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Strongly clean matrix rings over commutative local rings

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Abstract

We will completely characterize the commutative local rings for which $\mathbb{M}_n(R)$ is strongly clean, in terms of factorization in $R[t]$. We also obtain similar elementwise results which show additionally that for any monic polynomial $f \in R[t]$, the strong cleanness of the companion matrix of f is equivalent to the strong cleanness of all matrices with characteristic polynomial f.

MSC: 16S50

1. Introduction

An element of a ring is called *clean* if it can be written as the sum of a unit and an idempotent. A ring is clean if each of its elements is clean. This notion was introduced by Nicholson in [20] as a sufficient condition for a ring to have the exchange property. Camillo and Yu further proved in [6] that for rings

semiperfect ⇒ clean ⇒ exchange

with none of the implications reversible.

The authors of [5] define a clean module as one with a clean endomorphism ring. Prior to this, in [13], Han and Nicholson proved that if $e \in R$ is idempotent, then R is a clean ring provided eRe and $(1-e)R(1-e)$ are clean rings; a consequence of this is that if M_1 and M_2 are clean modules then $M_1 \oplus M_2$ is clean.

In [21], Nicholson also defined the notion of strong cleanness. An element of a ring is *strongly clean* if it is the sum of a unit and an idempotent which commute. A ring is strongly clean if each of its elements is strongly clean, and a module is strongly clean if its endomorphism ring is strongly clean. Local rings are strongly clean, and conversely, it follows from Nicholson's characterization of exchange rings in [20] and basic properties of local rings (e.g. [17, Section 19]) that an exchange ring with only the trivial idempotents must be local. This motivates our study of local rings; they are precisely the clean rings with only trivial idempotents, and, as such, provide a natural starting

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point for our investigation. Nicholson proved that strongly π-regular rings are strongly clean. (A ring R is strongly π-regular if all chains of the forms $aR \supseteq a^2R \supseteq \cdots$ and $Ra \supseteq Ra^2 \supseteq \cdots$ terminate.) Basic results on abelian regular rings (see [12, Chapter 3]) and strongly π-regular rings (see [11] or [18, Exercise 23.5]) yield that abelian regular rings and right (or left) perfect rings are strongly clean. In particular right (or left) artinian rings are strongly clean.

Nicholson provides the following useful characterization of strongly clean endomorphisms.

Lemma 1 (*[21]*). *Let M_R be a right module and $f \in \mathrm{End}(M_R)$. The following are equivalent:*

(1) *f is strongly clean.*
(2) *There exist f-invariant submodules A, B such that*

$$M = A \oplus B, \qquad f\colon A \xrightarrow{\sim} A, \quad \textit{and} \quad 1 - f\colon B \xrightarrow{\sim} B.$$

We will use this lemma frequently, and we shall refer to a pair (A, B) as in the lemma as an (A, B)-decomposition.

Although we will not need it for our purposes, there is an equally useful analogue of Lemma 1 for clean endomorphisms which can be found in [5].

In general, for a given $n > 1$, one may wish to characterize rings R for which $\mathbb{M}_n(R)$ is strongly clean. For example, it is easy to see that if R is left (or right) artinian, or if R is locally finite (see [14]; locally finite is Morita invariant, and is stronger than strongly π-regular), then $\mathbb{M}_n(R)$ is strongly clean for all n.

On the other hand, in [25, Example 1], Wang and Chen prove that $\mathbb{M}_2(\mathbb{Z}_{(2)})$ is not strongly clean by an *ad hoc* calculation involving a characterization of the idempotents in the ring $\mathbb{M}_2(\mathbb{Z}_{(2)})$.

The main goal of this paper is to characterize the commutative local rings R for which $\mathbb{M}_n(R)$ is strongly clean; to do so, we shall study factorization in $R[t]$. Indeed, restricting to R commutative local, it is already nontrivial to characterize when $\mathbb{M}_n(R)$ is strongly clean. Our results will place the example from [25] that $\mathbb{M}_2(\mathbb{Z}_{(2)})$ is not strongly clean in a broader context, and will, in particular, provide a conceptual proof of this result. Furthermore, our results will provide a large class of examples of semiperfect rings which are not strongly clean, giving further negative answers to [21, Question 5] (the first having been given in [25]).

During the time our results were being readied for publication, other papers have appeared which independently address certain special cases of the problem at hand: the characterization of when $\mathbb{M}_n(R)$, or an element thereof, is strongly clean. In parallel to our work, in [10], Chen, Yang and Zhou characterized the commutative local rings R for which $\mathbb{M}_2(R)$ is strongly clean, and in [19], Li completed the methods of [10] to characterize when an element $A \in \mathbb{M}_2(R)$ is strongly clean. Most of the results of [10,19] are special cases of Theorem 12 and Corollary 15. Proposition 30, as well as many of the results of Section 5, demonstrate how our methods obtain the results for the case $n = 2$ which also appear in [10,19]. Proposition 30 and Section 5 also help to place the $n = 2$ case in context, displaying some of the properties which are present in the $n = 2$ case, but which do not prevail for general n.

Following an earlier preprint of the present paper, Yang and Zhou have investigated, in [26], the class of n-SRC rings defined in Definition 5. Theorem 25 provides a natural result which encompasses many of the examples of [26].

2. Preliminaries and factorization in $R[t]$

Throughout the remainder of this paper, (R, J) will denote a commutative local ring R with maximal ideal J, and $\pi\colon R \longrightarrow R/J$ will denote the natural quotient ring homomorphism from R onto the field R/J. We will write $\pi(r) = \overline{r}$ and will have occasion to denote $\overline{R} = R/J$. The symbol π and bar-notation will denote the induced ring homomorphisms $R[t] \longrightarrow \overline{R}[t]$ and $\mathbb{M}_n(R) \longrightarrow \mathbb{M}_n(\overline{R})$.

If $\varphi \in \mathbb{M}_n(R)$, $\chi(\varphi) = \det(tI - \varphi) \in R[t]$ will denote the characteristic polynomial of φ, which is a monic polynomial of degree n. If $h \in R[t]$ is a monic polynomial of degree n, $C_h \in \mathbb{M}_n(R)$ will denote the companion matrix of h (as defined, e.g. [15, p. 358]). It is an easy exercise to show that $\chi(C_h) = h$, and that, over a field, the minimal polynomial of C_h is h. Finally, for any basis of R^n_R we will make the usual identification $\mathbb{M}_n(R) \cong \mathrm{End}_R(R^n)$. Recall that the characteristic polynomial of an endomorphism is independent of the choice of basis.

A basic result about local rings we shall need is:

Proposition 2 (*See, for Example, [17, Theorem 19.29], [16, Example 1.6(b)]*). *If (R, J) is a local ring (not necessarily commutative), then, any finitely generated projective right R-module P is free of unique rank. In particular, if $R^n \cong A \oplus B$ (as right R-modules), then A and B are free right R-modules, and* $\mathrm{rank}(A) + \mathrm{rank}(B) = n$.

We shall use this fact freely without further mention. Also we shall use the fact that if R is commutative, $\varphi \in \mathrm{End}(R^n)$ and $R^n = A \oplus B$ as R-free $R[\varphi]$-modules, then $\chi(\varphi) = \chi(\varphi|_A)\chi(\varphi|_B)$.

Definition 3. For $r \in R$, define

$$\mathbb{S}_r = \{f \in R[t] : f \text{ monic}, f(r) \in U(R)\}.$$

Recall that $U(R) = R \setminus J$ when R is local. Also, note that if f is monic, $f \in \mathbb{S}_r$ if and only if $\overline{f}(\overline{r}) \neq 0$.

Lemma 4. *For $r \in R$ and $\varphi \in \mathbb{M}_n(R)$, $\psi = rI - \varphi$ is invertible if and only if $\chi(\psi) \in \mathbb{S}_r$. In particular,*
(1) *φ is invertible if and only if $\chi(\varphi) \in \mathbb{S}_0$, and*
(2) *$1 - \varphi$ is invertible if and only if $\chi(\varphi) \in \mathbb{S}_1$.*

Proof. From standard facts concerning linear algebra over commutative rings, $rI - \varphi$ is invertible if and only if $\det(rI - \varphi)$ is a unit in R. But $\det(rI - \varphi) = \chi(\varphi)(r)$ (by definition of $\chi(\varphi)$), and $\chi(\varphi)(r)$ is a unit in R if and only if $\chi(\varphi) \in \mathbb{S}_r$. For the last two statements take $r = 0$ and $r = 1$, respectively. □

Definition 5. Let $x_0, \ldots, x_k \in R$. A factorization

$$h = g_{x_0} g_{x_1} \cdots g_{x_k}$$

in $R[t]$ of a monic polynomial h is said to be an SR factorization *relative to* $(x_0, \ldots, x_k)$ if $g_{x_i} \in \bigcap_{j \neq i} \mathbb{S}_{x_j}$. If, in addition, $\{\overline{g_{x_0}}, \ldots, \overline{g_{x_k}}\}$ are pairwise coprime in (the PID) $\overline{R}[t]$, the factorization will be said to be an SRC factorization *relative to* $(x_0, \ldots, x_k)$. An SR (resp. SRC) factorization will refer to the special case of an SR (resp. SRC) factorization relative to (1, 0), but with the subscripts modified as follows. For ease of notation, if $h(t) = f_1(t) f_0(t)$ is an SR (resp. SRC) factorization relative to (1, 0), we shall write $g_0 = f_1$ and $g_1 = f_0$. Explicitly, a factorization $h(t) = g_0(t)g_1(t)$ is an SR factorization if $g_0 \in \mathbb{S}_0$ and $g_1 \in \mathbb{S}_1$; if, in addition, g_0 and g_1 are coprime in $\overline{R}[t]$, then $h(t) = g_0(t)g_1(t)$ is an SRC factorization. We shall say that a ring R is an n-SRC ring (resp. n-SR ring) if every monic polynomial of degree n has an SRC factorization (resp. SR factorization).

An SR factorization of h is precisely a lift to $R[x]$ of a factorization $\overline{h} = F_0 F_1$ in $\overline{R}[t]$, $F_0(0) \neq 0$ and $F_1(1) \neq 0$. Therefore, if $h = g_0 g_1$ is an SR factorization and $\overline{h}(t) = t^r (t-1)^s F(t)$, where $F(0)$ and $F(1)$ are nonzero, then $\overline{g_0}(t) = (t-1)^s G_0(t)$ and $\overline{g_1}(t) = t^r G_1(t)$ for some $G_0, G_1 \in \overline{R}[t]$, where G_0 and G_1 are relatively prime to $t(t-1)$. This motivates our choice of the abbreviations SR (separates roots) and SRC (separates roots coprimely).

Recall that if R is a ring and $a, b \in R$, we say that (a, b) is right unimodular if $aR + bR = R$.

Remark 6. Since $\overline{R}[t]$ is a PID, $\overline{f}, \overline{g} \in \overline{R}[t]$ are coprime (have no nonunit common divisors) if and only if $(\overline{f}, \overline{g})$ is (right) unimodular in $\overline{R}[t]$ (e.g. by [15, Theorem 3.11(b)]).

In fact, for a commutative ring R, for $f, g \in R[t]$, with f monic, (f, g) are unimodular in $R[t]$ if and only if $(\overline{f}, \overline{g})$ is unimodular in $(R/J)[t]$. This follows easily from Nakayama's Lemma, as follows. Since f is monic, say of degree n, $R[t]/(f)$ is free of rank n, so the further quotient $R[t]/(f, g)$ is finitely generated. Since $(\overline{f}, \overline{g})$ is unimodular in $(R/J)[t]$, we see that $(f, g) + J(R) \cdot R[t] = R[t]$, and hence $(f, g) = R[t]$ by Nakayama's Lemma (e.g. [17, Theorem 4.22]), so (f, g) is unimodular. The reverse implication is obvious.

The following is an expected observation.

Lemma 7. *Suppose $m \geq 1$. If R is an m-SRC ring (resp. m-SR ring), then R is an n-SRC ring (resp. m-SR ring) for all $1 \leq n \leq m$.*

Proof. It suffices to show this result when $n = m - 1 \geq 1$. Let $h \in R[t]$ be a monic polynomial of degree n. We may assume that $h \notin \mathbb{S}_0 \cup \mathbb{S}_1$; otherwise, h trivially has an SRC factorization. Now, $t \cdot h(t)$ is a monic polynomial of degree $n + 1$, and hence, since R is an $(n + 1)$-SRC ring, $t \cdot h(t) = g_0(t)g_1(t)$ for some $g_i \in \mathbb{S}_i$ such that $\overline{g_0}$ and $\overline{g_1}$ are coprime in $\overline{R}[t]$. Now, $g_0(0)g_1(0) = 0$, and $g_0(0) \in R \setminus J = U(R)$ since $g_0 \in \mathbb{S}_0$, so $g_1(0) = 0$. Therefore, $g_1(t) = t f_1(t)$, for some $f_1(t) \in R[t]$. We can easily see that $f_1 \in \mathbb{S}_1$, since $f_1(1) = g_1(1) \notin J$. Also, it is obvious that $\overline{f_1}$ and $\overline{g_0}$ are coprime, since $\overline{f_1}$ is a divisor of $\overline{g_1}$, and $\overline{g_0}$ and $\overline{g_1}$ are coprime. Now, $t(h(t) - g_0(t) f_1(t)) = 0$, but t is a nonzerodivisor in $R[t]$, so we conclude that $h = g_0 f_1$, and hence that h has an SRC factorization. The proof for SR rings is similar, omitting all references to coprimeness. □

In the next section, we will show how SRC factorizations are related to strong cleanness of matrices over commutative local rings. In later sections we will examine the SRC property further (in particular, providing examples and non-examples of SRC rings in Section 4).

3. SRC factorization and strong cleanness in $\mathbb{M}_n(R)$

In this section, we develop the relationship between SR and SRC factorizations in $R[t]$ and the strong cleanness of elements in $\mathbb{M}_n(R)$.

Lemma 8. *If $h = h_0h_1 \in R[t]$ is an SR factorization, then the matrix $C_{h_0} \oplus C_{h_1} = \begin{bmatrix} C_{h_0} & 0 \\ 0 & C_{h_1} \end{bmatrix}$ is a strongly clean matrix that has characteristic polynomial h.*

Proof. By Lemma 4, C_{h_0} and $1 - C_{h_1}$ are invertible matrices. Thus, the matrix $E = \begin{bmatrix} 0_{\deg(h_0)} & 0 \\ 0 & 1_{\deg(h_1)} \end{bmatrix}$ is an idempotent with respect to which $C_{h_0} \oplus C_{h_1}$ is strongly clean. □

On the other hand, the following lemma shows that the characteristic polynomial of any strongly clean matrix has an SR factorization.

Lemma 9. *If $\varphi \in \mathbb{M}_n(R)$ is strongly clean, then $\chi(\varphi)$ has an SR factorization.*

Proof. By Lemma 1, there exist φ-invariant R-submodules A and B such that $R^n = A \oplus B$ and such that $\varphi|_A$ and $(1 - \varphi)|_B$ are both isomorphisms. Therefore, by Lemma 4, $\chi(\varphi) = \chi(\varphi|_A)\chi(\varphi|_B)$ is an SR factorization. □

In Lemma 9, one cannot replace the SR factorization with an SRC factorization, as the following example shows:

Example 10. Let $R = \mathbb{Z}_{(p)}$, with $p \equiv 3 (\mathrm{mod}\ 4)$, and consider the polynomial

$$h = h_0h_1 = [(t-1)(t^2+1) + p^2] \cdot [t(t^2+1) + p^2].$$

The given factorization is an SR factorization which is not an SRC factorization. Observe that $\overline{h} = t(t-1)(t^2+1)^2$, and that this is the factorization of $\overline{h}$ into irreducibles in $\overline{R}[t] \cong (\mathbb{Z}/p\mathbb{Z})[t]$, because -1 is not a square in $\mathbb{Z}/p\mathbb{Z}$. One can show that h has no roots in $\mathbb{Q}$, much less in R (e.g. by the rational root test). This precludes the existence of any other SR factorization of h; in particular, h has no SRC factorization. However, by Lemma 8, the matrix

$$X = C_{h_0} \oplus C_{h_1} \in \mathbb{M}_6(R)$$

is strongly clean, yet there is no SRC factorization for $\chi(X)$. In Theorem 12, we shall see that there exist matrices with characteristic polynomial h (in particular, the companion matrix of h) which are not strongly clean in $\mathbb{M}_6(R)$.

First, we need a lemma.

Lemma 11. *Let R be a ring, and M_R a right R-module. Suppose that $g, h \in \mathrm{End}(M_R)$ with $hg = gh = 0$, and that there exist $s, t \in \mathrm{End}(M_R)$ such that $sg + th = Id_{M_R}$, where $sh = hs$ and $tg = gt$. Then, $M = \ker(g) \oplus \ker(h)$ (as right R-modules).*

Proof. If $x \in \ker(g) \cap \ker(h)$, $x = s(g(x)) + t(h(x)) = 0 + 0 = 0$, so $\ker(g) \cap \ker(h) = 0$. On the other hand, for $x \in M$, we have $x = s(g(x)) + t(h(x))$. Note that $h(s(g(x))) = s(h(g(x))) = s((hg)(x)) = 0$ and $g(t(h(x))) = t(gh(x)) = 0$, and hence $M_R = \ker(g) + \ker(h)$. We conclude that $M_R = \ker(g) \oplus \ker(h)$. □

Theorem 12. *Suppose (R, J) is commutative local, $n \geq 2$, and $h \in R[t]$ is a monic polynomial of degree n. Then, the following are equivalent:*

(1) *For any $\varphi \in \mathbb{M}_n(R)$ with $\chi(\varphi) = h$, φ is strongly clean.*
(2) *The companion matrix C_h of h is strongly clean in $\mathbb{M}_n(R)$.*
(3) *There exists an SRC factorization of h in $R[t]$.*

Proof. (1) $\Longrightarrow$ (2): This implication is clear, since $\chi(C_h) = h$.

(2) $\Longrightarrow$ (3): By assumption, C_h is strongly clean, so, Lemma 1 guarantees the existence of a decomposition $R^n = A \oplus B$ into C_h-invariant R-submodules, such that $C_h|_A$ is invertible on A, and $(1 - C_h)|_B$ is invertible on B. By the discussion following Proposition 2 the characteristic polynomial h of C_h factors as $h = g_0 g_1$, where g_0 is the characteristic polynomial of $C_h|_A$ and g_1 is the characteristic polynomial of $C_h|_B$. But now, $g_0 \in \mathbb{S}_0$ and $g_1 \in \mathbb{S}_1$ by Lemma 4. We claim that $\overline{g_0}$ and $\overline{g_1}$ are coprime in the principal ideal domain $\overline{R}[t]$. Suppose, to obtain a contradiction, that $F \in \overline{R}[t]$ is a common factor of positive degree of $\overline{g_0}$ and $\overline{g_1}$. Since $\overline{R}$ is a field, the minimal polynomial of $\overline{C_h} = C_{\overline{h}}$ is $\overline{h}$, which has degree n. The minimal polynomial of $\overline{C_h}$ must also be the least common multiple of the minimal polynomials of $\overline{C_h}|_{\overline{A}}$ and $\overline{C_h}|_{\overline{B}}$, which must divide the least common multiple of $\overline{g_0}$ and $\overline{g_1}$ (since the minimal polynomials divide the characteristic polynomials). But the least common multiple of $\overline{g_0}$ and $\overline{g_1}$ will be a divisor of $\overline{g_0 g_1}/F$, which has degree less than n, which is a contradiction.

(3) $\Longrightarrow$ (1): Suppose that $\varphi \in \mathbb{M}_n(R)$ with $\chi(\varphi) = h$. By assumption, $h = g_0 g_1$ for some $g_i \in \mathbb{S}_i$, with $\overline{g_0}, \overline{g_1}$ coprime in $\overline{R}[t]$. Hence, since $\overline{R}[t]$ is a PID, there exist $a_0, a_1 \in R[t]$ such that $\overline{a_0 g_0} + \overline{a_1 g_1} = \overline{1}$. Then, $(a_0 g_0 + a_1 g_1)(\varphi) = (1 + \alpha)(\varphi)$, for some $\alpha \in J[t]$. But now, the right hand side is a matrix whose determinant is in $1 + J$, and is hence a unit. In fact, by the Cayley–Hamilton Theorem, the inverse of $(1 + \alpha)(\varphi)$ can be expressed as a polynomial $(1 + f)(\varphi)$ for some $f \in J[t]$. Therefore, defining matrices $f_i(\varphi) = a_i(1 + f)(\varphi)$ we have $f_0(\varphi)g_0(\varphi) + f_1(\varphi)g_1(\varphi) = Id_{R^n}$. Note that $g_0(\varphi)g_1(\varphi) = g_1(\varphi)g_0(\varphi) = h(\varphi) = 0$, by the Cayley–Hamilton Theorem. Also, observe that $f_0(\varphi)$, $g_0(\varphi)$, $f_1(\varphi)$, and $g_1(\varphi)$ each belong to $R[\varphi]$, which is a commutative subring of $\mathrm{End}(M_R)$. Therefore, by Lemma 11, $R^n = \ker(g_0(\varphi)) \oplus \ker(g_1(\varphi))$. Set $A = \ker(g_0(\varphi))$ and $B = \ker(g_1(\varphi))$. Clearly both A and B are φ-invariant, since φ commutes with $g_0(\varphi)$ and $g_1(\varphi)$.

We claim that φ is an isomorphism on A, and $1 - \varphi$ is an isomorphism on B. Set $\psi = \varphi|_A$. We know that $g_0(\psi) = 0$, by definition of A. The constant term of g_0 is a unit, since $g_0 \in \mathbb{S}_0$ and hence ψ must be invertible with inverse $-g_0(0)^{-1}(\psi^{m-1} + c_{m-1}\psi^{m-2} + \cdots + c_1 I)$, where $g_0 = t^m + c_{m-1}t^{m-1} + \cdots + c_1 t + c_0$. Similarly, we see that $(1 - \varphi)|_B$ is invertible on B. Therefore, by Lemma 1, φ is strongly clean. □

In particular, we see from Theorem 12 and Example 10 that there cannot exist a characterization of strong cleanness of a matrix involving only factorization properties of the characteristic polynomial. This is in contrast with the $n = 2$ case (e.g., see [19]).

Remark 13. Focusing on the set of matrices which have a fixed characteristic polynomial h, Theorem 12 shows that if the companion matrix C_h is strongly clean, then each of the other matrices in the set is strongly clean. This is reasonable to expect, however, for the following reasons. Note that $\varphi = \overline{C_h} = C_{\overline{h}}$ has minimal and characteristic polynomial $\overline{h}$. We know that $\overline{R}^n = \oplus_f K_f$, where each $K_f = \cup_{i \geq 0} \ker(f^i(\varphi))$ is the generalized eigenspace of φ corresponding to an irreducible factor f of $\overline{h}$ (cf. rational canonical form). For $\varphi = \overline{C_h}$, we claim that each K_f is indecomposable as an $R[\varphi]$-module (more generally, this conclusion holds whenever the matrix has minimal polynomial equal to its characteristic polynomial). To see this, suppose that $K_f = A \oplus B$ is a decomposition of K_f into nonzero $R[\varphi]$-modules. Then, the minimal polynomials of $\varphi|_A$ and $\varphi|_B$ are each powers of f (since $f^n(\varphi) = 0$, f is irreducible, and the minimal polynomials must each divide f^n), say f^{n_1} and f^{n_2}, respectively. Therefore, the minimal polynomial of $\varphi|_{K_f}$ is $f^{\max(n_1,n_2)}$. Thus, the minimal polynomial of φ has degree no more than $n - \min(n_1, n_2)$, and hence, the minimal and characteristic polynomials of φ cannot be equal, which is a contradiction. Thus, K_f is indecomposable as an $R[\varphi]$-module.

On the other hand, suppose that $R^n = A \oplus B$, as $R[C_h]$-modules, as in Lemma 1. Then, $\overline{A} \oplus \overline{B} = \overline{R}^n$, and intersecting this with the generalized eigenspaces, we find that each generalized eigenspace must belong entirely to $\overline{A}$ or $\overline{B}$ (note that when dealing with a strongly clean matrix whose minimal and characteristic polynomials are not equal, only $K_t \subseteq \overline{B}$ and $K_{t-1} \subseteq \overline{A}$ are forced). In fact, $\overline{A}$ and $\overline{B}$ are direct sums of generalized eigenspaces. This substantially restricts the possible SR factorizations corresponding to an (A, B)-decomposition as in Lemma 1. In fact, *every* (A, B)-decomposition that witnesses the strong cleanness of C_h corresponds to an SRC factorization, whereas, in general, an (A, B)-decomposition merely corresponds, by (the proof of) Lemma 9, to an SR factorization.

Suppose $\varphi \in \mathrm{End}(M_R)$ and f is an irreducible factor of $\chi(\varphi)$ which is not t nor $t - 1$. If the generalized eigenspace K_f is *decomposable* as an $R[\varphi]$-module, there is far greater flexibility in finding an (A, B)-decomposition.

Before stating the corollary characterizing the commutative local rings for which $\mathbb{M}_n(R)$ is strongly clean, let us further characterize n-SRC rings.

Proposition 14. *For R commutative local, the following are equivalent:*
(1) *R is an n-SRC ring (resp. n-SR ring).*
(2) *For every $x_0, x_1 \in R$ with $x_0 - x_1 \in U(R)$, every monic polynomial $h \in R[t]$ of degree n has an SRC factorization (resp. SR factorization) relative to (x_0, x_1).*
(3) *For every $x_0, \ldots, x_k \in R$, with $x_i - x_j \in U(R)$ whenever $i \neq j$, every monic polynomial $h \in R[t]$ of degree n has an SRC factorization (resp. SR factorization) relative to $(x_0, x_1, \ldots, x_k)$.*

Proof. We will only prove the statements involving SRC rings and SRC factorizations. Ignoring any reference to coprimeness we will prove the analogous statements for SR rings and SR factorizations.

(1) $\Longrightarrow$ (2): We will perform a linear change of variables. Suppose that $h \in R[t]$ is a monic polynomial of degree n. The polynomial

$$H(t) = (x_1 - x_0)^{-n} h((x_1 - x_0)t + x_0)$$

is a monic polynomial in $R[t]$, and, by hypothesis, there exists a factorization $H = G_0 G_1$, with $G_i \in \mathbb{S}_i$. Set

$$g_i(t) = (x_1 - x_0)^{\deg(g_i)} G_i((x_1 - x_0)^{-1} t - (x_1 - x_0)^{-1} x_0).$$

Then,

$$g_0(t) g_1(t) = (x_1 - x_0)^n H((x_1 - x_0)^{-1} t - (x_1 - x_0)^{-1} x_0) = h(t).$$

Observe that $g_i \in \mathbb{S}_{x_i}$, since $\overline{g_i}(x_i) = \overline{G_i}(i) \notin J$. Finally, note that g_0 and g_1 are obviously coprime, since any common factor would give rise to a common factor (of the same degree) of G_0 and G_1, upon performing the appropriate change of variables. Set $f_{x_0} = g_1$ and $f_{x_1} = g_0$. Therefore, $h = f_{x_0} f_{x_1}$ is an SRC factorization relative to (x_0, x_1).

(3) $\Longrightarrow$ (2) $\Longrightarrow$ (1): These implications are obvious specializations.

(2) $\Longrightarrow$ (3): We will prove this statement by induction on n. We may assume that $\overline{h}(x_i) = 0$ for all i; otherwise the factor corresponding to x_i can be chosen to be 1. For $n = 1$, condition (3) holds vacuously. Thus, supposing that $n > 1$ and that the implication (2) $\Longrightarrow$ (3) holds for $j < n$, let $h \in R[t]$ be a monic polynomial of degree n. By (2), there exists a factorization $h = g_{x_0} g_{x_1}$ which is an SRC factorization relative to (x_0, x_1). Now, $\deg(g_i) < \deg(h)$, since $\overline{h}(\overline{x_i}) = 0$ for $i \in \{0, 1\}$. By Lemma 7, R is a $\deg(g_i)$-SRC ring, and hence, by our inductive hypothesis, g_0 and g_1 each have SRC factorizations relative to $(x_0, x_1, \ldots, x_k)$, given by $g_{x_0} = s_{x_0} s_{x_1} \cdots s_{x_k}$ and $g_1 = r_{x_0} r_{x_1} \cdots r_{x_k}$. Thus, $h = (s_{x_0} r_{x_0})(s_{x_1} r_{x_1}) \cdots (s_{x_k} r_{x_k})$ is an SR factorization relative to $(x_0, \ldots, x_k)$, since $s_{x_i} r_{x_i}$ is obviously in $\bigcap_{j \neq i} \mathbb{S}_{x_j}$ because each $\mathbb{S}_{x_j}$ is closed under multiplication. It is easy to check that $\{\overline{s_{x_i} r_{x_i}}\}_i$ is pairwise coprime, using the fact that $h = g_{x_0} g_{x_1}$ is an SRC factorization relative to (x_0, x_1) and that $g_{x_0} = s_{x_0} s_{x_1} \cdots s_{x_k}$ and $g_{x_1} = r_{x_0} r_{x_1} \cdots r_{x_k}$ are each SRC factorizations relative to $\{x_0, x_1, \ldots, x_k\}$. Therefore, $h = (s_0 r_0) \cdots (s_k r_k)$ is an SRC factorization relative to $(x_0, \ldots, x_k)$. □

Corollary 15. *For a commutative local ring R and $n \geq 1$, the following are equivalent:*
(1) *$\mathbb{M}_n(R)$ is strongly clean.*
(2) *Every companion matrix in $\mathbb{M}_n(R)$ is strongly clean.*
(3) *R is an n-SRC ring.*
(4) *For every $x_0, x_1 \in R$ with $x_0 - x_1 \in U(R)$, every monic polynomial $h \in R[t]$ of degree n has an SRC factorization relative to (x_0, x_1).*
(5) *For every $x_0, \ldots, x_k \in R$, with $x_i - x_j \in U(R)$ whenever $i \neq j$, every monic polynomial $h \in R[t]$ of degree n has an SRC factorization relative to $(x_0, x_1, \ldots, x_k)$.*

Proof. The equivalence of the first three statements follows immediately from Theorem 12. The equivalence of the last three statements is as in Proposition 14. □

Remark 16. By Lemma 7 and Corollary 15, we can now see directly that (for R commutative local) if $\mathbb{M}_m(R)$ is strongly clean and $n \leq m$ then $\mathbb{M}_n(R)$ is strongly clean. On the other hand, it is true in general that if S is a strongly clean ring and $e^2 = e \in S$, then eSe is a strongly clean ring (personal communication from T.Y. Lam; independently, this was proved by Sánchez Campos, and also appears in [8, Theorem 2.4]). This observation together with Corollary 15 yields an alternate proof of Lemma 7.

Another consequence of Proposition 14 is the following.

Proposition 17. *For $n < 6$, R is an n-SR ring if and only if R is an n-SRC ring.*

Proof. Only the forward implication needs proof. Suppose $n < 6$ and that R is an n-SR ring. Let $h(t) \in R[t]$ be a monic polynomial of degree n. We may assume $h(0), h(1) \in J$, for otherwise, h has a trivial SRC factorization. Write $\overline{h}(t) = H(t)\prod_{i=0}^{m}(t-\overline{r_i})^{k_i}$, where $\overline{r_i} \neq \overline{r_j}$ if $i \neq j$, $k_i \geq 1$, and $H \in \overline{R}[t]$ has no roots in $\overline{R}$. We may assume $r_0 = 0$ and $r_1 = 1$, since $\overline{h}(0) = \overline{h}(1) = 0$. By Proposition 14, there is an SR factorization $h(t) = g_{r_0} \cdots g_{r_m}$ relative to $(r_0, \ldots, r_m)$. Set $f_0 = g_{r_1}$ and $f_1 = g_{r_0}\prod_{j=2}^{m} g_{r_j}$. Then, $h = f_0 f_1$ is an SR factorization, since $g_{r_1} \in \bigcap_{j\neq i}\mathbb{S}_{r_j} \subseteq \mathbb{S}_0$, and $g_{r_0}\prod_{j=2}^{m} g_{r_j} \in \mathbb{S}_1$. Suppose that $\overline{f_0}$ and $\overline{f_1}$ have a common factor. We know that $\overline{f_0} = F_0(t)(t-1)^{k_1}$ and $\overline{f_1} = F_1(t)t^{k_0}\prod_{i=2}^{m}(t-r_i)^{k_i}$ for some $F_0, F_1 \in \overline{R}[t]$ which have no roots in $\overline{R}$. Therefore, if $\overline{f_0}$ and $\overline{f_1}$ have a common factor $D \in \overline{R}[t]$ of positive degree, then D must be a common factor of F_0 and F_1. But F_0 and F_1 have no roots in $\overline{R}$, so $\deg(D) \geq 2$. Therefore, $\deg(f_0) \geq 3$ and $\deg(f_1) \geq m+2 \geq 3$. Therefore, $n = \deg(f_0) + \deg(f_1) \geq 6$, which is a contradiction. □

Problem 18. For $n \geq 6$, do there exist examples of commutative local rings which are n-SR rings but are not n-SRC rings?

4. Examples of SRC rings

The notion of an *SRC* factorization is quite similar to the situation of Hensel's Lemma.

Example 19. A Henselian local ring is a commutative local ring (R, J) which satisfies Hensel's Lemma, which says that if $f \in R[t]$ is monic and satisfies $\overline{f} = GH$, with $G, H \in \overline{R}[t]$ monic and coprime, then $f = gh$ for some monic polynomials $g, h \in R[t]$, with $\overline{g} = G$ and $\overline{h} = H$. Requiring that Hensel's Lemma be satisfied for polynomials of degree $\leq n$ in $R[t]$ implies that R is an n-SRC ring. Hensel's Lemma can be used to lift the factorization $\overline{f} = t^i(t-1)^j H(t)$, where H is coprime to t and to $(t-1)$, to a factorization $f = g_0 g_1 h$ (which is more than is needed for an SRC factorization). Therefore, by Corollary 15, any Henselian local ring is an n-SRC ring for all n. In particular, any local ring which is complete with respect to its maximal ideal is Henselian, and in particular, must be an n-SRC ring for all n.

In fact, a much more general result is true for Henselian rings.

Theorem 20. *Let R be a Henselian local ring. Then, any algebraic R-algebra is strongly clean* (*here, an associative ring A is said to be an R-algebra if A is equipped with a unital ring homomorphism $\varphi : R \longrightarrow A$ mapping into the center of A; A is algebraic over R if each $a \in A$ satisfies a monic polynomial with coefficients in $\varphi(R)$*).

Proof. Let $x \in A$ satisfy a monic polynomial of degree n, with coefficients in $\varphi(R)$. The unital R-subalgebra S generated by x in A is commutative and is finitely generated as an R-module (generated by $1, x, \ldots, x^{n-1}$). By [4, Exercise III.4.3], S is a direct product of local rings; it follows that S is strongly clean, and in particular that $x = e + u$ with $eu = ue$ for some $e^2 = e \in S$ and $u \in U(S)$. Certainly e is idempotent in A and u is a unit in A since the identity element of S is the same as the identity element of A. We conclude that x is strongly clean as an element of A. Therefore, A is strongly clean. □

Corollary 21. *If R is a Henselian local ring, then any unital R-subalgebra of $\mathbb{M}_n(R)$ is strongly clean.*

Proof. By the Cayley–Hamilton Theorem, any element of $\mathbb{M}_n(R)$ satisfies a monic polynomial with coefficients in R. In particular, any unital R-subalgebra of $\mathbb{M}_n(R)$ is algebraic over R. □

Remark 22. Looking ahead to Section 7, any incidence ring over a finite preordered set with coefficients in a Henselian ring R embeds as an R-subalgebra of a full matrix ring, and hence must be strongly clean.

A weaker result which can be seen several times means the following.

Corollary 23. *If R is a commutative local artinian ring, then R is an n-SRC ring for all n.*

Proof. For such rings and all n, $\mathbb{M}_n(R)$ is artinian, and hence $\mathbb{M}_n(R)$ is strongly clean for all n. Alternatively, R is certainly complete with respect to its nilpotent maximal ideal, and hence R is Henselian. □

To state the next theorem, it will be useful to have the following relative notion of the property that R is Henselian.

Definition 24. Let R be a commutative local ring, let I be an ideal of R, let $\overline{R} = R/I$ and let bar-notation denote the usual quotient map. We say that R is n-Henselian with respect to I if whenever $f \in R[t]$ is monic of degree n and satisfies $\overline{f} = GH$, with $G, H \in (R/I)[t]$ monic and unimodular, then $f = gh$ for some monic polynomials $g, h \in R[t]$, with $\overline{g} = G$, and $\overline{h} = H$. We shall say that R is Henselian with respect to I if R is n-Henselian with respect to I, for each $n \in \mathbb{N}$.

In particular, one can show that if R is a local ring which is complete with respect to an ideal $I \subseteq J(R)$, then R is Henselian with respect to I (e.g. see [4, Section III.4.3, Theorem 1 (p. 215)]).

The following is then a natural generalization of the observation that Henselian local rings (in fact, the n-Henselian local rings) are n-SRC rings.

Theorem 25. *Let R be a commutative local ring, and let $I \subseteq J(R)$ be an ideal, such that R is n-Henselian with respect to I. Then, R is an n-SRC ring if and only if R/I is an n-SRC ring.*

Proof. In the light of Corollary 15 (or directly from the definition), the forward implication is obvious, since, for instance, $\mathbb{M}_n(R)$ strongly clean implies that its quotient $\mathbb{M}_n(R/I)$ is strongly clean.

For the reverse implication, suppose that R/I is an n-SRC ring, and let $f \in R[t]$ be a monic polynomial of degree n. We will obtain a factorization of f in two steps: first we obtain an SRC factorization of the image of f in $(R/I)[t]$ using the fact that R/I is an n-SRC ring; next, we use the condition that R is Henselian with respect to I to lift that factorization to $R[t]$.

We have two quotient rings under consideration. The use of a subscript 1 will denote the image in $(R/I)[t]$, and the subscript 2 will denote the image in $(R/J)[t]$.

By hypothesis R/I is a (local) n-SRC ring with maximal ideal J/I, so $f_1 \in (R/I)[t]$ has an SRC factorization $f_1 = GH$. That is, G and H are monic, their images are unimodular in $(R/J)[t]$, and finally, $G(0)$ and $H(1)$ are units of R/I. Since R is n-Henselian with respect to I, the factorization $f_1 = GH$ lifts to a factorization $f = gh$ in $R[t]$, where g and h are monic polynomials for which $g_1 = G$ and $h_1 = H$. We claim that $f = gh$ is an SRC factorization. Indeed, $g_2 = G_2$ and $h_2 = H_2$ are coprime in $(R/J)[t]$, and also $G(0) = g_1(0)$ and $H(1) = h_1(1)$ are units in R/I. Thus, the images of $g(0)$ and $h(1)$ in R/I are units. Since $I \subseteq J(R)$, it follows that $g(0)$ and $h(1)$ are units. We conclude that $f = gh$ is an SRC factorization, as desired. □

Theorem 25 can be used to provide simpler, more conceptual proofs of some of the results of [26]. For instance, [26, Theorem 2.6] is immediate because $R[[x]]$ is complete with respect to the ideal (x), and $R[[x]]/(x) \cong R$. In addition, we can generalize [26, Theorem 2.9] as follows (which deals with the case when $p = 2$), and provide a much simpler proof thereto.

Corollary 26. *Let p be a prime number, G a finite abelian p-group, and let R be a commutative local ring for which $p = 0$ in R. Then, RG is n-SRC if and only if R is n-SRC.*

Proof. We claim that the augmentation ideal $\Delta(RG)$ is nilpotent, and hence that RG is complete with respect to $\Delta(RG)$. If $g \in G$, then $(1 - g)^{|G|} = 1 - g^{|G|}$, since $p = 0$ in R, so $(1 - g)^{|G|} = 0$. Now, $\Delta(RG)$ is generated, as an ideal, by the finite set $\{1 - g : g \in G\}$, all of whose elements are nilpotent. Since RG is commutative, we conclude that $\Delta(RG)$ is nilpotent. It follows that RG is complete with respect to $\Delta(RG)$; in particular, RG is Henselian with respect to $\Delta(RG)$. Since $RG/\Delta(RG) \cong R$, we conclude from Theorem 25 that RG is n-SRC if and only if R is n-SRC. □

We now turn our attention to an example. We thank Pace Nielsen for permitting us to include the following construction of a 2-SRC ring which is not a 3-SRC ring.

Example 27 (*P. Nielsen, Personal Communication*). Let K be the quadratic closure of $\mathbb{Q}$. Let S be the ring of integers in K (i.e. S is the set of elements of K which satisfy monic polynomials over $\mathbb{Z}$). Now, let p be a prime in $\mathbb{Z}$. Then p^{-1} is not an integral element, hence it does not lie in S. Let I be a prime ideal of S containing p (which exists since p is not invertible in S). For ease, we will set $p = 3$.

Define R to be the ring S localized at I. Notice that R naturally embeds in K. The polynomial $f(x) = x^2(x-1)+3$ is irreducible over $\mathbb{Q}$, and hence over K, since the degree of every finite extension of $\mathbb{Q}$ contained in K is a power of 2. Therefore, R is not a 3-SRC ring.

On the other hand, R is a 2-SRC ring. To see this, let $f(x) = x^2 - ax + b \in R[t]$ with $f(0), f(1) \in J$. Then, f certainly has two roots in K, since K is the quadratically closed, and these roots are integral over R. But now, S is integrally closed in K, since it is the integral closure of $\mathbb{Z}$ in K, and hence, R is integrally closed in K. Therefore, f must factor into linear polynomials over R, and hence, by Corollary 15, we conclude that R is a 2-SRC ring.

Problem 28. For each $n > 2$, does there exist a commutative local ring R such that R is an n-SRC ring, but R is not an $(n+1)$-SRC ring? Equivalently, does there exist a commutative local ring R for which $\mathbb{M}_n(R)$ is strongly clean but $\mathbb{M}_{n+1}(R)$ is not strongly clean?

Problem 29. If R is a commutative local ring which is an n-SRC ring for all $n \in \mathbb{N}$, must R be a Henselian local ring?

The following Proposition will allow us to draw the conclusions of [25, Example 1] more simply, and to extend their observations to odd primes. It will also provide a simpler description of 2-SRC rings. Many of the main results in [19,10] are special cases of this proposition.

Proposition 30. *Let (R, J) be a commutative local ring. Then, the following are equivalent:*

(1) $\mathbb{M}_2(R)$ *is strongly clean,*
(2) *R is a 2-SRC ring,*
(3) *The polynomial $t^2 - t + j \in R[t]$ has a root for every $j \in J$.*

If $\operatorname{char}(R/J) \neq 2$ *(equivalently,* $2 \in U(R)$*), the above are equivalent to*

(4) *Every element of the multiplicative group $1+J$ has a square root (in $1+J$).*
(5) *Every element of $1+J$ has a square root in R.*

Proof. The first two statements are equivalent by Theorem 12.

(2) $\Longrightarrow$ (3): Assume R is a 2-SRC ring, and let $h(t) = g_0g_1$ be an SRC factorization of $h(t) = t^2 - t + j$, where $j \in J$. Since $h(0) = h(1) = j \in J$, whereas $g_0(0)$ and $g_1(1)$ are not in J, we conclude that $\deg(g_0) = \deg(g_1) = 1$, and hence that $g_0(t) = t - a$ and $g_1(t) = t - b$ for some $a, b \in R$.

(3) $\Longrightarrow$ (2): Let $h(t) \in R[t]$ be a monic polynomial of degree 2. We seek an SRC factorization for h. We may assume that $h \notin \mathbb{S}_0 \cup \mathbb{S}_1$, and hence $h(0), h(1) \in J$, since otherwise a trivial SRC factorization exists. Writing $h(t) = t^2 - at + b$, we have $b = h(0) \in J$, and $1 - a + b = h(1) \in J$, so a is a unit in $1+J$. Consider $g(t) = a^{-2}h(at) = t^2 - t + a^{-2}b$. By assumption, since $a^{-2}b \in J$, $g(t)$ has a root, and hence there is a factorization $g(t) = (t-r)(t-s)$. Since $rs \in J$, $r + s = 1$, and J is maximal, we conclude that either $r \in J$ or $s \in J$. Without loss of generality, $r \in J$, $s \in 1+J$, so $ar \in J$ and $as \in 1+J$. Therefore, $h(t) = a^2g(a^{-1}t) = (t-ar)(t-as)$ is an SRC factorization.

For the remainder of the proof, suppose that $\operatorname{char}(R/J) \neq 2$; equivalently, that 2 is a unit in R.

(3) $\Longrightarrow$ (4): Let $j \in J$, and consider the polynomial $h(t) = t^2 - t + \frac{j}{4}$. By assumption, there exists $r \in R$ such that $h(r) = 0$. We see easily that $h(1-r) = 0$ as well. Since $r(r-1) = \frac{-j}{4} \in J$, we may assume, without loss of generality, that $r \in J$. Now, $(1-2r)^2 = 1 - 4r + 4r^2 = 1 - 4r(r-1) = 1 - 4\frac{-j}{4} = 1 + j$. Since $1 - 2r \in 1 + J$, we conclude that every element of $1+J$ has a square root in $1+J$.

(4) $\Longrightarrow$ (5): This is an obvious weakening.

(5) $\Longrightarrow$ (4): Suppose that $x^2 = 1 + b$, where $b \in J$. Then, $\overline{x}^2 = 1$, so $\overline{x} \in \{1, -1\}$. Therefore, either $x \in 1+J$, or else $-x \in 1+J$, and $x^2 = (-x)^2 = 1 + b$.

(4) $\Longrightarrow$ (3): Suppose that $j \in J$. We seek a root of $h(t) = t^2 - t + j$. By hypothesis there is a square root $x \in 1+J$ of $1 - 4j \in 1+J$. Consider $r_1 = 2^{-1}(1+x)$ and $r_2 = 2^{-1}(1-x)$. Note that $r_1 \in 1+J$ and $r_2 \in J$. Observe that $r_1 + r_2 = 1$ and $r_1r_2 = 4^{-1}(1-x^2) = j$. Therefore, $h(t) = (t-r_1)(t-r_2)$ is an SRC factorization, since $r_1 \in 1+J$ and $r_2 \in J$. $\square$

In [25, Example 1], Wang and Chen show that $\begin{bmatrix} 8 & 6 \\ 3 & 7 \end{bmatrix} \in \mathbb{M}_2(\mathbb{Z}_{(2)})$ is not strongly clean. Proposition 30 shows, in fact, for any prime $p \in \mathbb{Z}$, the ring $\mathbb{Z}_{(p)}$ of integers localized at the prime ideal (p) is not a 2-SRC ring, so $\mathbb{M}_2(\mathbb{Z}_{(p)})$ is not strongly clean, because $t^2 - t + p$ has no roots in $\mathbb{Q}$ (much less in $\mathbb{Z}_{(p)} \subseteq \mathbb{Q}$) since its discriminant is $1 - 4p < 0$. Specifically, for any prime p, the matrix $\begin{bmatrix} 1 & p \\ -1 & 0 \end{bmatrix} \in \mathbb{M}_2(\mathbb{Z}_{(p)})$ is not strongly clean.

5. Other types of factorizations

In this section, we will examine the relationship between SRC factorization and various other types of factorizations. Note that many of these types of factorizations collapse to the same concept if $n = 2$.

Definition 31. Let $h \in R[t]$ be a monic polynomial of degree n. A factorization $h = g_0 g_1 f$ is said to be a PR factorization if $\overline{g_0} = t^m$, $\overline{g_1} = (t-1)^k$, for some $m, k \in \mathbb{N}$, and $\overline{f}(0), \overline{f}(1)$ are both nonzero (equivalently, $f \in \mathbb{S}_0 \cap \mathbb{S}_1$).

Remark 32. The terminology 'PR' factorization refers to 'powers of roots'. Example 19 shows that any monic polynomial over a Henselian local ring has a PR factorization.

Lemma 33. *Let $h \in R[t]$ be a monic polynomial of degree n. Consider the following conditions*

(1) *h factors as a product of linear polynomials in $R[t]$,*
(2) *h has a PR factorization,*
(3) *h has an SRC factorization.*

In general we have (1) $\Longrightarrow$ (2) $\Longrightarrow$ (3). *If $n = 2$ and if $h \notin \mathbb{S}_0 \cup \mathbb{S}_1$, then additionally, we have* (3) $\Longrightarrow$ (1).

Proof. (1) $\Longrightarrow$ (2): This is immediate, grouping together the linear factors corresponding to roots belonging to J, $1 + J$ and $R \setminus (J \cup (1 + J))$ respectively.

(2) $\Longrightarrow$ (3): If $h = g_0 g_1 f$ is a PR factorization, it is straightforward to check that $h = (g_1)(g_0 f) = (g_1 f)g_0$ are each SRC factorizations.

If $n = 2$ and $h \notin \mathbb{S}_0 \cup \mathbb{S}_1$, then any SRC factorization gives h as a product of linear polynomials. □

Thus, we have the following corollary:

Corollary 34. *Let (R, J) be a commutative local ring, and suppose that $h \in R[t]$ is a monic polynomial of degree 2, with $h \notin \mathbb{S}_0 \cup \mathbb{S}_1$. Then, the following are equivalent:*

(1) *Every $\varphi \in \mathbb{M}_2(R)$ with $\chi(\varphi) = h$ is strongly clean.*
(2) *C_h is strongly clean.*
(3) *h has an SRC factorization in $R[t]$.*
(4) *h factors as a product of linear polynomials in $R[t]$.*
(5) *There exist $r_1 \in J$ and $r_2 \in 1 + J$ such that $h(r_i) = 0$.*
(6) *There exists $r \in R$ such that $h(r) = 0$.*
(7) *h has a PR factorization in $R[t]$.*

Proof. The equivalence of the first three statements is exactly the statements in Theorem 12 when $n = 2$. Observe that the condition that $h \notin \mathbb{S}_0 \cup \mathbb{S}_1$ is precisely the condition that $h(0), h(1) \in J$.

(3) $\Longrightarrow$ (4): This is proved in Lemma 33.

(4) $\Longrightarrow$ (5): By hypothesis, $h = (t - r_1)(t - r_2)$. Hence $r_1 r_2 = h(0) \in J$ and $h(1) = (1 - r_1)(1 - r_2) \in J$. Since R is local, we conclude (without loss of generality) that $r_1 \in J$, and $r_2 \in 1 + J$.

(5) $\Longrightarrow$ (6): This implication is immediate.

(6) $\Longrightarrow$ (4): Suppose that $h(r) = 0$. Write $h(t) = t^2 + at + b$. Observe that

$$(t - (-a - r))(t - r) = t^2 + at - r(a + r) = t^2 + at + (b - h(r)) = t^2 + at + b.$$

(4) $\Longleftrightarrow$ (7): As in Lemma 33. □

We thank T.Y. Lam for kindly pointing out the argument used in (4) $\Longleftrightarrow$ (5) $\Longleftrightarrow$ (6).

Corollary 35. *Let (R, J) be a commutative local ring. Then, $\mathbb{M}_2(R)$ is strongly clean if and only if every degree 2 monic polynomial $h \notin \mathbb{S}_0 \cup \mathbb{S}_1$ has a root.*

Proof. Let $\varphi \in \mathbb{M}_2(R)$, and set $h = \chi(\varphi)$. If $h \in \mathbb{S}_0 \cup \mathbb{S}_1$, either φ or $1 - \varphi$ is a unit, and hence φ is strongly clean. Otherwise, we appeal to Corollary 34. □

Remark 36. Note that even if $\mathbb{M}_2(R)$ is strongly clean, there still may exist monic polynomials of degree 2 in $\mathbb{S}_0$ which have no roots. For instance, the ring $\mathbb{Z}_p$ of p-adic integers is a Henselian local ring, hence a 2-SRC ring, but it is easy to see that the polynomial $t^2 - p \in \mathbb{S}_0$ has no roots in $\mathbb{Z}_p$. We thank George Bergman for bringing this example to our attention.

Theorem 12 gave conditions on a polynomial h that are equivalent to strong cleanness of all matrices with characteristic polynomial h. Let us now examine how the strong cleanness of a particular matrix relates to the factorization properties of its characteristic polynomial.

Theorem 37. *Let $\varphi \in \mathbb{M}_n(R)$ such that neither φ nor $1 - \varphi$ is a unit, with characteristic polynomial $h = \chi(\varphi)$. Consider the following conditions:*

(1) *h factors as a product of linear polynomials.*
(2) *h factors as a product $L \cdot h_1(t)$, where $h_1 \in \mathbb{S}_0 \cap \mathbb{S}_1$, and L is a product of linear polynomials.*
(3) *There exists a PR factorization for h.*
(4) *There exists an SRC factorization for h*
(5) *φ is strongly clean.*
(6) *There exists an SR factorization for h.*

For every $n \geq 2$, (1) $\Longrightarrow$ (2) $\Longrightarrow$ (3) $\Longrightarrow$ (4) $\Longrightarrow$ (5) $\Longrightarrow$ (6). For $n = 2$, (6) $\Longrightarrow$ (1). In general, none of these implications are reversible, even for φ for which φ and $1 - \varphi$ are nonunits.

Proof. (1) $\Longrightarrow$ (2): This is an obvious weakening.

(2) $\Longrightarrow$ (3): This implication is straightforward and is similar to (1) $\Longrightarrow$ (2) from Lemma 33.

(3) $\Longrightarrow$ (4): This is done in Lemma 33.

(4) $\Longrightarrow$ (5): This is the implication (3) $\Longrightarrow$ (1) from Theorem 12.

(5) $\Longrightarrow$ (6): This is the content of Lemma 9.

(2) $\not\Longrightarrow$ (1): Over the real numbers, the polynomial $x(x - 1)(x^2 + 1)$ has a factorization as in (2), but has no factorization into linear polynomials.

(3) $\not\Longrightarrow$ (2): Suppose that (R, J) is a commutative local domain with an element $j \in J$ which is not a square in R. For instance, take $R = \mathbb{Z}_{(2)}$, with $j = 2$. The factorization $h(t) = (t^2 + j)((t + 1)^2 + j)$ is a PR factorization, but this factors no further, since there are no square roots of j in R.

(4) $\not\Longrightarrow$ (3): This is similar to Example 10. Let $R = \mathbb{Z}_{(p)}$, with $p \equiv 3 (\mathrm{mod}\ 4)$, and now consider SRC factorization

$$h(t) = h_0 h_1 = [(t - 1)(t^2 + 1) + p^2] \cdot t.$$

This factorization, for the same reasons as given in Example 10, cannot be refined. In particular, the factorization is an SRC factorization, but it is not a PR factorization. (The irreducible polynomial $(t - 1)(t^2 + 1) + p^2$, is also an example, but its companion matrix is a unit.)

(5) $\not\Longrightarrow$ (4): See Example 10.

(6) $\not\Longrightarrow$ (5): Take the polynomial in Example 10, and let φ be its companion matrix, which is not strongly clean by Theorem 12. □

6. Strongly π-regular matrix rings

We now address the question of when a matrix ring (or an element thereof) over a commutative local ring is strongly π-regular (which, as noted in the introduction, is a stronger property than strongly cleanness). As we shall now demonstrate, it follows from results in the literature that if R is a commutative (not necessarily local) ring, then $\mathbb{M}_n(R)$ is strongly π-regular if and only if R is strongly π-regular. The purpose of this section is to give a new way to look at this result, using the techniques of the previous sections.

By [1, Theorem 1.1], for any ring R, $\mathbb{M}_n(R)$ is a strongly π-regular ring if and only if for every finitely generated left (right) R-module M, injective endomorphisms of M are isomorphisms (i.e. M is cohopfian). In [24], Vasconcelos proved that for a commutative ring R, injective endomorphisms of finitely generated R-modules are isomorphisms if and only if every prime ideal of R is maximal. By [17, Ex. 4.15], for a commutative ring R, every prime ideal of R is maximal if and only if R is strongly π-regular. Combining these results, we see that, for any commutative ring, $\mathbb{M}_n(R)$ is strongly π-regular if and only if R is strongly π-regular. In particular,

Proposition 38. *Let R be a commutative local ring. Then, the following are equivalent (see Lemma 39 below):*

(1) *$J(R)$ is nil,*
(2) *R is strongly π-regular,*
(3) *$\mathbb{M}_n(R)$ is strongly π-regular.*

On the other hand, there exists a noncommutative local ring (R, J) with J locally nilpotent (in particular, R is strongly π-regular) for which $\mathbb{M}_2(R)$ is not strongly π-regular (see [22,7]).

As we shall see in this section, our previous methods apply to the study of strongly π-regular endomorphisms as well. Using these methods, we shall obtain an elementwise characterization of strongly π-regular endomorphisms in terms of factorization, and in turn an alternate proof of the fact that for R commutative local, $\mathbb{M}_n(R)$ is strongly π-regular if and only if $J(R)$ is nil.

Recall that an element $a \in R$ is called left π-regular if the chain

$$Ra \supseteq Ra^2 \supseteq Ra^3 \supseteq \cdots$$

terminates, and right π-regular if the chain

$$aR \supseteq a^2R \supseteq a^3R \supseteq \cdots$$

terminates, and is called strongly π-regular if it is both right and left π-regular. Dischinger proved [11] that if every element of R is right π-regular, then every element of R is left π-regular.

Let us recall some well-known facts about strongly π-regular rings, whose basic proofs are left to the reader.

Lemma 39. (1) *If R is a strongly π-regular ring, and $e^2 = e \in R$, then eRe is strongly π-regular.*
(2) *For any ring R, every nilpotent element and every unit of R is strongly π-regular.*
(3) *If $a \in J(R)$, then a is strongly π-regular if and only if a is nilpotent.*
(4) *A local ring R is strongly π-regular if and only if $J(R)$ is nil.*

It follows that if $\mathbb{M}_n(R)$ is strongly π-regular, then R is strongly π-regular by Lemma 39, and hence $J(R)$ must be nil.

The following lemma which is similar to Lemma 1 also appeared in [21],[1] and allows us to apply our previous methods to the study of strong π-regularity.

Lemma 40 (*[21]*). *Let R be a ring, M_R a right R-module, and $\alpha \in \mathrm{End}(M_R)$. Then, the following are equivalent*

(1) *α is strongly π-regular in $\mathrm{End}(M_R)$.*
(2) *There exists a decomposition $M = A \oplus B$ (as R-modules), where A and B are α-invariant, $\alpha|_A \in \mathrm{End}(A_R)$ is an isomorphism, and $\alpha|_B \in \mathrm{End}(B_R)$ is nilpotent.*

We shall need one useful general proposition.

Proposition 41. *Let R be a nonzero commutative ring. Then, $\varphi \in \mathbb{M}_n(R)$ is nilpotent if and only if $\chi(\varphi) \equiv t^n (\mathrm{mod}\ \mathrm{Nil}(R))$, where $\mathrm{Nil}(R)$ is the nilradical of R.*

[1] Note that in [21], Nicholson asserts that Lemma 40 above, with other equivalent conditions, is true, by an argument (which he omits) similar to the proof of [21, Theorem 3]. Nicholson attributes the implication (1) $\Longleftrightarrow$ (2) to [1, Proposition 2.3]. However, [1, Proposition 2.3] contains only the ring theoretic version of Lemma 40, as opposed to the elementwise version which Nicholson states. See [2] for a complete proof of the equivalent conditions asserted in [21].

Proof. For the reverse implication, suppose that $\chi(\varphi) = t^n - a_{n-1}t^{n-1} - \cdots - a_0$, where $a_i \in \mathrm{Nil}(R)$ for $0 \le i \le n-1$. By the Cayley–Hamilton Theorem, $\phi^n = a_{n-1}\phi^{n-1} + \cdots + a_1\phi + a_0 I$. Consider the (commutative) subring S of $\mathbb{M}_n(R)$ generated by $a_{n-1}\phi^{n-1}, \ldots, a_1\phi, a_0 I$. For $0 \le i \le n-1$, $a_i\phi^i$ is nilpotent. Since S is commutative, and is generated by a finite collection of nilpotent elements, we conclude that every element of S is nilpotent. In particular, $\phi^n \in S$ is nilpotent.

To prove the forward implication, we will make a series of reductions. We shall first prove the result assuming that R is a domain. In this case, we may embed R in an algebraic closure F of its quotient field, and we may thus view φ as a nilpotent element of $\mathbb{M}_n(F)$. Since F is algebraically closed, it is clear that the characteristic polynomial of φ over F is t^n, since φ clearly has no nonzero eigenvalues. The characteristic polynomial of φ is unchanged by embedding R in F, and hence the characteristic polynomial of φ over R is t^n. Suppose that R is any nonzero commutative ring, and that $\varphi \in \mathbb{M}_n(R)$ is nilpotent. Write $\chi(\varphi) = t^n + a_{n-1}t^{n-1} + \cdots + a_0$. For any prime ideal P of R, the image φ' of φ in $\mathbb{M}_n(R/P)$ is nilpotent. But R/P is a domain, so the characteristic polynomial of φ' is t^n, and hence we conclude that $a_i \in P$ for $0 \le i \le n-1$. Since

$$\mathrm{Nil}(R) = \bigcap_{P \in \mathrm{Spec}\,R} P,$$

we conclude that $a_i \in \mathrm{Nil}(R)$ for $0 \le i \le n-1$, as desired. □

Definition 42. For $r \in R$, define

$$\mathbb{P}_r = \{f \in R[t] : f \text{ monic, and } f - (t-r)^{\deg(f)} \in \mathrm{Nil}(R)[t]\}.$$

Thus, by Proposition 41, $\varphi - rI$ is nilpotent if and only if $\chi(\varphi) \in \mathbb{P}_r$, if and only if $h(\varphi) = 0$ for some monic polynomial $h \in \mathbb{P}_r$. Before stating our results, let us define another type of factorization.

Definition 43. A factorization $h = h_0 p_0$ is an SP factorization if $h_0 \in \mathbb{S}_0$ and $p_0 \in \mathbb{P}_0$.

Since $\mathrm{Nil}(R) \subseteq J$, it is immediate that $\overline{h_0}, \overline{p_0} \in (R/J)[t]$ are coprime, since $\overline{p_0}$ is a power of t, and $\overline{h_0}(0) \neq 0$. An SP factorization is a lift of a factorization $H(t)t^n \in (R/J)[t]$, where $H(0) \neq 0$, with the restriction that p_0 (which is a monic lift of t^n to $R[t]$) satisfies $p_0 - t^n \in \mathrm{Nil}(R)$ (rather than simply $p_0 - t^n \in J$).

In analogy with Theorem 12, we have the following result, whose proof is similar to the proof of that of Theorem 12.

Proposition 44. *Let R be a commutative local ring and let $h \in R[t]$ be a monic polynomial of degree n. Then, the following are equivalent:*

(1) *Every $\varphi \in \mathbb{M}_n(R)$ with $\chi(\varphi) = h$ is strongly π-regular.*
(2) *There exists $\varphi \in \mathbb{M}_n(R)$ with $\chi(\varphi) = h$ such that φ is strongly π-regular.*
(3) *h has an SP factorization.*

Proof. (1) $\Longrightarrow$ (2): This is an obvious weakening.

(2) $\Longrightarrow$ (3): By Lemma 40 there exists a decomposition $R^n = A \oplus B$ as $R[\varphi]$-modules such that $\varphi|_A$ is invertible on A, and $\varphi|_B$ is nilpotent on B. The characteristic polynomial of φ then factors as $h = h_0 p_0$, where $h_0 \in \mathbb{S}_0$ (by Lemma 4) and $p_0 \in \mathbb{P}_0$ (by the remarks after Definition 42).

(3) $\Longrightarrow$ (1): Suppose that $\varphi \in \mathbb{M}_n(R)$, and suppose $\chi(\varphi) = h = h_0 p_0$ is an SP factorization. Note that $\overline{h_0}$ and $\overline{p_0}$ are coprime in $\overline{R}[t]$ (since $\mathrm{Nil}(R) \subseteq J$).

As in the proof of Theorem 12, there exist polynomials f_0 and f_1 such that $f_0(\varphi)h_0(\varphi) + f_1(\varphi)p_0(\varphi) = Id_{R^n}$. By the Cayley–Hamilton Theorem, we have $h_0(\varphi)p_0(\varphi) = p_0(\varphi)h_0(\varphi) = h(\varphi) = 0$. Also, $f_0(\varphi), h_0(\varphi), f_1(\varphi)$, and $p_0(\varphi)$ are contained in $R[\varphi]$, which is a commutative subring of $\mathrm{End}(R^n_R)$. By Lemma 11, $R^n = \ker(h_0(\varphi)) \oplus \ker(p_0(\varphi))$ (as right R-modules). As in the proof of Theorem 12, it is easy to see that $\ker(h_0(\varphi))$ and $\ker(p_0(\varphi))$ are φ-invariant submodules of R^n.

Finally, we claim that φ is an isomorphism on $\ker(h_0(\varphi))$ (cf. Theorem 12 for the proof), and φ is nilpotent on $\ker(p_0(\varphi))$. For the second statement, we know that $p_0(\varphi|_{\ker(p_0(\varphi))}) = 0$, where $p_0 \in \mathbb{P}_0$. By the remark following Definition 42, $\varphi|_{\ker(p_0(\varphi))}$ is nilpotent. By Lemma 40, φ is strongly π-regular, with $A = \ker(h_0(\varphi))$ and $B = \ker(p_0(\varphi))$. □

The coprimeness forced by the SP factorization makes this result easier to prove than its analogue; here we do not need to deal with companion matrices. The following corollary is immediate.

Corollary 45. *Let R be a commutative local ring. Then, the following are equivalent.*

(1) $\mathbb{M}_n(R)$ *is strongly π-regular,*
(2) *Every monic polynomial of degree n in $R[t]$ has an SP factorization.*

We shall now study condition (2) of Corollary 45 to obtain an alternate proof of Proposition 38. The implications (3) $\Longrightarrow$ (2) $\Longrightarrow$ (1) follow from Lemma 39. Thus, it suffices to show that if $J(R)$ is nil, then every monic polynomial of $R[t]$ has an SP factorization. We shall show this with the aid of one general lemma.

Lemma 46. *Let R be a ring, let $I \subseteq J(R)$ be an ideal of R, and let bar-notation denote the natural quotient map from R to $\overline{R} = R/I$. For $a, b \in R$, (a, b) is right unimodular in R if and only if $(\overline{a}, \overline{b})$ is right unimodular in $\overline{R}$.*

Proof. The forward implication is immediate, and does not require the hypothesis $I \subseteq J(R)$. Conversely, if $\overline{a}\overline{R} + \overline{b}\overline{R} = \overline{R}$, there must exist $x, y \in R$ such that $z = ax + by \in 1 + I$. But $1 + I \subseteq 1 + J(R) \subseteq U(R)$, so z is a unit, and hence $a(xz^{-1}) + b(yz^{-1}) = 1$. Therefore, $aR + bR = R$. □

Corollary 47. *Let (R, J) be a commutative local ring with J nil. Then, R is a Henselian local ring. Furthermore, every monic polynomial $h \in R[t]$ has an SP factorization.*

Proof. Let $f \in R[t]$ be monic, and pick a factorization $\overline{f} = GH \in (R/J)[t]$ where (G, H) is unimodular in $(R/J)[t]$. Take any monic lifts $g_1, h_1 \in R[t]$ of G and H, respectively, and consider the ideal I generated by the coefficients of $f - g_1h_1$. It is clear that the coefficients of $f - g_1h_1$ lie in J, since $\overline{g_1} = G$, $\overline{h_1} = H$ and $\overline{f} = GH$. Now, I is a finitely generated ideal contained in the nil ideal J, since R is commutative, it follows that I is nilpotent. Thus, R is Henselian with respect to I. The image f' of f in $(R/I)[t]$ factors as $g_1'h_1'$, where the prime notation denotes the image in $(R/I)[t]$. By Snapper's Theorem [17, Theorem 5.1], $J(R[t]) = \mathrm{Nil}(R)[t] = J(R)[t]$. It follows that (g_1, h_1) is unimodular in $R[t]$ by Lemma 46, and hence that (g_1', h_1') is unimodular in $(R/I)[t]$. Since R is Henselian with respect to I, we can lift the factorization $f' = g_1'h_1'$ to a factorization $f = gh$ in $R[t]$. It is clear that $\overline{g} = G$ and $\overline{h} = H$ (since g and h agree with g_1 and h_1 modulo I). Thus, R is Henselian. The last remark follows by lifting the factorization $\overline{f} = t^m H(t) \in (R/J)[t]$, where $H(0) \neq 0$ in R/J, and using the fact that $J(R) = \mathrm{Nil}(R)$, since J is nil. □

Note that Example 27 gives a local ring R (necessarily with $J(R) \neq \mathrm{Nil}(R)$) for which $\mathbb{M}_2(R)$ is strongly clean but is not strongly π-regular.

7. Further questions

The related question of when $\mathbb{T}_n(R)$, the ring of upper triangular matrices over R, is strongly clean is addressed in [21] (for $n = 2$, R local), [25] (for $R = \mathbb{Z}_{(2)}$), in [9], and by the present authors in [3] (for arbitrary n, R local).

In particular, if R is a commutative local ring, $\mathbb{T}_n(R)$ is always strongly clean, whereas we have seen that $\mathbb{M}_n(R)$ need not be strongly clean. From this small piece of evidence (as well as some other special cases for small values of n), we ask the following questions (primarily for R local):

Problem 48. If $\mathbb{M}_n(R)$ is strongly clean, must $\mathbb{T}_n(R)$ be strongly clean?

A stronger elementwise question is the following.

Problem 49. If $\varphi \in \mathbb{T}_n(R) \subseteq \mathbb{M}_n(R)$ is strongly clean as an element of $\mathbb{M}_n(R)$, must it be strongly clean as an element of $\mathbb{T}_n(R)$?

Certainly, an affirmative answer to Problem 49 would yield an affirmative answer to Problem 48.

There is a more general context in which one can state natural generalizations of Question 48 and 49. Let R be a ring and let $P = (X, \leq)$ be a locally finite preordered set; that is, $\leq$ is reflexive and transitive, and for any $x, y \in X$, the interval $[x, y] = \{a \in X \mid x \leq a \leq y\}$ is finite. We shall define the incidence ring $I(P, R)$. Before doing so, we direct the interested reader to [23], which is an accessible reference for the standard facts about incidence rings. As a set,

$$I(P, R) = \{f : P \times P \longrightarrow R \mid f(x, y) = 0 \text{ if } x \not\leq y\}.$$

Addition is usual addition of functions with range R and multiplication is defined by

$$(f \cdot g)(x, y) = \sum_{z \in [x,y]} f(x, z)g(z, y).$$

The identity and zero elements are the diagonal maps with value 1 and 0, respectively. It is easy to check that $I(P, R)$ is an associative ring.

Incidence rings form a natural generalization of both full matrix rings and upper triangular matrix rings. To see this, let $X = \{1, 2, \ldots, n\}$. If P is the usual total order on X, then $I(P, R) \cong \mathbb{T}_n(R)$. On the other hand, if P is the full preorder on X, where $i \leq j$ for all $i, j \in X$, then $I(P, R) \cong \mathbb{M}_n(R)$. If P is the trivial preorder on X (where $i \leq j$ if and only if $i = j$), then $I(X, R) \cong R \times R \times \cdots \times R$, the direct product of n copies of R.

Suppose that X is a set, and suppose that $P = (X, \leq)$ and $P' = (X, \leq')$ are preorders on X, such that P' refines $P(x \leq y \Longrightarrow x \leq' y)$. Then, there is a natural embedding $I((X, P), R) \subseteq I((X, P'), R)$.

We ask (again, primarily for R local):

Problem 50. If $I(P', R)$ is strongly clean, must $I(P, R)$ be strongly clean?

Its stronger elementwise counterpart is:

Problem 51. If $\alpha \in I(P, R) \subseteq I(P', R)$ is strongly clean as an element of $I(P', R)$, must it be strongly clean as an element of $I(P, R)$?

In the case that $|X| = n$, P' is the full preorder, and P is a total order on X, $I(P, R) \cong \mathbb{T}_n(R)$ and $I(P', R) \cong \mathbb{M}_n(R)$ (and the embedding of $I(P, R)$ into $I(P', R)$ is compatible with these isomorphisms). Thus, these special cases of Problems 50 and 51 are precisely Problems 48 and 49.

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