in terms of b's. $F_\alpha(Q_i{}^p Q_j{}^q) = a(i,j) = b(w(i,j))$; $F_\alpha(Q_k{}^p Q_j{}^q) = a(k,j) = b(w(k,j))$; $F_\alpha(Q^p_{g(i)} Q^q_{g(j)}) = a(g(i), g(j)) = b(w(g(i), g(j)))$; $F_\alpha(Q^p_{g(k)} Q^q_{g(j)}) = a(g(k), g(j)) = b(w(g(k), g(j)))$. Hence (7) implies

$$b(w(i,j))\, b(w(g(k), g(j))) = b(w(k,j))\, b(w(g(i), g(j))). \tag{9}$$

Now assume that w has been so chosen that (9) is the same as (8). In (8), any b on one side of the equation has a subscript (either row or column) in common with each b on the other side. Hence, in particular, in (9) $b(w(i, j))$ and $b(w(k, j))$ should belong either to the same row or to the same column. Since the equation (8) is invariant under an interchange of rows and columns, we may assume that $w(i, j)$ and $w(k, j)$ belong to the same column. Hence $w_2(i, j) = w_2(k, j)$. Since this is true for all i, $k \in N$, w_2 is independent of the first variable, and we may write $w_2(i, j) = w_2(j)$. If the second subscript of $b(w(i, j))$ matches the second subscript of $b(w(k, j))$, then by (8) the first subscript of $b(w(i, j))$ should match the first subscript of $b(w(g(i), g(j)))$. That is,

$$w_1(i, j) = w_1(g(i), g(j)). \tag{10a}$$

Similarly,

$$w_2(g(j)) = w_2(g(j)) \tag{10b}$$

$$w_1(g(k), g(j)) = w_1(k, j). \tag{10c}$$

The new column labels are a permutation $w_2(j)$ of the old column labels. But the new row labels $w_1(i, j)$ are a function both of i and of j. For fixed i', if $w_1(i_1, j) = w_1(i_2, j) = i'$, then $w(i_1, j) = w(i_2, j)$, hence $i_1 = i_2$. Hence the i such that $i' = w_1(i, j)$ is uniquely determined by i' and j. Let $i = h_{i'}(j)$ be this function, such that $i' = w_1(h_{i'}(j), j)$.

Now $w_1(i,j) = i_1$ where $h_{i_1}(j) = i$; $w_1(g(i), g(j)) = i_2$ where $h_{i_2}(g(j)) = g(i)$; $w_1(g(k), g(j)) = i_3$ where $h_{i_3}g(j) = g(k)$; $w_1(k, j) = i_4$ where $h_{i_4}(j) = k$. Equation (10) is equivalent to $i_1 = i_2$, $i_3 = i_4$. Hence $h_{i_1} = h_{i_2}$, $h_{i_3} = h_{i_4}$. This means that $h_{i_1}g(j) = gh_{i_1}(j)$, $h_{i_3}g(j) = gh_{i_3}(j)$. Hence the h's must satisfy $gh_i = h_ig$. Since $g \in G$ is arbitrary, h_i must commute with every element of G. However, we have not yet shown that $h_i : N \to N$ is a 1–1 function. Suppose $h_i(j_1) = h_i(j_2)$. Pick $g_1 \in G$ such that $g_1(j_1) = j_2$ (always possible). Then $g_1h_i(j_1) = h_ig_1(j_1) = h_i(j_2) = h_i(j_1)$. But all the elements of G except 1 (the identity permutation) have no fixed point. Hence $g_1 = 1$, $j_1 = j_2$, and h_i is 1–1. Hence h_i is a permutation in the centralizer of G. Hence $h_i \in H$.

6. Generalization to n-Person Conversation

The above analyses can be extended without too much difficulty to various types of man–person conversations. Let n persons be engaged in a conversation or a number of conversations. Let $M = \{1, 2,..., n\}$. A snapshot of the conversation should indicate who is speaking to whom. We may suppose that between any two persons A and B in the conversation, one of the following relations holds: (1) A is addressing B, (2) B is addressing A, (3) B is listening to A (but A is not directly addressing B), (4) A is listening to B, or (5) no relation at all. Denote these five relations by 1, 2, 3, 4, and 0, respectively. Let $P = \{1, 2, 3, 4, 0\}$. Then a snapshot will be a map j of $M \times M - \{(x, x) \mid x \in M\}$ into P, subject to the following restrictions:

(i) $j(x, y) = R(j(y, x))$, where $R(1) = 2$, $R(2) = 1$, $R(3) = 4$, $R(4) = 3$, $R(0) = 0$. (Thus the relation of A to B is the converse of the relation of B to A.)

(ii) any $x \in M$ can be either a speaker, an addressee, or a listener, but not two of these at the same time; i.e., $j(x, y) = 1$ and $j(y, z) = 1$, $j(x, y) = 1$ and $j(y, z) = 3$, $j(x, y) = 3$ and $j(y, z) = 1$, $j(x, y) = 3$ and $j(y, z) = 3$, $j(x, y) = 1$ and $j(z, y) = 3$ are all impossible.

If desired, further restrictions can be placed on j. Let N be the set of all such snapshots j. Let S_0 be the group of all permutations of N. Then sociolinguistically relevant transformations form a subset of S_0. Moreover, each permutation σ: $M \to M$ of participants $x \in M$ induces a permutation $g_\sigma \in S_0$ of snapshots in the obvious way. The set of all such permutations $\{g_\sigma \mid \sigma$ a permutation of $M\}$ forms a subgroup G of S_0, isomorphic to the symmetric group on n letters.

Assume now that the conversation type is invariant under interchange of all or perhaps only some participant roles. Let G_1 be the subgroup of G consisting of permutations that leave the conversation type invariant. As in Sections 2–5, this assumption implies an identity involving elements $g \in G_1$. The sociolinguistically relevant transformations $h \in S_0$ are the elements of the centralizer subgroup H of G_1 in S_0.

7. Possible Application to Phonology and Grammar

One of the advantages of the mode of argument developed in Section 5 is that it can be applied to problems superficially quite remote from speaker–addressee relationships. According to the Lemma of Section 5, the validity of $b(i, j)\, b(k, l) = b(i, l)\, b(k, j)$ implies the independence of events associated with rows and columns. Though *exact* independence may never be found in events associated with linguistics, a situation *approximating* independence is a common occurrence. As illustrations, we take two cases from English grammar.

Consider first a situation of free variation. The English bitransitive active clause has two variant forms,

Mary gave John the book.
Mary gave the book to John.

Let Q_1^0 be the question, "Does the indirect object come before the direct object in the given bitransitive clause?" Let Q_2^0 be the question, "Does the indirect object follow the direct object?" Let Q_j^1, for j ranging over an index set J, be a set of questions concerning the context of the clause, or concerning the subject, the verb, and the NPs constituting the indirect object and the direct object. Let $a(i, j) = F_\alpha(Q_i^0 Q_j^1)$, i.e., the probability of joint "yes" answers to Q_i^0 and Q_j^1. That $A = \| a(i, j)\|$ is of rank 1 (or "close to" a matrix of rank 1) is an assertion that the order of indirect and direct object is a matter of free variation. Thus free variation can be defined in terms of independence of two sets of questions.

Next, consider as an example of grammatically conditioned grammatical variation agreement of subject and verb in English clauses. Let $I = \{1, 2, 3, 4\}$. Let Q_i^0 $(i \in I)$ be questions concerning person and number of the subject. Q_1^0 (Q_2^0, Q_3^0) are, respectively, "Is the subject first (second, third) person singular?" Q_4^0 is "Is the subject plural?" Since plural verb forms are always alike, we need not subdivide Q_4^0 by person. Let Q_j^1 $(j \in I)$ be the corresponding questions concerning the verb phrase. Of course, only when the first verb in the verb phrase is a form of 'to be' will the verbal forms for first and second person singular be different. In other cases, when we deal with homophonous forms, we assume that the distinction between questions Q_1^1, Q_2^1, and Q_4^1 is determined by referring to the subject.

Let $a(i, j) = F_\alpha(Q_i^0 Q_j^1)$ be the probability of joint "yes" answers to Q_i^0 and Q_j^1. That, in general, $a(i, j)\ a(k, l) \neq a(i, l)\ a(k, j)$ indicates that agreement between subject and predicate is occurring (the variation is not "free"). However, invariance of the overall clause structure under changes of person and number places restrictions on the quantities $a(i, j)$. Let G be the group of permutations of I. Perfect invariance of structure would mean $a(i, j)\ a(g(i), g(k)) = a(i, k)\ a(g(i), g(j))$ for all $g \in G$. As a matter of fact, because of the occurrence of semologically influenced exceptions such as "The crowd are coming," the equality above will be only approximate. $A = \| a(i, j)\|$ can now be reordered by setting $b(i, j) = a(w(i, j))$ for some permutation $w \in S_{16}$ of ordered pairs (i, j). The entries of B will be properly alined when, say, the row labels $i \in I$ indicate the person and number, and the column labels $h \in H$ are in the centralizer of G in G. In this case the centralizer is the center, the identity subgroup. The column label $h = 1$

heads the column of all nonnegligible probabilities $a(i, i)$. Thus h is the "agreement" column, while other columns need no labels, as they are columns of negligible probabilities of "disagreement."

Even this analysis has a curious exception in dialects using the sequence 'Aren't *I*.' The occurrence of 'Aren't *I*' will show up as an irregularity in the matrix $A_1 = \| a_1(i, j) \|$ of probabilities $a_1(i, j) = F_\alpha(Q_i^0 Q_j^1 Q^2 Q^3)$, where Q^2 asks whether the clause in question is interrogative and Q^3 asks whether the verb phrase is negative. When either Q^2 or Q^3 is answered "no," the corresponding matrix is regular.

In any case, an instance of conditioned variation can be treated as an independence of row and column questions where the column questions have been rechosen to ask questions of correlation ("Does x_1 occurring in slot y_1 correlate with x_2 occurring in slot y_2?") rather than questions of occurrence ("Does x_1 occur in slot y_1?").

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