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Generalized intersection bodies are not equivalent

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Abstract

In [A. Koldobsky, A functional analytic approach to intersection bodies, Geom. Funct. Anal. 10 (2000) 1507–1526], A. Koldobsky asked whether two types of generalizations of the notion of an intersection body are in fact equivalent. The structures of these two types of generalized intersection bodies have been studied by the author in [E. Milman, Generalized intersection bodies, J. Funct. Anal. 240 (2) (2006) 530–567], providing substantial evidence for a positive answer to this question. The purpose of this note is to construct a counter-example, which provides a surprising negative answer to this question in a strong sense. This implies the existence of non-trivial non-negative functions in the range of the spherical Radon transform, and the existence of non-trivial spaces which embed in L_p for certain negative values of p.

Keywords: Generalized intersection bodies; Generalized Busemann–Petty problem; Non-trivial non-negative range of Radon transform; Non-trivial spaces which embed in L_p for negative values of p

1. Introduction

Let $\mathrm{Vol}(L)$ denote the Lebesgue measure of a set $L \subset \mathbb{R}^n$ in its affine hull, and let $G(n,k)$ denote the Grassmann manifold of k-dimensional subspaces of $\mathbb{R}^n$. Let D_n denote the Euclidean unit ball, and S^{n-1} the Euclidean sphere. All of the bodies considered in this note will be assumed to be centrally-symmetric star-bodies (even if the central-symmetry assumption is omitted). A centrally-symmetric star-body K is a compact set with non-empty interior such that $K = -K$,

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$tK \subset K$ for all $t \in [0,1]$, and such that its radial function $\rho_K(\theta) = \max\{r \geqslant 0 \mid r\theta \in K\}$ for $\theta \in S^{n-1}$ is an even continuous function on S^{n-1}. We denote the spaces of continuous and even continuous functions on S^{n-1} by $C(S^{n-1})$ and $C^e(S^{n-1})$, respectively.

This note concerns two generalizations of the notion of an *intersection body*, first introduced by E. Lutwak in [27] (see also [28]).

Definition. A star-body K is said to be an intersection body of a star-body L, if $\rho_K(\theta) = \mathrm{Vol}(L \cap \theta^\perp)$ for every $\theta \in S^{n-1}$, where $\theta^\perp$ is the hyperplane perpendicular to θ. K is said to be an intersection body, if it is the limit in the radial metric d_r of intersection bodies $\{K_i\}$ of star-bodies $\{L_i\}$, where $d_r(K_1, K_2) = \sup_{\theta \in S^{n-1}} |\rho_{K_1}(\theta) - \rho_{K_2}(\theta)|$.

We remark that the distinction in the above definition between the notion of intersection body of a star-body and the more general notion of intersection body is indeed essential. For instance, it was shown in [5,40] that convex polytopes in $\mathbb{R}^4$ are not intersection bodies of star-bodies, whereas all centrally-symmetric convex bodies in $\mathbb{R}^4$ *are* in fact intersection bodies, as shown in [41].

Let $R : C(S^{n-1}) \to C(S^{n-1})$ denote the Spherical Radon Transform, defined by

$$R(f)(\theta) = \int\limits_{S^{n-1} \cap \theta^\perp} f(\xi)\, d\sigma_\theta(\xi) \tag{1.1}$$

for $f \in C(S^{n-1})$, where σ_θ denotes the Haar probability measure on $S^{n-1} \cap \theta^\perp$. Let R^* denote the dual transform (as in (1.2) below). We will use the following characterization of intersection bodies (see [6,28]) as an equivalent definition:

Equivalent Definition. A star-body K is an intersection body iff $\rho_K = R^*(d\mu)$, where μ is a non-negative Borel measure on S^{n-1}.

The notion of an intersection body has been shown to be fundamentally connected to the Busemann–Petty problem (first posed in [3]), which asks whether two centrally-symmetric convex bodies K and L in $\mathbb{R}^n$ satisfying

$$\mathrm{Vol}(K \cap H) \leqslant \mathrm{Vol}(L \cap H) \quad \forall H \in G(n, n-1)$$

necessarily satisfy $\mathrm{Vol}(K) \leqslant \mathrm{Vol}(L)$. It was shown in [6,28,38] that the answer is equivalent to whether all centrally-symmetric convex bodies in $\mathbb{R}^n$ are intersection bodies, and in a series of results [1,2,6,7,9,12,18,24,31,41] that this is true for $n \leqslant 4$, but false for $n \geqslant 5$. We comment in passing that intersection bodies are still objects of significant current interest, see for example, [8,15,17,26,37].

In [39], G. Zhang considered a generalization of the Busemann–Petty problem, the so-called *generalized k-codimensional problem*, asking whether two centrally-symmetric convex bodies K and L in $\mathbb{R}^n$ satisfying

$$\mathrm{Vol}(K \cap H) \leqslant \mathrm{Vol}(L \cap H) \quad \forall H \in G(n, n-k)$$

for some fixed integer k between 1 and $n-1$, necessarily satisfy $\mathrm{Vol}(K) \leqslant \mathrm{Vol}(L)$. Zhang showed that this generalized problem is also naturally associated to a class of generalized intersection bodies, which will be referred to as *k-Busemann–Petty bodies*, and that the generalized k-codimensional problem is equivalent to whether all centrally-symmetric convex bodies in $\mathbb{R}^n$ are k-Busemann–Petty bodies. It was shown in [4] (see also [33]), and later in [21], that the answer is negative for $k < n-3$, but the cases $k = n-3$ and $k = n-2$ remain open (the case $k = n-1$ is obviously true).

Several partial answers to these unresolved cases are known, all in the positive direction. It was shown in [39] (see also [33]) that when K is a centrally-symmetric convex body of revolution and $k = n-2$ or $k = n-3$, then the answer is positive for the pair K, L, with any star-body L. This was recently extended in [32] (see also [23] for a related result), by showing that the same statement holds when K has more general axial symmetries. When $k = n-2$, it was shown in [4] that the answer is positive if L is a Euclidean ball and K is convex and sufficiently close to L. This was extended in [30], where it was shown that this is again true for $k = n-2$ and $k = n-3$, when L is an arbitrary star-body and K is sufficiently close to a Euclidean ball (but to an extent depending on its curvature). Several other generalizations of the Busemann–Petty problem were treated in [33,35,36,42].

Before defining the class of k-Busemann–Petty bodies we shall need to introduce the m-dimensional Spherical Radon Transform, acting on spaces of continuous functions as follows:

$$\begin{aligned} R_m : C\big(S^{n-1}\big) &\longrightarrow C\big(G(n,m)\big), \\ R_m(f)(E) &= \int\limits_{S^{n-1}\cap E} f(\theta)\, d\sigma_E(\theta), \end{aligned}$$

where σ_E is the Haar probability measure on $S^{n-1} \cap E$. It is well known (e.g. [16]) that as an operator on *even* continuous functions, R_m is injective. The dual transform is defined on spaces of *signed* Borel measures $\mathcal{M}$ by

$$\begin{gathered} R_m^* : \mathcal{M}\big(G(n,m)\big) \longrightarrow \mathcal{M}\big(S^{n-1}\big), \\ \int\limits_{S^{n-1}} f R_m^*(d\mu) = \int\limits_{G(n,m)} R_m(f)\, d\mu \quad \forall f \in C\big(S^{n-1}\big), \end{gathered} \tag{1.2}$$

and for a measure μ with continuous density g, the transform may be explicitly written in terms of g (see [39]):

$$R_m^* g(\theta) = \int\limits_{\theta\in E\in G(n,m)} g(E)\, d\nu_{m,\theta}(E),$$

where $\nu_{m,\theta}$ is the Haar probability measure on the homogeneous space $\{E \in G(n,m) \mid \theta \in E\}$.

Definition. A star-body K in $\mathbb{R}^n$ is called a k-Busemann–Petty body if $\rho_K^k = R_{n-k}^*(d\mu)$ as measures in $\mathcal{M}(S^{n-1})$, where μ is a non-negative Borel measure on $G(n, n-k)$. This class of bodies is denoted by $\mathcal{BP}_k^n$.

Choosing $k = 1$, for which $G(n, n-1)$ is isometric to S^{n-1}/Z_2 by mapping H to $S^{n-1} \cap H^\perp$, and noticing that R is equivalent to R_{n-1} under this map, we see that $\mathcal{BP}_1^n$ is exactly the class of intersection bodies.

In [21], a second generalization of the notion of an intersection body was introduced by A. Koldobsky, who studied a different analytic generalization of the Busemann–Petty problem.

Definition. A centrally-symmetric star-body K is said to be a k-intersection body of a star-body L, if $\mathrm{Vol}(K \cap H^\perp) = \mathrm{Vol}(L \cap H)$ for every $H \in G(n, n-k)$. K is said to be a k-intersection body, if it is the limit in the radial metric d_r of k-intersection bodies $\{K_i\}$ of star-bodies $\{L_i\}$. We shall denote the class of such bodies by $\mathcal{I}_k^n$.

Again, choosing $k = 1$, we see that $\mathcal{I}_1^n$ is exactly the class of intersection bodies.

In [21], Koldobsky considered the relationship between these two types of generalizations, $\mathcal{BP}_k^n$ and $\mathcal{I}_k^n$, and proved that $\mathcal{BP}_k^n \subset \mathcal{I}_k^n$ (see also [29]). Koldobsky also asked whether the opposite inclusion is equally true for all k between 2 and $n-2$ (for 1 and $n-1$ this is true):

Question. (See [21].) Is it true that $\mathcal{BP}_k^n = \mathcal{I}_k^n$ for $n \geqslant 4$ and $2 \leqslant k \leqslant n-2$?

If this were true, as remarked by Koldobsky, a positive answer to the generalized k-codimensional Busemann–Petty problem for $k \geqslant n-3$ would follow, since for those values of k any centrally-symmetric convex body in $\mathbb{R}^n$ is known to be a k-intersection body [19–21].

In [29], it was shown that these two classes $\mathcal{BP}_k^n$ and $\mathcal{I}_k^n$ share many identical structural properties, suggesting that it is indeed reasonable to believe that $\mathcal{BP}_k^n = \mathcal{I}_k^n$. Using techniques from integral geometry for the class $\mathcal{BP}_k^n$ and Fourier transform of distributions techniques for the class $\mathcal{I}_k^n$, the following structure theorem was established (see [29] for an account of particular cases which were known before). We define the *k-radial sum* of two star-bodies L_1, L_2 as the star-body L satisfying $\rho_L^k = \rho_{L_1}^k + \rho_{L_2}^k$.

Structure Theorem. *(See [29].) Let $\mathcal{C} = \mathcal{I}$ or $\mathcal{C} = \mathcal{BP}$ and $k, l = 1, \dots, n-1$. Then:*

(1) *$\mathcal{C}_k^n$ is closed under full-rank linear transformations, k-radial sums and taking limit in the radial metric.*
(2) *$\mathcal{C}_1^n$ is the class of intersection bodies in $\mathbb{R}^n$, and $\mathcal{C}_{n-1}^n$ is the class of all symmetric star-bodies in $\mathbb{R}^n$.*
(3) *Let $K_1 \in \mathcal{C}_{k_1}^n$, $K_2 \in \mathcal{C}_{k_2}^n$ and $l = k_1 + k_2 \leqslant n-1$. Then the star-body L defined by $\rho_L^l = \rho_{K_1}^{k_1} \rho_{K_2}^{k_2}$ satisfies $L \in \mathcal{C}_l^n$. As corollaries:*
   (a) *$\mathcal{C}_{k_1}^n \cap \mathcal{C}_{k_2}^n \subset \mathcal{C}_{k_1+k_2}^n$ if $k_1 + k_2 \leqslant n-1$.*
   (b) *$\mathcal{C}_k^n \subset \mathcal{C}_l^n$ if k divides l.*
   (c) *If $K \in \mathcal{C}_k^n$ then the star-body L defined by $\rho_L = \rho_K^{k/l}$ satisfies $L \in \mathcal{C}_l^n$ for $l \geqslant k$.*
(4) *If $K \in \mathcal{C}_k^n$ then any m-dimensional central section L of K (for $m > k$) satisfies $L \in \mathcal{C}_k^m$.*

Despite this and other evidence from [29] for a positive answer to Koldobsky's question, we give the following negative answer. Let $O(n)$ denote the orthogonal group on $\mathbb{R}^n$. Recall that a star-body K is called a body of revolution if its radial function $\rho_K \in C(S^{n-1})$ is invariant under the natural action of $O(n-1)$ identified as some subgroup of $O(n)$.

Theorem 1.1. *Let $n \geqslant 4$ and $2 \leqslant k \leqslant n-2$. Then there exists an infinitely smooth centrally-symmetric body of revolution K such that $K \in \mathcal{I}_k^n$ but $K \notin \mathcal{BP}_k^n$.*

Note that Theorem 1.1 does not imply a negative answer to the unresolved cases $k = n-2$, $n-3$ (for $n \geqslant 5$) of the generalized Busemann–Petty problem, which pertains to *convex* bodies. Indeed, the K we construct cannot be a convex body in those ranges of k, since as already mentioned, convex bodies of revolution are known ([39], see also [33]) to belong to $\mathcal{BP}_{n-2}^n$ and $\mathcal{BP}_{n-3}^n$. Theorem 1.1 does however imply that if one wishes to prove a positive answer to these unresolved cases by means of comparing k-intersection bodies to k-Busemann–Petty bodies, it is essential to restrict one's attention to convex bodies.

Let $I : C(G(n,k)) \to C(G(n,n-k))$ denote the operator defined by $I(f)(E) = f(E^\perp)$ for all $E \in G(n,n-k)$. Let $R_{n-k}(C(S^{n-1})) = \operatorname{Im} R_{n-k}$ denote the range of R_{n-k}. As explained in Section 2, Theorem 1.1 can be equivalently reformulated as follows:

Theorem 1.2. *Let $n \geqslant 4$ and $2 \leqslant k \leqslant n-2$. Then there exists an infinitely smooth function $g \in C(G(n,n-k))$ such that $R_{n-k}^*(g) \geqslant 1$ and $(I \circ R_k)^*(g) \geqslant 1$ as functions in $C(S^{n-1})$, but g is not non-negative as a functional on $R_{n-k}(C(S^{n-1}))$. In other words, there exists a non-negative $h \in R_{n-k}(C(S^{n-1}))$ such that $\int_{G(n,n-k)} g(E)h(E)\,d\eta_{n,n-k}(E) < 0$, where $\eta_{n,n-k}$ is the Haar probability measure on $G(n,n-k)$. Moreover, both g and h can be chosen to be invariant under the action of $O(n-1)$.*

In [29], several equivalent formulations to Koldobsky's question were obtained using cone-duality and the Hahn–Banach theorem. Let $C_+(S^{n-1})$ denote the cone of non-negative continuous functions on the sphere, and let $R_{n-k}(C(S^{n-1}))_+$ denote the cone of non-negative functions in the image of R_{n-k}. Let $\overline{A}$ denote the closure of a set A in the corresponding normed space. Note that by the results from [29], $\overline{\operatorname{Im} I \circ R_k} = \overline{\operatorname{Im} R_{n-k}}$, and hence:

$$\overline{R_{n-k}\big(C\big(S^{n-1}\big)\big)_+} \supset \overline{R_{n-k}\big(C_+\big(S^{n-1}\big)\big) + I \circ R_k\big(C_+\big(S^{n-1}\big)\big)}.$$

As formally verified in [29], the dual formulation to Theorem 1.2 then reads:

Theorem 1.3. *Let $n \geqslant 4$ and $2 \leqslant k \leqslant n-2$. Then:*

$$R_{n-k}\big(C\big(S^{n-1}\big)\big)_+ \setminus \overline{R_{n-k}\big(C_+\big(S^{n-1}\big)\big) + I \circ R_k\big(C_+\big(S^{n-1}\big)\big)} \neq \emptyset.$$

In other words, there exists an (infinitely smooth) function $f \in R_{n-k}(C(S^{n-1}))_+$ which cannot be approximated (in $C(G(n,n-k))$) by functions of the form $R_{n-k}(g) + I \circ R_k(h)$ with $g, h \in C_+(S^{n-1})$.

Other equivalent formulations using the language of Fourier transforms of homogeneous distributions are given in Section 5. We comment here that one such formulation pertains to embeddings in L_p for negative values of p. The definition of embedding into such a space (for $-n < p < 0$) was given by Koldobsky in [21] by means of analytic continuation of the usual definition for $p > 0$. It is known (see Section 5) that for $p \geqslant -1$ ($p \neq 0$) and for $-n < p \leqslant -n+1$, any star-body K such that $(\mathbb{R}^n, \|\cdot\|_K)$ embeds in L_p can be generated in a "trivial" manner, by starting with the Euclidean ball D_n, applying full-rank linear transformations, $(-p)$-radial sums and taking the limit in the radial metric. Our results imply that $p = -1$ and $p = -n+1$ are critical values for this property, and that this is no longer true for $p=-k$, $2 \leqslant k \leqslant n-2$. In other words:

Theorem 1.4. *There exist "non-trivial" n-dimensional spaces which embed in L_{-k} for $2 \leqslant k \leqslant n-2$.*

The rest of this note is organized as follows. In Section 2, we provide some additional background which is required to see why Theorem 1.2 implies Theorems 1.1 and 1.3. In Section 3, we develop several formulas for the Spherical Radon Transform and its dual for functions of revolution, i.e. functions invariant under the action of $O(n-1)$. In Section 4, we use these formulas to prove Theorem 1.2, thereby constructing the desired counter-example to Koldobsky's question. In Section 5, we give several additional equivalent formulations to Theorem 1.1 using the language of Fourier transforms of homogeneous distributions.

2. Additional background

In this section, we summarize the relevant results needed for this note. We also explain why Theorems 1.1 and 1.3 follow from Theorem 1.2. We refer to [29] for more details.

For a star-body K (not necessarily convex), we define its Minkowski functional as $\|x\|_K = \min\{t \geqslant 0 \mid x \in tK\}$. When K is a centrally-symmetric convex body, this of course coincides with the natural norm associated with it. Obviously $\rho_K(\theta) = \|\theta\|_K^{-1}$ for $\theta \in S^{n-1}$.

It was shown by Koldobsky in [21] that for a star-body K in $\mathbb{R}^n$, $K \in \mathcal{I}_k^n$ iff $\|\cdot\|_K^{-k}$ is a positive definite distribution on $\mathbb{R}^n$, meaning that its Fourier transform (as a distribution) $(\|\cdot\|_K^{-k})^\wedge$ is a non-negative Borel measure on $\mathbb{R}^n$. We refer the reader to Section 5 for more on Fourier transforms of homogeneous distributions, as this will not be of essence in the ensuing discussion. To translate this result to the language of Radon transforms, it was shown in [29, Corollary 4.2] that for an infinitely smooth star-body K and a (signed) Borel measure $\mu \in \mathcal{M}(G(n,n-k))$:

$$\|\cdot\|_K^{-k} = R_{n-k}^*(d\mu) \quad \text{iff} \quad \left(\|\cdot\|_K^{-k}\right)^\wedge = c(n,k)(I \circ R)_k^*(d\mu), \tag{2.1}$$

where $c(n,k)$ is some explicit positive constant and the equalities above are interpreted as equalities between measures on S^{n-1}. Hence, it follows [29, Lemma 5.3] that for an infinitely smooth star-body K in $\mathbb{R}^n$, $K \in I_k^n$ iff there exists a (possibly signed) Borel measure $\mu \in \mathcal{M}(G(n,n-k))$, such that as measures $\rho_K^k = R_{n-k}^*(d\mu) \geqslant 0$ and $(I \circ R_k)^*(d\mu) \geqslant 0$.

This should be compared with the definition of k-Busemann–Petty bodies: $K \in \mathcal{BP}_k^n$ iff $\rho_K^k = R_{n-k}^*(d\mu)$ as measures on S^{n-1} for a *non-negative* Borel measure $\mu \in \mathcal{M}(G(n,n-k))$. Since for such a measure, $(I \circ R_k)^*(d\mu) \geqslant 0$, it follows that every infinitely smooth k-Busemann–Petty body is also a k-intersection body, and this easily implies (see [29, Corollary 4.4]) that $\mathcal{BP}_k^n \subset \mathcal{I}_k^n$ in general, as first shown by Koldobsky in [21].

R_{n-k} is known (e.g. [16]) to be injective on the space of *even* functions in $C(S^{n-1})$, so by duality R_{n-k}^* is onto a dense subset of *even* measures in $\mathcal{M}(S^{n-1})$, which is known to include even measures with infinitely smooth densities. However, it is important to note that for $2 \leqslant k \leqslant n-2$, the image of R_{n-k} is not dense in $C(G(n,n-k))$, and equivalently, R_{n-k}^* has a non-trivial kernel. The above implies that for any infinitely smooth star-body K, we can find a measure μ such that $\rho_K^k = R_{n-k}^*(d\mu)$, but if $2 \leqslant k \leqslant n-2$ this measure will not be unique. Nevertheless, as a functional on $R_{n-k}(C(S^{n-1}))$, such a measure μ is determined uniquely. The conclusion is that if we need to determine whether $K \in \mathcal{BP}_k^n$ given a representation $\rho_K^k = R_{n-k}^*(d\mu)$ for some measure $\mu \in \mathcal{M}(G(n,n-k))$, a necessary and sufficient condition is that μ is a non-negative functional on $R_{n-k}(C(S^{n-1}))$, i.e. $\int_{G(n,n-k)} R_{n-k}(h)(E)\,d\mu(E) \geqslant 0$ for any $h \in C(S^{n-1})$ such that $R_{n-k}(h) \geqslant 0$. Indeed, any non-negative functional on $R_{n-k}(C(S^{n-1}))$ can be extended to a non-negative functional on $C(G(n,n-k))$ by a version of the Hahn–Banach theorem (see the remarks before [29, Lemma 5.2] for more details).

The above discussion explains why Theorem 1.1 is an immediate consequence of Theorem 1.2. Given the infinitely smooth function g provided by Theorem 1.2, we define the centrally-symmetric star-body K given by $\rho_K^k = R_{n-k}^*(g)$. Note that this indeed defines a star-body since $R_{n-k}^*(g) \geqslant 0$. In fact, K is an infinitely smooth star-body since it is known (e.g. [10]) that $R_{n-k}^*(g)$ is an infinitely smooth function on S^{n-1} if g is infinitely smooth; and since $\rho_K^k = R_{n-k}^*(g) \geqslant 1$, it follows that ρ_K itself is infinitely smooth. In addition $K \in \mathcal{I}_k^n$ since $(I \circ R_k)^*(g) \geqslant 0$. But since g is not a non-negative functional on $R_{n-k}(C(S^{n-1}))$, if follows that $K \notin \mathcal{BP}_k^n$.

To explain why Theorem 1.1 is equivalent to Theorem 1.3, we recall another result from [29]. Denote $\mathcal{M} = \mathcal{M}(G(n,n-k))$ for short, and let

$$\mathcal{M}\big(\mathcal{BP}_k^n\big) = \big\{\mu \in \mathcal{M};\ \mu \text{ is a non-negative functional on } R_{n-k}\big(C\big(S^{n-1}\big)\big)\big\},$$

and

$$\mathcal{M}\big(\mathcal{I}_k^n\big) = \big\{\mu \in \mathcal{M};\ R_{n-k}^*(d\mu) \geqslant 0 \text{ and } (I \circ R_k)^*(d\mu) \geqslant 0\big\}.$$

It should already be clear from the above discussion that the statement $\mathcal{BP}_k^n = \mathcal{I}_k^n$ is equivalent to the statement $\mathcal{M}(\mathcal{BP}_k^n) = \mathcal{M}(\mathcal{I}_k^n)$. By the Hahn–Banach theorem for convex cones, it is not hard to see [29, Theorem 5.6] that the latter statement is dual to

$$\overline{R_{n-k}\big(C\big(S^{n-1}\big)\big)_+} = \overline{R_{n-k}\big(C_+\big(S^{n-1}\big)\big) + I \circ R_k\big(C_+\big(S^{n-1}\big)\big)}. \tag{2.2}$$

As follows from (2.1), $\operatorname{Ker} R_{n-k}^* = \operatorname{Ker}(I \circ R_k)^*$, and therefore $\overline{\operatorname{Im} R_{n-k}} = \overline{\operatorname{Im} I \circ R_k}$. This explains why the right-hand side of (2.2) is always a subset of the left. Theorem 1.1 shows that it is a proper subset, implying Theorem 1.3. Since this theorem is attained using a convex separation argument, we have no constructive way of finding the function f of the theorem. Albeit, we can always find an infinitely smooth f, since the subspace of infinitely smooth functions in $R_{n-k}(C(S^{n-1}))$ is known to be dense in $R_{n-k}(C(S^{n-1}))$, and hence in $\overline{R_{n-k}(C(S^{n-1}))}$.

3. Radon transform for functions of revolution

Fix $n \geqslant 3$ and $\xi_0 \in S^{n-1}$. We denote by $O_{\xi_0}(n-1)$ the subgroup of $O(n)$ whose natural action on S^{n-1} leaves ξ_0 invariant, and by $C_{\xi_0}(S^{n-1})$ the linear subspace of functions in $C^e(S^{n-1})$ invariant under $O_{\xi_0}(n-1)$. Clearly $O_{\xi_0}(n-1)$ is isometric to $O(n-1)$. We refer to members of $C_{\xi_0}(S^{n-1})$ as spherical functions of revolution. For $\xi_1, \xi_2 \in S^{n-1}$, let $\measuredangle(\xi_1,\xi_2)$ denote the angle in $[0,\pi/2]$ between ξ_1 and ξ_2, i.e. $\cos\measuredangle(\xi_1,\xi_2) = |\langle\xi_1,\xi_2\rangle|$. We also denote $\measuredangle(\xi_1,0) = \pi/2$. Clearly $F \in C_{\xi_0}(S^{n-1})$ iff $F(\xi) = f(\measuredangle(\xi,\xi_0))$ for $f \in C([0,\pi/2])$. In that case, we denote by $\tilde{f} \in C([0,1])$ the function given by $\tilde{f}(\cos\theta) = f(\theta)$, so $F(\xi) = \tilde{f}(|\langle\xi,\xi_0\rangle|)$. We denote the operator $T : C([0,\pi/2]) \to C([0,1])$ defined by $T(f) = \tilde{f}$, for future reference. It is well known by polar integration (e.g. [34]), that

$$\int_{S^{n-1}} F(\xi)\, d\sigma_n(\xi) = c_n \int_0^{\pi/2} f(\theta) \sin^{n-2}(\theta)\, d\theta = c_n \int_0^1 \tilde{f}(t)\big(1-t^2\big)^{\frac{n-3}{2}}\, dt, \tag{3.1}$$

where σ_n is the Haar probability measure on S^{n-1}, and c_n is a constant whose value may be deduced by using $F \equiv f \equiv \tilde{f} \equiv 1$.

For $E \in G(n,k)$ and $\xi \in S^{n-1}$, denote by $\operatorname{Proj}_E \xi$ the orthogonal projection of ξ onto E, and by $\overline{\operatorname{Proj}}_E \xi := \operatorname{Proj}_E \xi / |\operatorname{Proj}_E \xi|$ if $\operatorname{Proj}_E \xi \neq 0$, and $\overline{\operatorname{Proj}}_E \xi := 0$ otherwise. When $E = \mathit{span}(\xi_1)$ for $\xi_1 \in S^{n-1}$, we may sometimes replace E by ξ_1 in Proj_E and $\overline{\operatorname{Proj}}_E$. Denote by $\measuredangle(\xi, E) = \measuredangle(\xi, \overline{\operatorname{Proj}}_E \xi)$ if $\overline{\operatorname{Proj}}_E \xi \neq 0$ and $\measuredangle(\xi, E) = \pi/2$ otherwise.

Since the natural action of $O(n)$ on $C(G(n,k))$ and $C^e(S^{n-1})$ commutes with R_k, and since $O_{\xi_0}(n-1)$ acts transitively on all $E \in G(n,k)$ such that $\measuredangle(\xi_0, E)$ is fixed, it clearly follows that if $F \in C_{\xi_0}(S^{n-1})$ then $R_k(F)(E)$ only depends on $\measuredangle(\xi_0, E)$. Hence, if $F(\xi) = f(\measuredangle(\xi, \xi_0))$ for $f \in C([0,\pi/2])$, we denote (abusing notation) by $R_k(f) \in C([0,\pi/2])$ the function given by $R_k(f)(\measuredangle(\xi_0, E)) = R_k(F)(E)$. Similarly, we define $\tilde{R}_k : C([0,1]) \to C([0,1])$ as $\tilde{R}_k = T \circ R_k \circ T^{-1}$.

The following lemma was essentially stated in [39]. We provide a simple proof for completeness:

Lemma 3.1. *Let $f \in C([0,\pi/2])$ and $2 \leqslant k \leqslant n-1$. Then:*

$$R_k(f)(\phi) = c_k \int_0^{\pi/2} f\big(\cos^{-1}(\cos\phi \cos\theta)\big) \sin^{k-2}\theta\, d\theta,$$

where the value of c_k is found by using $f \equiv 1$, in which case $R_k(f) \equiv 1$.

Remark 3.2. This lemma, together with the subsequent ones, extend to the case $k = 1$, if we properly interpret the (formally) diverging integral as integration with respect to an appropriate delta-measure. Note also that the value c_k is consistent with the one used in (3.1).

Proof of Lemma 3.1. Let $F \in C_{\xi_0}(S^{n-1})$ be given by $F(\xi) = f(\measuredangle(\xi, \xi_0))$. Let $E \in G(n,k)$ be such that $\measuredangle(\xi_0, E) = \phi$. Hence, if $\xi_1 = \overline{\operatorname{Proj}}_E \xi_0$ then $\measuredangle(\xi_0, \xi_1) = \phi$. For $\xi \in S^{n-1} \cap E$, since $\xi \operatorname{Proj}_{\xi_1} \xi$ and $\xi_0 - \operatorname{Proj}_{\xi_1} \xi_0$ are orthogonal, it follows that $\langle \xi, \xi_0 \rangle = \langle \operatorname{Proj}_{\xi_1} \xi, \xi_0 \rangle$, i.e. $\operatorname{Proj}_{\xi_0} \xi = \operatorname{Proj}_{\xi_0}(\operatorname{Proj}_{\xi_1} \xi)$. Hence $\cos\measuredangle(\xi, \xi_0) = \cos\measuredangle(\xi, \xi_1)\cos\measuredangle(\xi_1, \xi_0) = \cos\measuredangle(\xi, \xi_1)\cos\phi$. Since the function F is even, a standard polar integration formula then gives:

$$\begin{aligned} R_k(f)(\phi) = R_k(F)(E) &= \int_{S^{n-1}\cap E} F(\xi)\, d\mu_E(\xi) \\ &= \int_{S^{n-1}\cap E} f\big(\measuredangle(\xi, \xi_0)\big)\, d\mu_E(\xi) \end{aligned}$$

$$= \int_{S^{n-1}\cap E} f\bigl(\cos^{-1}\bigl(\cos\measuredangle(\xi,\xi_1)\cos\phi\bigr)\bigr)\,d\mu_E(\xi)$$

$$= c_k \int_0^{\pi/2} f\bigl(\cos^{-1}(\cos\phi\cos\theta)\bigr)\sin^{k-2}\theta\,d\theta. \qquad \square$$

Performing the change of variables $t = \cos\theta$, $s = \cos\phi$ above, we immediately have:

Corollary 3.3. *Let $\tilde{f} \in C([0,1])$ and $2 \leqslant k \leqslant n-1$. Then:*

$$\tilde{R}_k(\tilde{f})(s) = c_k \int_0^1 \tilde{f}(st)\bigl(1-t^2\bigr)^{\frac{k-3}{2}}\,dt,$$

where the value of c_k is the same as in Lemma 3.1.

Next, we introduce $C_{\xi_0}(G(n,k))$, the linear subspace of all functions in $C(G(n,k))$ invariant under the action of $O_{\xi_0}(n-1)$. We refer to members of $C_{\xi_0}(G(n,k))$ as functions of revolution on the Grassmannian. As before, it is clear that $G \in C_{\xi_0}(G(n,k))$ iff $G(E) = g(\measuredangle(\xi_0,E))$ for $g \in C([0,\pi/2])$. We have the following:

Lemma 3.4. *Let $G \in C_{\xi_0}(G(n,k))$ such that $G(E) = g(\measuredangle(\xi_0,E))$, and let $\tilde{g} = g \circ T^{-1}$. Then:*

$$\int_{G(n,k)} G(E)\,d\eta_{n,k}(E) = b_{n,k}\int_0^{\pi/2} g(\phi)\sin^{n-k-1}\phi\cos^{k-1}\phi\,d\phi$$

$$= b_{n,k}\int_0^1 \tilde{g}(s)\bigl(1-s^2\bigr)^{\frac{n-k-2}{2}} s^{k-1}\,ds,$$

where $\eta_{n,k}$ is the Haar probability measure on $G(n,k)$, and the value of $b_{n,k}$ may be deduced by using $G \equiv g \equiv \tilde{g} \equiv 1$.

Proof. Clearly:

$$\int_{G(n,k)} G(E)\,d\eta_{n,k}(E) = \int_0^{\pi/2} g(\phi)\,d\bigl(\eta_{n,k}\bigl\{E \in G(n,k);\ \measuredangle(\xi_0,E) \leqslant \phi\bigr\}\bigr).$$

Since σ_n and $\eta_{n,k}$ are both rotation-invariant, it follows that:

$$\eta_{n,k}\bigl(\bigl\{E \in G(n,k); \measuredangle(\xi_0,E) \leqslant \phi\bigr\}\bigr) = \sigma_n\bigl\{\xi \in S^{n-1};\ \measuredangle(\xi,E_0) \leqslant \phi\bigr\}$$

for any $E_0 \in G(n,k)$. Using bi-polar coordinates (e.g. [34, Chapter IX]), it is easy to see that:

$$d\sigma_n\bigl\{\xi \in S^{n-1}; \measuredangle(\xi,E_0) \leqslant \phi\bigr\} = b_{n,k} sin^{n-k-1}\phi\cos^{k-1}\phi\,d\phi,$$

for some $b_{n,k}$. This concludes the proof of the first equality of the lemma, and the second one follows by the change of variables $s = \cos(\phi)$. $\square$

Next, we find an expression for the dual spherical Radon transform of a function in $C_{\xi_0}(G(n,k))$. As before, it is clear that if $F \in C_{\xi_0}(S^{n-1})$ then $R_k(F) \in C_{\xi_0}(G(n,k))$, and that if $G \in C_{\xi_0}(G(n,k))$ then $R_k^*(G) \in C_{\xi_0}(S^{n-1})$. If $G \in C_{\xi_0}(G(n,k))$ is given by $G(E) = g(\measuredangle(\xi_0, E))$, we denote by $R_k^*(g) \in C([0,\pi/2])$ the function given by $R_k^*(g)(\measuredangle(\xi,\xi_0)) = R_k^*(G)(\xi)$. As usual, we define $\tilde{R}_k^* : C([0,1]) \to C([0,1])$ by $\tilde{R}_k^* = T \circ R_k^* \circ T^{-1}$. The standard duality relation

$$\int_{S^{n-1}} R_k^*(G)(\xi) F(\xi)\, d\sigma_n(\xi) = \int_{G(n,k)} G(E) R_k(F)(E)\, d\eta_{n,k}(E)$$

is immediately translated using (3.1) and Lemma 3.4 into the following duality relation between $\tilde{R}_k$ and $\tilde{R}_k^*$ on $C([0,1])$:

Lemma 3.5. *Let $\tilde{f}, \tilde{g} \in C([0,1])$ and $1 \leqslant k \leqslant n-1$. Then:*

$$\int_0^1 \tilde{R}_k^*(\tilde{g})(t) \tilde{f}(t) \left(1-t^2\right)^{\frac{n-3}{2}} dt = d_{n,k} \int_0^1 \tilde{g}(s) \tilde{R}_k(\tilde{f})(s) \left(1-s^2\right)^{\frac{n-k-2}{2}} s^{k-1}\, ds$$

where the value of $d_{n,k}$ is found by using $\tilde{f}, \tilde{g} \equiv 1$, in which case $\tilde{R}_k(\tilde{f}), \tilde{R}_k^*(\tilde{g}) \equiv 1$.

We can now deduce an expression for $\tilde{R}_k^*$:

Lemma 3.6. *Let $\tilde{g} \in C([0,1])$ and $2 \leqslant k \leqslant n-1$. Then:*

$$\tilde{R}_k^*(\tilde{g})(t) = e_{n,k} \int_0^1 \tilde{g}\left(\sqrt{1-s^2\left(1-t^2\right)}\right) \left(1-s^2\right)^{\frac{k-3}{2}} s^{n-k-1}\, ds,$$

where the value of $e_{n,k}$ is found by using $\tilde{g} \equiv 1$, in which case $\tilde{R}_k^*(\tilde{g}) \equiv 1$.

Proof. We start with Lemma 3.5 and use the formula for $\tilde{R}_k$ given in Corollary 3.3:

$$\begin{aligned}
\int_0^1 \tilde{R}_k^*(\tilde{g})(t) \tilde{f}(t) \left(1-t^2\right)^{\frac{n-3}{2}} dt &= d_{n,k} \int_0^1 \tilde{g}(s) \tilde{R}_k(\tilde{f})(s) \left(1-s^2\right)^{\frac{n-k-2}{2}} s^{k-1}\, ds \\
&= d_{n,k} c_k \int_0^1 \tilde{g}(s) \int_0^1 \tilde{f}(st) \left(1-t^2\right)^{\frac{k-3}{2}} dt \left(1-s^2\right)^{\frac{n-k-2}{2}} s^{k-1}\, ds \\
&= d_{n,k} c_k \int_0^1 \tilde{f}(v) \int_v^1 \tilde{g}(s) \left(1-\frac{v^2}{s^2}\right)^{\frac{k-3}{2}} \left(1-s^2\right)^{\frac{n-k-2}{2}} s^{k-2}\, ds\, dv.
\end{aligned}$$

Since this is true for any $\tilde{f} \in C([0,1])$, setting $e_{n,k} = d_{n,k} c_k$, we conclude that:

$$\tilde{R}_k^*(\tilde{g})(t) = e_{n,k}\big(1-t^2\big)^{-\frac{n-3}{2}} \int_t^1 \tilde{g}(s)\left(1-\frac{t^2}{s^2}\right)^{\frac{k-3}{2}} \big(1-s^2\big)^{\frac{n-k-2}{2}} s^{k-2}\, ds.$$

By the change of variable $s = \sqrt{1-(s')^2(1-t^2)}$, one easily checks that the assertion of the lemma is obtained. $\square$

We now recall the definition of the "perp" operator I from the Introduction, and extend it to the context of functions of revolution. For every $k = 1, \dots, n-1$, we define $I : C(G(n,k)) \to C(G(n,n-k))$ as $I(f)(E) = f(E^\perp)$ for all $E \in G(n,n-k)$, without specifying the index k. With these notations, I is obviously "self-adjoint":

$$\int_{G(n,n-k)} I(F)(H)G(H)\, d\eta_{n-k}(H) = \int_{G(n,k)} F(E)I(G)(E)\, d\eta_k(E),$$

for all $F \in C(G(n,k))$ and $G \in C(G(n,n-k))$, where η_m denotes the Haar probability measure on $G(n,m)$.

Since $\measuredangle(\xi_0, E) = \pi/2 - \measuredangle(\xi_0, E^\perp)$, it is clear that for $G \in C_{\xi_0}(G(n,k))$ such that $G(E) = g(\measuredangle(\xi_0, E))$ for every $E \in G(n,k)$, $I(G)(H) = g(\pi/2 - \measuredangle(\xi_0, H))$ for every $H \in G(n,n-k)$. We therefore define $I : C([0,\pi/2]) \to C([0,\pi/2])$ as $I(g)(\phi) = g(\pi/2 - \phi)$. Similarly, for $\tilde{g} \in C([0,1])$, we define $I(\tilde{g})(s) = \tilde{g}(\sqrt{1-s^2})$. Clearly, if $G(E) = \tilde{g}(\cos(\measuredangle(\xi_0, E)))$ then $I(G)(H) = I(\tilde{g})(\cos(\measuredangle(\xi_0, H)))$. Hence in both cases I must be self-adjoint, and this can be also verified directly. As an immediate corollary of Lemma 3.6, we have:

Corollary 3.7. *Let $\tilde{g} \in C([0,1])$ and $2 \leqslant k \leqslant n-1$. Then:*

$$(I \circ \tilde{R}_k)^*(\tilde{g})(t) = e_{n,k} \int_0^1 \tilde{g}\big(s\sqrt{1-t^2}\big)\big(1-s^2\big)^{\frac{k-3}{2}} s^{n-k-1}\, ds,$$

where the value of $e_{n,k}$ is the same as in Lemma 3.6.

We are now ready to construct the counter-example to Koldobsky's question, as described in the next section.

4. The construction

The main step in the proof of Theorem 1.2 is the following:

Proposition 4.1. *For any $n \geqslant 4$, $2 \leqslant k \leqslant n-2$ and $s_0 \in (0,1)$, there exists an infinitely smooth function $\tilde{g} \in C([0,1])$ such that:*

(1) *For all* $t \in [0,1]$*:*

$$\tilde{R}^*_{n-k}(\tilde{g})(t) = e_{n,n-k} \int_0^1 \tilde{g}\left(\sqrt{1-s^2\left(1-t^2\right)}\right)\left(1-s^2\right)^{\frac{n-k-3}{2}} s^{k-1}\, ds \geqslant 1.$$

(2) *For all* $t \in [0,1]$*:*

$$(I \circ \tilde{R}_k)^*(\tilde{g})(t) = e_{n,k} \int_0^1 \tilde{g}\left(s\sqrt{1-t^2}\right)\left(1-s^2\right)^{\frac{k-3}{2}} s^{n-k-1}\, ds \geqslant 1.$$

(3) $\tilde{g}(s_0) = -1$.
(4) *All the derivatives* $\tilde{g}^{(l)}$ *vanish at 0 and 1 for* $l \geqslant 1$.

Proof. Let $\varepsilon > 0$ be such that $[s_0 - 2\varepsilon, s_0 + 2\varepsilon] \subset (0,1)$. Let $T_t, T'_t \in C([0,1])$ be defined by $T_t(s) = \sqrt{1-s^2(1-t^2)}$ and $T'_t(s) = s\sqrt{1-t^2}$, and let λ denote the Lebesgue measure on $\mathbb{R}$. It is elementary to check that the maximum of $\lambda\{T_t^{-1}[s_0 - 2\varepsilon, s_0 + 2\varepsilon]\}$ over $t \in [0,1]$ is attained at $t = s_0 - 2\varepsilon$, in which case it is equal to

$$\delta_1 := \max_{t \in [0,1]} \lambda\left\{T_t^{-1}[s_0 - 2\varepsilon, s_0 + 2\varepsilon]\right\} = 1 - \sqrt{\frac{1-(s_0+2\varepsilon)^2}{1-(s_0-2\varepsilon)^2}} < 1.$$

An analogous computation shows that the maximum of $\lambda\{(T'_t)^{-1}[s_0 - 2\varepsilon, s_0 + 2\varepsilon]\}$ over $t \in [0,1]$ is attained at $t = \sqrt{1-(s_0+2\varepsilon)^2}$, in which case it is equal to

$$\delta_2 := \max_{t \in [0,1]} \lambda\left\{\left(T'_t\right)^{-1}[s_0 - 2\varepsilon, s_0 + 2\varepsilon]\right\} = \frac{4\varepsilon}{s_0 + 2\varepsilon} < 1.$$

Set $\delta := \max(\delta_1, \delta_2) < 1$. Now denote by $\mu_{n,m}$ the measure $e_{n,m}(1-s^2)^{\frac{m-3}{2}} s^{n-m-1} ds$ on $[0,1]$, for $2 \leqslant m \leqslant n-2$. These are probability measures, as witnessed by using $\tilde{g} \equiv 1$ in Lemma 3.6, in which case $\tilde{R}^*_k(\tilde{g}) \equiv 1$. Since their densities (with respect to λ) are absolutely continuous and do not vanish on $(0,1)$, a compactness argument shows that (fixing n)

$$\gamma := \sup_{v \in [0,1],\, 2 \leqslant m \leqslant n-2} \mu_{n,m}\left([v, v+\delta]\right) < 1.$$

Set $\gamma^* = \frac{1+\gamma}{1-\gamma}$. We conclude by constructing $\tilde{g}$ as follows. Set $\tilde{g}(s) = -1$ for $s \in [s_0 - \varepsilon, s_0 + \varepsilon]$, $\tilde{g}(s) = \gamma^*$ for $s \in [0,1] \setminus [s_0 - 2\varepsilon, s_0 + 2\varepsilon]$, and for $s \in [s_0 - 2\varepsilon, s_0 + 2\varepsilon] \setminus [s_0 - \varepsilon, s_0 + \varepsilon]$ set $\tilde{g}(s) \in [-1, \gamma^*]$ so that the resulting function $\tilde{g} \in C([0,1])$ is in fact infinitely smooth (using standard methods). Clearly the derivatives of $\tilde{g}$ vanish at 0 and 1 as required. Setting

$$\beta_1(t) := \mu_{n,n-k}\left\{s \in [0,1];\ T_t(s) \in [s_0 - 2\varepsilon, s_0 + 2\varepsilon]\right\},$$

the definition of γ and δ imply that $\beta_1(t) \leqslant \gamma$ for all $t \in [0,1]$, hence

$$\int_0^1 \tilde{g}\left(\sqrt{1-s^2\left(1-t^2\right)}\right) d\mu_{n,n-k}(s) \geqslant \gamma^*\left(1-\beta_1(t)\right) - \beta_1(t) \geqslant 1$$

for all $t \in [0,1]$. Similarly, setting

$$\beta_2(t) := \mu_{n,k}\left\{s \in [0,1];\ T_t'(s) \in [s_0 - 2\varepsilon, s_0 + 2\varepsilon]\right\},$$

we have $\beta_2(t) \leqslant \gamma$ for all $t \in [0,1]$, and

$$\int_0^1 \tilde{g}\left(s\sqrt{1-t^2}\right) d\mu_{n,k}(s) \geqslant \gamma^*\left(1-\beta_2(t)\right) - \beta_2(t) \geqslant 1$$

for all $t \in [0,1]$. This concludes the proof. $\square$

Remark 4.2. Note that for $k = 1$ and $k = n-1$ the above reasoning fails, as the measure $\mu_{n,1}$ is a singular measure.

Remark 4.3. Note also that the function $\tilde{g}$ we have constructed in fact satisfies the claims (1) and (2) for *all* values of k in the range $2 \leqslant k \leqslant n-2$.

We can now almost conclude the proof of Theorem 1.2. We still need one last observation, since a priori, the fact that $\tilde{g}(s_0) < 0$ does not guarantee that the function $G \in C(G(n, n-k))$ defined as $G(E) = \tilde{g}(\cos(\measuredangle(\xi_0, E)))$, is not a non-negative functional on $R_{n-k}(C(S^{n-1}))$. This is resolved by the following:

Lemma 4.4. *The polynomials on* $[0,1]$ *are in the range of* $\tilde{R}_{n-k}(C([0,1]))$.

Proof. This is immediate by Corollary 3.3, because if $\tilde{p}(t) = t^m$ ($m \geqslant 0$), then

$$\tilde{R}_{n-k}(\tilde{p})(s) = c_{n-k} \int_0^1 \tilde{p}(st)\left(1-t^2\right)^{\frac{n-k-3}{2}} dt = d_{n-k,m} s^m,$$

with $d_{n-k,m} > 0$. Hence polynomials are mapped to polynomials by $\tilde{R}_{n-k}$, and any polynomial in the range may be obtained. $\square$

By the Weierstrass approximation theorem, if follows that:

Corollary 4.5. *The range of* $\tilde{R}_{n-k}$ *is dense in* $C([0,1])$.

We can now turn to the proof of Theorem 1.2.

Proof of Theorem 1.2. Let $\tilde{g} \in C([0,1])$ be the infinitely smooth function constructed in Proposition 4.1, with, say $s_0 = 1/2$. Fix some $\xi_0 \in S^{n-1}$, and let $G \in C_{\xi_0}(G(n,n-k))$ be defined by $G(E) = \tilde{g}(\cos(\measuredangle(\xi_0, E)))$ for every $E \in G(n,n-k)$. Of course the functions $\tilde{g}$ and cos are infinitely smooth on their corresponding domains, and so is $\measuredangle(\xi_0,\cdot)$, except at the points $\xi \in S^{n-1}$ where $\measuredangle(\xi_0,\xi)$ attains the value 0 or $\pi/2$. Nevertheless, as the composition of these functions, G is indeed infinitely smooth on all of $G(n,n-k)$, since we required that the derivatives of $\tilde{g}$ vanish at 0 and 1, which takes care of the singularities of $\measuredangle(\xi_0,\cdot)$ at the "stitching" points. By the construction of $\tilde{g}$ and the compatibility of R^*_{n-k} and $(I \circ R_k)^*$ with $\tilde{R}^*_{n-k}$ and $(I \circ \tilde{R}_k)^*$, respectively, it follows that $R^*_{n-k}(G) = \tilde{R}^*_{n-k}(\tilde{g}) \geqslant 1$ and $(I \circ R_k)^*(G) = (I \circ \tilde{R}_k)^*(\tilde{g}) \geqslant 1$. It remains to show that G is not a non-negative functional on $R_{n-k}(C(S^{n-1}))$. Let $H \in C_{\xi_0}(S^{n-1})$ be such that $H(\xi) = \tilde{h}(\cos(\measuredangle(\xi_0,\xi))$ for some $\tilde{h} \in C([0,1])$. Then by Lemma 3.4:

$$\int\limits_{G(n,n-k)} G(E) R_{n-k}(H)(E)\, d\eta_{n,n-k} = b_{n,n-k} \int\limits_0^1 \tilde{g}(s) \tilde{R}_{n-k}(\tilde{h})(s) \big(1-s^2\big)^{\frac{k-2}{2}} s^{n-k-1}\, ds. \tag{4.1}$$

Since $\tilde{g}(s)(1-s^2)^{\frac{k-2}{2}} s^{n-k-1}$ is a continuous function on $[0,1]$ whose value at s_0 is negative, by Corollary 4.5 we can find a function $\tilde{h} \in C([0,1])$ such that $\tilde{R}_{n-k}(\tilde{h})$ (and therefore $R_{n-k}(H)$) is non-negative, but the integral in (4.1) *is* negative. This demonstrates that G is not a non-negative functional on $R_{n-k}(C(S^{n-1}))$ and concludes the proof. $\square$

5. Additional formulations

In this section, we provide several additional equivalent formulations to the main result of this note, using the language of Fourier transforms of homogeneous distributions (we refer the reader to [22] for more on this subject).

We denote by $\mathcal{S}(\mathbb{R}^n)$ the Schwartz space of test functions on $\mathbb{R}^n$ (i.e. infinitely differentiable functions whose partial derivatives of any order decay to 0 faster than any polynomial), and by $\mathcal{S}'(\mathbb{R}^n)$ the space of distributions over $\mathcal{S}(\mathbb{R}^n)$. The Fourier transform $\hat{f}$ of a distribution $f \in \mathcal{S}'(\mathbb{R}^n)$ is defined by $\langle \hat{f}, \phi\rangle = \langle f, \hat{\phi}\rangle$ for every test function ϕ, where $\hat{\phi}(y) = \int \phi(x)\exp(-i\langle x,y\rangle)\,dx$. A distribution f is called homogeneous of degree $p \in \mathbb{R}$ if $\langle f, \phi(\cdot/t)\rangle = |t|^{n+p}\langle f,\phi\rangle$ for every $t>0$, and it is called even if the same is true for $t=-1$. An even distribution f always satisfies $(\hat{f})^\wedge = (2\pi)^n f$. The Fourier transform of an even homogeneous distribution of degree p is an even homogeneous distribution of degree $-n-p$. A distribution f is called positive if $\langle f,\phi\rangle \geqslant 0$ for every $\phi \geqslant 0$, implying that f is necessarily a non-negative Borel measure on $\mathbb{R}^n$. We use Schwartz's generalization of Bochner's theorem [11] as a definition, and call a homogeneous distribution positive-definite if its Fourier transform is a positive distribution.

5.1. *Embeddings in L_p*

Recall the following:

Definition. A normed space $(\mathbb{R}^n, \|\cdot\|)$ is said to embed in L_p $(p \geqslant 1)$ iff there exists a basis $x_1,\dots,x_n \in \mathbb{R}^n$ and functions $f_1,\dots,f_n \in L_p([0,1])$ such that $\|\sum_{i=1}^n a_i x_i\|^p = \int |\sum_{i=1}^n a_i f_i(t)|^p\,dt$ for all scalars $\{a_i\}$.

This definition may be extended to the range $0 < p < 1$, in which case $\|\cdot\|$ is no longer necessarily a norm, but rather the Minkowski functional of some star-body. The following classical result of P. Lévy [25] provides an equivalent definition. Note that this definition makes sense for $p > -1$ (and $p \neq 0$, the case $p = 0$ requires passing to the limit).

Equivalent Definition. $(\mathbb{R}^n, \|\cdot\|)$ embeds in L_p ($p > -1$, $p \neq 0$) iff

$$\|x\|^p = \int_{S^{n-1}} \left|\langle x, \theta\rangle\right|^p \, d\mu(\theta), \tag{5.1}$$

for some $\mu \in \mathcal{M}_+(S^{n-1})$, the cone of non-negative Borel measures on S^{n-1}.

Unfortunately, this characterization breaks down at $p = -1$ since the above integral no longer converges. However, A. Koldobsky showed that it is possible to regularize this integral by using Fourier transforms of distributions, and gave the following definition in [21]:

Definition. $(\mathbb{R}^n, \|\cdot\|)$ embeds in L_{-p} for $0 < p < n$ iff there exists a measure $\mu \in \mathcal{M}_+(S^{n-1})$ such that for any even test-function ϕ:

$$\int_{\mathbb{R}^n} \|x\|^{-p} \phi(x)\, dx = \int_{S^{n-1}} \int_0^\infty t^{p-1} \hat{\phi}(t\theta)\, dt\, d\mu(\theta). \tag{5.2}$$

Consequently, the following characterization was given in [21]:

Theorem 5.1 *(Koldobsky). The following are equivalent for a centrally-symmetric star-body K in $\mathbb{R}^n$:*

(1) $K \in \mathcal{I}_k^n$.
(2) $\|\cdot\|_K^{-k}$ *is a positive definite distribution on* $\mathbb{R}^n$.
(3) *The space* $(\mathbb{R}^n, \|\cdot\|_K)$ *embeds in* L_{-k}.

In addition to the characterization (3) in Theorem 5.1 of $\mathcal{I}_k^n$ as the class of unit-balls of subspaces of *scalar* L_{-k} spaces, a functional analytic characterization of $\mathcal{BP}_k^n$ as the class of unit-balls of subspaces of certain *vector-valued* L_{-k} spaces was given in [21]. To explain this better, we state the definition given by Koldobsky:

Definition. $(\mathbb{R}^n, \|\cdot\|)$ embeds in $L_{-p}(\mathbb{R}^k)$ for $0 < p < n$ iff there exists a measure $\mu \in \mathcal{M}_+(\mathbb{R}^{nk})$ such that for any even test-function ϕ:

$$\int_{\mathbb{R}^n} \|x\|^{-p} \phi(x)\, dx = \int_{\mathbb{R}^{nk}} \int_{\mathbb{R}^k} \|v\|_2^{p-k} \hat{\phi}\left(\sum_{i=1}^k v_i \xi_i\right) dv\, d\mu(\xi). \tag{5.3}$$

For $k = 1$ it is easy to see that this coincides with the definition of embedding in L_{-p}. Using this definition, the following was shown in [21]:

Theorem 5.2 *(Koldobsky). $K \in \mathcal{BP}_k^n$ iff $(\mathbb{R}^n, \|\cdot\|_K)$ embeds in $L_{-k}(\mathbb{R}^k)$.*

For $p > 0$, it is known that every separable vector-valued L_p space is isometric to a subspace of a scalar L_p space and vice versa. Translating Theorem 1.1 into the language of L_p spaces, we see that this is no longer true when $p = -k$, $2 \leqslant k \leqslant n-2$:

Corollary 5.3. *Let $n \geqslant 4$ and $2 \leqslant k \leqslant n-2$. Then there exists an infinitely smooth centrally-symmetric body of revolution K such that $(\mathbb{R}^n, \|\cdot\|_K)$ embeds in L_{-k} but does not embed in $L_{-k}(\mathbb{R}^k)$.*

5.2. Non-trivial spaces which embed in L_p ($p < -1$)

We proceed to describe another property of L_p spaces which breaks down when passing the critical value of $p = -1$.

Definition. Let SL_p^n ($p \neq 0$) denote the set of all star-bodies K in $\mathbb{R}^n$ for which $(\mathbb{R}^n, \|\cdot\|_K)$ embeds in L_p.

For $p \neq 0$, let the *p-norm sum* of two bodies L_1, L_2 be defined as the body L satisfying $\|\cdot\|_L^p = \|\cdot\|_{L_1}^p + \|\cdot\|_{L_2}^p$. Obviously, the p-norm sum coincides with the $(-p)$-radial sum, defined in the Introduction (before the Structure Theorem).

Definition. Let D_p^n ($p \neq 0$) denote the class of bodies created from the Euclidean ball D_n by applying full-rank linear transformations, p-norm sums, and taking the limit in the radial metric d_r.

Using the characterization in (5.1), it is easy to show (e.g. [14, Theorem 6.13]) that for $p > -1$ ($p \neq 0$), $SL_p^n = D_p^n$. In order to understand what happens when $p \leqslant -1$, we turn to the following characterization of $\mathcal{BP}_k^n$, first proved by Goodey and Weil in [13] for intersection bodies (the case $k = 1$), and extended to general k by Grinberg and Zhang in [14]:

Theorem 5.4 *(Grinberg and Zhang). $\mathcal{BP}_k^n = D_{-k}^n$ for $k = 1, \dots, n-1$.*

Recall that $\mathcal{I}_1^n = \mathcal{BP}_1^n$ is the class of all intersection bodies in $\mathbb{R}^n$ and $\mathcal{I}_{n-1}^n = \mathcal{BP}_{n-1}^n$ is the class of all centrally-symmetric star-bodies in $\mathbb{R}^n$ (this is clear from the definitions, see also the Structure Theorem from the Introduction). Since $\mathcal{I}_k^n = SL_{-k}^n$ by characterization (3) of Theorem 5.1, we see that $SL_{-k}^n = D_{-k}^n$ for $k = 1$ and $k = n-1$. However, Theorem 1.1 implies that this is no longer true for $2 \leqslant k \leqslant n-2$:

Corollary 5.5. *Let $n \geqslant 4$ and $2 \leqslant k \leqslant n-2$. Then $SL_{-k}^n \setminus D_{-k}^n \neq \emptyset$.*

Note that since $\mathcal{BP}_k^n \subset \mathcal{I}_k^n$, it is always true that $D_{-k}^n \subset SL_{-k}^n$ (in fact, this is straightforward to check directly, implying that $\mathcal{BP}_k^n \subset \mathcal{I}_k^n$ by using Theorems 5.1 and 5.4). In some sense, the members of D_{-k}^n are the "trivial" elements of SL_{-k}^n, since obviously $D_n \in SL_{-k}^n$, and SL_{-k}^n is closed under taking full-rank linear transformations, $(-k)$-norm sums and limit in the radial-metric. Corollary 5.5 should therefore be interpreted as stating that there are also "non-trivial" elements in SL_{-k}^n, for $2 \leqslant k \leqslant n-2$.

5.3. Non-trivial positive-definite homogeneous distributions

We conclude by translating Corollary 5.5 into the language of Fourier transforms of homogeneous distributions.

Notation. Given an even $f \in C(S^{n-1})$, we denote by $E_p(f)$ its homogeneous extension of degree p onto $\mathbb{R}^n$ (formally excluding $\{0\}$ if $p < 0$), i.e. $E_p(f)(t\theta) = t^p f(\theta)$ for $t > 0$ and $\theta \in S^{n-1}$. We denote by $E_p^{\wedge}(f)$ the Fourier transform of $E_p(f)$ as a distribution. Given a full-rank linear transformation T on $\mathbb{R}^n$, we denote $T(E_p(f)) = E_p(f) \circ T^{-1}$.

Note that $E_p^{\wedge}(f)$ need not necessarily be a continuous function on $\mathbb{R}^n \setminus \{0\}$, nor even a measure on $\mathbb{R}^n$. In order to ensure that $E_p^{\wedge}(f)$ is a continuous function, we need to add some smoothness assumptions on f [22]. We remark that for an infinitely smooth function $f \in C(S^{n-1})$, $E_p^{\wedge}(f)$ is infinitely smooth on $\mathbb{R}^n \setminus \{0\}$ for any $p \in (-n, 0)$. Whenever $E_p^{\wedge}(f)$ is continuous on $\mathbb{R}^n \setminus \{0\}$, it is uniquely determined by its value on S^{n-1} (by homogeneity), so we identify (abusing notation) between $E_p^{\wedge}(f)$ and its restriction to S^{n-1}. Clearly $E_{-k}(\rho_K^k) = \|\cdot\|_K^{-k}$ for a star-body K, hence $T(E_{-k}(\rho_K^k)) = E_{-k}(\rho_{T(K)}^k)$. Again, we identify (by homogeneity) between $T(E_p(f))$ and its restriction to S^{n-1}.

It is easy to check (e.g. [29]) that for any infinitely smooth $K \in D_{-k}^n$, we have $E_{-k}^{\wedge}(\rho_K^k) \geqslant 0$ (and clearly $\rho_K^k \geqslant 0$). In fact, this immediately follows from the fact that this is true for $D_n \in D_{-k}^n$, the linearity of the Fourier transform, and its behavior under full-rank linear transformations. With Theorem 5.4 and characterization (2) of Theorem 5.1 in mind, asking whether $\mathcal{BP}_k^n = \mathcal{I}_k^n$ is equivalent to asking whether the only infinitely smooth functions $f \in C(S^{n-1})$ such that $f \geqslant 0$ and $E_{-k}^{\wedge}(f) \geqslant 0$, are the ones such that $f = \rho_K^k$ for some $K \in D_{-k}^n$. In other words, whether every such f can be approximated (in the topology induced by the maximum norm on $C(S^{n-1})$, which is clearly the same for f and for $f^{1/k}$) by functions of the form $\sum_{i=1}^m T_i(E_{-k}(1))$, where T_i are full-rank linear transformations. The following is thus an immediate consequence of Theorem 1.1:

Corollary 5.6. *Let $n \geqslant 4$ and $2 \leqslant k \leqslant n-2$. Then there exists a "non-trivial" infinitely smooth function of revolution $f \in C(S^{n-1})$ such that $f \geqslant 0$ and $E_{-k}^{\wedge}(f) \geqslant 0$. By "non-trivial," we mean that f cannot be approximated in the maximum norm topology on $C(S^{n-1})$ by functions of the form $\sum_{i=1}^m T_i(E_{-k}(1))$, where $\{T_i\}$ are full-rank linear transformations in $\mathbb{R}^n$.*

5.4. Concluding remark

To conclude, we comment that although the original definitions of $\mathcal{BP}_k^n$ and $\mathcal{I}_k^n$ make sense only for integer values of k (between 1 and $n-1$), some of the alternative characterizations of these classes stated in this section make sense for arbitrary real-valued k, for $0 < k < n$. In particular, characterizations (2) and (3) of Theorem 5.1 for the class $\mathcal{I}_k^n$ and Theorem 5.4 for the class $\mathcal{BP}_k^n$ may be taken as definitions for these classes of star-bodies in this extended range of k. It then makes sense to ask whether Theorem 1.1 also holds for any non-integer $1 < k < n-1$. Although we do not proceed in this direction, the answer should be positive, since our construction of the function $\tilde{g}$ in Proposition 4.1 is purely analytic, and everything still works for arbitrary real-valued k, for $1 < k < n-1$.

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