Modular automorphisms preserving idempotence
and Jordan isomorphisms of triangular matrices
over commutative rings

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Abstract

Suppose \( \mathbb{R} \) is a commutative ring with 1 and 2 being the units of \( \mathbb{R} \). Let \( T_n(\mathbb{R}) \) be the \( n \times n \) upper triangular matrix modular over \( \mathbb{R} \), and let \( L(\mathbb{R}) \) be the set of all \( \mathbb{R} \)-module automorphisms on \( T_n(\mathbb{R}) \) that preserve idempotence. The main result in this paper is that \( f \in L(\mathbb{R}) \) if and only if there exist an invertible matrix \( U \in T_n(\mathbb{R}) \) and an idempotence \( e \in \mathbb{R} \) such that \( f(X) = U(eX + (1 - e)X^\delta)U^{-1} \) for any \( X = (x_{ij}) \in T_n(\mathbb{R}) \), where \( X^\delta = (x_{n+1-j \ n+1-j}) \). As applications, we determine all Jordan isomorphisms of \( T_n(\mathbb{R}) \) over the ring \( \mathbb{R} \). © 2001 Elsevier Science Inc. All rights reserved.

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1. Introduction

Suppose \( \mathbb{R} \) is a commutative ring with 1 and \( n \ (> 1) \) a positive integer. Let \( M_n(\mathbb{R}) \) and \( T_n(\mathbb{R}) \) be the \( n \times n \) full matrix modular and upper triangular matrix modular over \( \mathbb{R} \), respectively.

In the past several decades, many authors have studied Linear Preserver Problems (LPPs) on \( M_n(\mathbb{R}) \) that satisfy various properties (e.g., [1,2,4,8,9]). But LPPs on

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$T_n(\mathbb{R})$ are only found in several papers (e.g., [5]). In this paper, using localization techniques from commutative algebra, we characterize all $\mathbb{R}$-module automorphisms on $T_n(\mathbb{R})$ that preserve idempotence when 2 is a unit of $\mathbb{R}$. As applications, we also determine all Jordan isomorphisms of $T_n(\mathbb{R})$ over $\mathbb{R}$, and therefore obtain the result as in [10]: if 1 and 2 are the units of $\mathbb{R}$, then $\mathbb{R}$ contains no idempotents except 0 and 1 if and only if every Jordan isomorphism on $T_n(\mathbb{R})$ over $\mathbb{R}$ is either an automorphism or an anti-automorphism.

Set

$$
\Gamma_n(\mathbb{R}) = \{A \in T_n(\mathbb{R}) | A^2 = A\}, $
$$
$$A_n(\mathbb{R}) = \{A \in \Gamma_n(\mathbb{R}) | \text{all entries of } A \text{ are 0 or } \pm 1\}.
$$

An $\mathbb{R}$-module automorphism $f : T_n(\mathbb{R}) \to T_n(\mathbb{R})$ is called an idempotence preserver if it satisfies $f(\Gamma_n(\mathbb{R})) = \Gamma_n(\mathbb{R})$. Let $\mathcal{L}(\mathbb{R})$ denote the sets of idempotence preservers of $T_n(\mathbb{R})$.

Let $E_{ij}$ denote the matrix with 1 at the $(i, j)$ position and 0 elsewhere and $I_n$ the $n \times n$ identity matrix over $\mathbb{R}$. We denote by $\mathbb{R}^*$, $T_n^*(\mathbb{R})$, and $[1, n]$ the sets of all units of $\mathbb{R}$, all units of $T_n(\mathbb{R})$, and $\{1, 2, \ldots, n\}$, respectively. For a prime ideal $q$ in Spec $\mathbb{R}$, the prime spectrum of $\mathbb{R}$, $\mathbb{R}_q$ denotes the localization of $\mathbb{R}$ at $q$. For a matrix $A = (a_{ij}) \in T_n(\mathbb{R})$, $A_q = (a_{ij}/1) \in T_n(\mathbb{R}_q)$ denotes the localization of $A$ at $q$, and $A^q$ the matrix $(a_{n+1−j n+1−i})$.

2. Some lemmas

**Lemma 2.1.** Suppose $\mathbb{R}$ is a commutative ring with 1.

(i) If $A, B \in T_n(\mathbb{R})$ satisfy $A_q = B_q$ for some $q \in \text{Spec } \mathbb{R}$, then there exists $s \in \mathbb{R} - q$ such that $sA = sB$.

(ii) If $A = (a_{ij}), B = (b_{ij}) \in T_n(\mathbb{R})$, then $A = B$ if and only if $A_q = B_q$ for any $q \in \text{Spec } \mathbb{R}$.

(iii) If $X \in T_n(\mathbb{R}_q)$, then there exist $A \in T_n(\mathbb{R})$ and $s \in \mathbb{R} - q$ such that $X = \frac{1}{s}A_q$.

**Proof.** The proof is easy by applying definitions. □

For each $f \in L_i(\mathbb{R})$ ($i \in [1, 3]$), we denote

$$
f_q : \frac{1}{s}A_q \mapsto \frac{1}{s}(f(A))_q \quad \forall X = \frac{1}{s}A_q \in T_n(\mathbb{R}_q).
$$

Obviously, $f_q$ is an $\mathbb{R}_q$-module automorphism of $T_n(\mathbb{R}_q)$ and $f_q(A_q) = (f(A))_q$.

**Lemma 2.2.** Suppose $\mathbb{R}$ is a commutative ring with 1, $f$ is a linear isomorphism of $T_n(\mathbb{R})$ satisfying for each $q \in \text{Spec } \mathbb{R}$, and there exists $P(q) \in T_n^*(\mathbb{R}_q)$ such that $f_q(X) = P(q)X(P(q))^{-1}$ for any $X \in T_n(\mathbb{R}_q)$. Then $f$ is an $\mathbb{R}$-algebra inner automorphism of $T_n(\mathbb{R})$, i.e., there exists $Q \in T_n^*(\mathbb{R})$ such that $f(A) = QAQ^{-1}$ for any $A \in T_n(\mathbb{R})$. 


Proof. It follows from [6] that $f$ is an $\mathbb{R}$-module automorphism of $T_n(\mathbb{R})$. For $A, B \in T_n(\mathbb{R})$, applying

$$f_q((AB)_q) = P(q)A_qB_q(P(q))^{-1} = f_q(A_q)f_q(B_q) = (f(A)f(B))_q$$

for any $q \in \text{Spec } \mathbb{R}$ and condition (ii) of Lemma 2.1, we have $f(AB) = f(A)f(B)$, i.e., $f$ is an $\mathbb{R}$-algebra automorphism of $T_n(\mathbb{R})$. Again applying [7], the lemma follows. □

Lemma 2.3. Suppose $\mathbb{R}$ is a commutative ring with 1.

(i) $A = (a_{ij}) \in \Gamma_n(\mathbb{R})$ if and only if $A_q = (a_{ij}/1) \in \Gamma_n(\mathbb{R}_q)$ for any $q \in \text{Spec } \mathbb{R}$.

(ii) If $f \in \mathcal{L}(\mathbb{R})$, then $f_q(\Lambda_n(\mathbb{R}_q)) \subseteq \Gamma_n(\mathbb{R}_q)$.

Proof. Condition (i) is obtained from the definition of $f_q$ and condition (ii) of Lemma 2.1. If $f \in \mathcal{L}(\mathbb{R})$, then $f(\Lambda_n(\mathbb{R})) \subseteq \Gamma_n(\mathbb{R})$, and hence $f_q(\Lambda_n(\mathbb{R}_q)) \subseteq \Gamma_n(\mathbb{R}_q)$ from (i), i.e., (ii) holds. □

Lemma 2.4. Suppose $\mathbb{R}$ is a commutative local ring with 1 and $M$ is the maximal ideal of $\mathbb{R}$. Let $\sigma : \mathbb{R} \mapsto \mathbb{R}/M$ be defined by $\sigma(x) = \overline{x} = x + M$. Then:

(i) $\sigma$ induces a multiplicative surjective homomorphism from group $T_n^*(\mathbb{R})$ to group $T_n^*(\mathbb{R}/M)$, i.e., $A = (a_{ij}) \mapsto \overline{A} = (\overline{a_{ij}})$ for any $A \in T_n^*(\mathbb{R})$.

(ii) $\sigma$ induces a surjective homomorphism from ring $T_n(\mathbb{R})$ to ring $T_n(\mathbb{R}/M)$, i.e., $A \mapsto \overline{A}$ for any $A \in T_n(\mathbb{R})$.

(iii) If $f$ is an $\mathbb{R}$-module homomorphism of $T_n(\mathbb{R})$, then $\overline{f} : \overline{X} \mapsto \overline{f(\overline{X})}$ for any $X \in T_n(\mathbb{R})$ is an $\mathbb{R}/M$-module homomorphism of $T_n(\mathbb{R}/M)$. Moreover, if $f$ is invertible, then $\overline{f}$ is also.

(iv) If $f \in \mathcal{L}(\mathbb{R})$ for some $k \in [1, 3]$, then $\overline{f}(\Lambda_n(\mathbb{R}/M)) \subseteq \Gamma_n(\mathbb{R}/M)$.

Proof. The proof is easy by applying definitions. □

Lemma 2.5. Suppose $\mathbb{R}$ is a commutative local ring with 1. If $A \in \Gamma_n(\mathbb{R})$, then there exists $P \in T_n^*(\mathbb{R})$ such that $A = P \text{ diag}(a_1, \ldots, a_n)P^{-1}$, where $a_i = 1$ or 0 for all $i \in [1, n]$.

Proof. See the proof of [7, pp. 208–209]. □

3. Modular automorphisms preserving idempotence

3.1. The case that $\mathbb{R}$ is a commutative local ring

In this section, let $\mathbb{R}$ be a commutative local ring with 1, $2 \in \mathbb{R}^*$ and $M$ the unique maximal ideal of $\mathbb{R}$. 
Lemma 3.1.

(i) Suppose \( A_1, A_2, \ldots, A_n \in \Gamma_n(\mathbb{R}) \) satisfy \( A_i A_j = O \) for any distinct \( i \) and \( j \), and \( A_i \neq O \) for any \( i \). Then there exists \( P \in T_n^*(\mathbb{R}) \) such that \( A_i = P E_{\tau(i)\tau(i)} P^{-1} \) for all \( i \in [1, n] \), where \( \tau \) is a bijective map of \([1, n]\).

(ii) Suppose \( f \) is an \( \mathbb{R} \)-module automorphism of \( T_n(\mathbb{R}) \) satisfying \( f(\Delta_n(\mathbb{R})) \subseteq \Gamma_n(\mathbb{R}) \). Then there exists \( P \in T_n^*(\mathbb{R}) \) such that
\[
 f(E_{ii}) = P E_{\tau(i)\tau(i)} P^{-1} \forall i \in [1, n],
\]
where \( \tau \) is as in (i).

Proof. (i) The proof is by induction on \( n \), the size of the upper triangular matrices. When \( n = 1 \), the result follows from Lemma 2.5. We may assume that the result holds for \( n - 1 \) \((n \geq 2)\).

Without loss of generality, we may assume from Lemma 2.5 that \( A_1 = I_{r_1} \oplus O \), where \( r_1 \in [1, n] \) and \( I_{r_1} \) lies in the \( i_1 \)th, \( i_2 \)th, \ldots, \( i_{r_1} \)th rows and columns. Applying \( A_1 A_j = A_j A_1 = O \), we have
\[
 A_j = O_{r_1} \oplus B_j \quad \forall j \in [2, n],
\]
where \( B_j \in \Gamma_{n-r_1}(\mathbb{R}) \). Clearly, \( B_2, \ldots, B_n \in \Gamma_{n-r_1}(\mathbb{R}) \) satisfy \( B_i B_j = O \) for any \( 2 \leq i < j \leq n \) and \( B_j \neq O \) for any \( j \). By the inductive hypothesis, condition (i) holds.

(ii) For any distinct \( i \) and \( j \), it follows that \( f(E_{ii} + E_{jj}), f(E_{ii}), f(E_{jj}) \in \Gamma_n(\mathbb{R}) \) from \( E_{ii} + E_{jj}, E_{ii}, E_{jj} \in \Delta_n(\mathbb{R}) \). By a direct computation, we have
\[
 f(E_{ii}) f(E_{jj}) + f(E_{jj}) f(E_{ii}) = O \quad \forall 1 \leq i, j \leq n,
\]
and hence
\[
 f(E_{ii}) f(E_{jj}) = -f(E_{jj}) f(E_{ii}) f(E_{jj}) = f(E_{jj}) f(E_{ii}).
\]
Again applying \( 2 \in \mathbb{R}^* \) and (1), we obtain that \( f(E_{ii}) f(E_{jj}) = O \) for any distinct \( i \) and \( j \), again \( f(E_{ii}) \neq O \) by \( f \) is automorphism. Hence (ii) holds by letting \( A_i = f(E_{ii}) \) in condition (i). \( \Box \)

Lemma 3.2. Suppose \( f \) is an \( \mathbb{R} \)-module automorphism of \( T_n(\mathbb{R}) \) satisfying \( f(\Delta_n(\mathbb{R})) \subseteq \Gamma_n(\mathbb{R}) \). Then either
\[
 (i) \quad f(E_{ij}) = P(d_{\tau(j)} E_{\tau(i)\tau(j)}) P^{-1} \quad \text{if } \tau(i) < \tau(j)
\]
or
\[
 (ii) \quad f(E_{ij}) = P(d_{\tau(i)} E_{\tau(j)\tau(i)}) P^{-1} \quad \text{if } \tau(i) > \tau(j)
\]
for any \( i < j \), where \( P \) and \( \tau \) are as in Lemma 3.1.

Proof. Let \( f(E_{ij}) = P D^{(ij)} P^{-1} \) for any \( i < j \), where \( P \) is as in Lemma 3.1. Then
\[
 D^{(ij)} = D^{(ij)} E_{\tau(i)\tau(i)} + E_{\tau(i)\tau(i)} D^{(ij)}
\]
and

\[ D^{(ij)} = D^{(ij)} E_{\tau(j)\tau(j)} + E_{\tau(j)\tau(j)} D^{(ij)} \]

from Lemma 3.1 and

\[ f(E_{ii} \pm E_{ij}), f(E_{jj} \pm E_{ij}) \in \Gamma_n(\mathbb{R}). \]

By a direct computation, the lemma follows. \( \square \)

**Theorem 3.1.** Suppose \( f \) is an \( \mathbb{R} \)-module automorphism of \( T_n(\mathbb{R}) \) which satisfies \( f(\Delta_n(\mathbb{R})) \subseteq \Gamma_n(\mathbb{R}) \). Then \( f \) has one of the following forms:

(i) there exists \( P \in T_n^*(\mathbb{R}) \) such that \( f(A) = PAP^{-1} \) for all \( A \in T_n(\mathbb{R}) \),

(ii) there exists \( P \in T_n^*(\mathbb{R}) \) such that \( f(A) = PAP^\delta P^{-1} \) for all \( A \in T_n(\mathbb{R}) \).

**Proof.** For \( i \leq j \), it follows from Lemmas 3.1 and 3.2 that \( f(E_{ij}) = d_{kl} P_1 E_{kl} P_1^{-1} \) for some \( P_1 \in T_n^*(\mathbb{R}) \), where \( k = \min(\tau(i), \tau(j)) \) and \( l = \max(\tau(i), \tau(j)) \). Let \( \epsilon_i \) be the \( i \)th row and \( e_i \) be the \( i \)th column of \( I_n \). Then \( 1 = \epsilon_i E_{ij} e_j = d_{kl}(\epsilon_i f^{-1}(P_1 E_{kl} P_1^{-1})) e_j \) because \( f \) is invertible, and hence \( d_{kl} \) is invertible, i.e., \( d_{\tau(j)\tau(i)}, d_{\tau(i)\tau(j)} \in \mathbb{R}^* \) in Lemma 3.2.

When \( n = 2 \), it is easy to see that \( f(E_{12}) = d_{12} P_1 E_{12} P_1^{-1} \). If \( f(E_{11}) = P_1 E_{11} P_1^{-1} \), then \( P(f(E_{1j})P^{-1} = E_{ij} \) for any \( 1 \leq i \leq j \leq 2 \) by letting \( P = P_1 \) diag(1, \( d_{12}^{-1} \)), i.e., condition (i) holds. If \( f(E_{11}) = P_1 E_{22} P_1^{-1} \), by a similar argument, then (ii) holds.

When \( n \geq 3 \), for any \( i < j < k \):

(I) Suppose \( \tau(j) < \tau(i) < \tau(k) \). Since \( E_{ii} + E_{ij} + E_{ik} \in \Delta_n(\mathbb{R}) \), we have

\[ f(E_{ii} + E_{ij} + E_{ik}) = P_1 (E_{\tau(i)\tau(i)}) + d_{\tau(j)\tau(i)} E_{\tau(j)\tau(i)} + d_{\tau(i)\tau(k)} E_{\tau(i)\tau(k)} P_1^{-1} \in \Gamma_n(\mathbb{R}) \]

and thus \( d_{\tau(j)\tau(i)} d_{\tau(i)\tau(k)} = 0 \). Further, \( d_{\tau(j)\tau(i)} d_{\tau(i)\tau(k)} = 0 \). Hence \( d_{\tau(j)\tau(i)} = 0 \) or \( d_{\tau(i)\tau(k)} = 0 \), this contradicts condition (iii) of Lemma 2.4.

(II) Suppose \( \tau(k) < \tau(i) < \tau(j) \). The proof is similar to (I).

(III) Suppose \( \tau(i) < \tau(k) < \tau(j) \) or \( \tau(j) < \tau(k) < \tau(i) \). We can obtain the contradiction from \( T = f(E_{ii} + E_{ij} - E_{ik} + E_{jk} + E_{kk}) \in \Gamma_n(\mathbb{R}) \).

(IV) Suppose \( \tau(i) < \tau(j) < \tau(k) \). Then \( \tau(1) < \tau(2) < \cdots < \tau(n) \), i.e., \( \tau(i) = i \) for all \( i \), and hence \( f(E_{ij}) = P_1 d_{ij} E_{ij} P_1^{-1} \) for any \( i \leq j \) from Lemma 3.2. Applying \( T \in \Gamma_n(\mathbb{R}) \), we have \( d_{ik} = d_{ij} d_{jk} \) for any \( 1 \leq i < j < k \leq n \). Let \( P = P_1 \) diag(1, \( d_{12}^{-1}, \ldots, d_{nn}^{-1} \)). Then \( f \) is the form (i).

(V) Suppose \( \tau(i) > \tau(j) > \tau(k) \). Then \( f \) is the form (ii) by a similar argument to (IV). \( \square \)
3.2. The case that $\mathbb{R}$ is a commutative ring with 1

**Lemma 3.3.** Suppose $f$ is an $\mathbb{R}$-module automorphism of $T_n(\mathbb{R})$ and $q \in \text{Spec } \mathbb{R}$.

(i) If there exists $P_1 \in T_n^+(\mathbb{R}_q)$ such that $f_q(X) = P_1 X P_1^{-1}$ for any $X \in T_n(\mathbb{R}_q)$, then there exist $s \in \mathbb{R} - q$ and $P, Q \in T_n(\mathbb{R})$ with $(PQ)_q = sI_n$ such that $sf(A) = PAQ$ for all $A \in T_n(\mathbb{R})$.

(ii) If there exists $P_1 \in T_n^+(\mathbb{R}_q)$ such that $f_q(X) = P_1 X^\delta P_1^{-1}$ for any $X \in T_n(\mathbb{R}_q)$, then there exist $s \in \mathbb{R} - q$ and $P, Q \in T_n(\mathbb{R})$ with $(PQ)_q = sI_n$ such that $sf(A) = PA^\delta Q$ for all $A \in T_n(\mathbb{R})$.

**Proof.** (i) It follows from (iii) of Lemma 2.1 that there exist $s_1, s_2 \in \mathbb{R} - q$ and $A, B \in T_n(\mathbb{R})$ such that $P_1 = (1/s_1)Aq$, $P_1^{-1} = (1/s_2)Bq$ and $f_q(E_{ij}) = (1/s_1 s_2)(A E_{ij} B)_q$ for any $i \leq j$. Again applying (i) of Lemma 2.1, there exists $s_{ij} \in \mathbb{R} - q$ such that $s_{ij} s_1 s_2 f(E_{ij}) = s_{ij} A E_{ij} B$ for any $i \leq j$. Letting

$$s = s_1 s_2 \prod_{1 \leq i \leq j \leq n} s_{ij}, \quad P = \left( \prod_{1 \leq i \leq j \leq n} s_{ij} \right) A$$

and $Q = B$, the lemma follows.

(ii) The proof is similar to (i). \[\square\]

**Theorem 3.2.** Suppose $2 \in \mathbb{R}^*$. Then $f \in \mathcal{L}(\mathbb{R})$ if and only if there exist $U \in T_n^+(\mathbb{R})$ and an idempotent element $e \in \mathbb{R}$ such that $f(X) = U(eX + (1 - e)X^\delta)U^{-1}$ for all $X \in T_n(\mathbb{R})$.

**Proof.** The “if” part is obvious. Now we prove the “only if” part.

It follows from (ii) of Lemma 2.3 that $f_q(E_n(\mathbb{R}_q)) \subseteq T_n(\mathbb{R}_q)$ for any $q \in \text{Spec } \mathbb{R}$, and thus $f_q$ has one of the forms in Theorem 3.1. Again applying Lemma 3.3, there exist $s \in \mathbb{R} - q$ and $P, Q \in T_n(\mathbb{R})$ with $(PQ)_q = sI_n$ such that either

(I) $sf(A) = PXQ$ for any $X \in T_n(\mathbb{R})$
or

(II) $sf(A) = PX^\delta Q$ for any $X \in T_n(\mathbb{R})$.

Let

$$S_1 = \{s \det(PQ) \mid s, P, Q \text{ satisfy (I)} \},$$

$$S_2 = \{s \det(PQ) \mid s, P, Q \text{ satisfy (II)} \}$$

and

$$D(S_i) = \{q \in \text{Spec } \mathbb{R} \mid S_i \not\subseteq q \} \text{ for } i = 1, 2.$$

Obviously, $D(S_1) \cap D(S_2) = \emptyset$ and $D(S_1) \cup D(S_2) = \text{Spec } \mathbb{R}$, i.e., $D(S_1)$ and $D(S_2)$ are open–closed sets under Zariski topology, and then from [6] it follows that there
exists an idempotent element $e \in \mathbb{R}$ such that $D(S_1) = D(e)$, $D(S_2) = D(1 - e)$ and $\mathbb{R} = \mathbb{R}e \oplus \mathbb{R}(1 - e)$. Clearly $f$ has only the form (I) if $q \in D(S_1)$. Similarly, $f$ has only the form (II) if $q \in D(S_2)$.

Let $\Omega_e(X) = eX + (1 - e)X^\delta$ for any $X \in T_n(\mathbb{R})$ and $\Theta = f\Omega_e$. Then $\Theta_q(X_q) = Q_q^{-1}X_qQ_q$ for any $X \in T_n(\mathbb{R})$ and any $q \in \text{Spec } \mathbb{R}$. Hence there exists $U \in T_n^*(\mathbb{R})$ such that $\Theta(X) = f\Omega_e(X) = UXU^{-1}$ for any $X \in T_n(\mathbb{R})$ from Lemma 2.2. Again applying $\Omega_e^2 = 1$, we have $f = \Theta\Omega_e$. The theorem follows. □

4. Applications

4.1. The Jordan isomomorphisms on $T_n(\mathbb{R})$

Recall that a bijective $\mathbb{R}$-linear map $f : T_n(\mathbb{R}) \rightarrow T_n(\mathbb{R})$ is called a Jordan isomorphism on $T_n(\mathbb{R})$ if $f(AB + BA) = f(A)f(B) + f(B)f(A)$ $\forall A, B \in T_n(\mathbb{R})$.

**Theorem 4.1.** Suppose $\mathbb{R}$ is a commutative ring with $1, 2 \in \mathbb{R}^\ast$. Then following three conditions are equivalent:

(i) $f$ is a Jordan isomorphism on $T_n(\mathbb{R})$ over the ring $\mathbb{R}$.

(ii) $f \in L(\mathbb{R})$.

(iii) There exist an invertible matrix $P \in T_n(\mathbb{R})$ and an idempotence $e \in \mathbb{R}$ such that $f(X) = P(eX + (1 - e)X^\delta)P^{-1}$ for any $X = (x_{ij}) \in T_n(\mathbb{R})$, where $X^\delta = (x_{n+1-j, n+1-i})$.

**Proof.** The equivalence of (ii) and (iii) is obvious from Theorem 3.2.

Assume that condition (i) holds, that is, $f(AB + BA) = f(A)f(B) + f(B)f(A)$ $\forall A, B \in T_n(\mathbb{R})$. Let $A = B$ and note $2 \in \mathbb{R}^\ast$. We have $f(A^2) = f(A)^2$. Hence $f \in L(\mathbb{R})$, and so (ii) holds. From (iii), condition (i) is easily attained. Hence Theorem 4.1 is proved. □

**Theorem 4.2.** Suppose $\mathbb{R}$ is a commutative ring with $1, 2 \in \mathbb{R}^\ast$. Then following two conditions are equivalent:

(i) $\mathbb{R}$ is a connected ring;

(ii) every Jordan isomorphism on $T_n(\mathbb{R})$, $n > 1$, is either an automorphism or an anti-automorphism.

**Proof.** (i) ⇒ (ii) Let $f$ be a Jordan isomorphism on $T_n(\mathbb{R})$, $n > 1$, and $\mathbb{R}$ be a connected ring. Then $\mathbb{R}$ contains no idempotents except 0 and 1. So $f(X) = PXP^{-1}$ or $f(X) = PX^\delta P^{-1}$ from Theorem 4.1, and hence (ii) holds.

(ii) ⇒ (i) If $\varepsilon \neq 0, 1$ is idempotent on $\mathbb{R}$, then $f(X) = \varepsilon X + (1 - \varepsilon)X^\delta$ is a Jordan isomorphism, but it is neither an automorphism nor an anti-automorphism.
The result similar to [3] can be attained from it easily, that is: automorphisms and anti-automorphisms are the only Jordan automorphisms of $T_n(\mathbb{R})$, where $\mathbb{R}$ is a connected commutative ring with units 1 and 2. □

As a corollary to Theorem 3.2, we obtain the following well-known result.

**Corollary 4.3.** Every algebra automorphism of $T_n(\mathbb{R})$ is inner.

We also get the following companion result.

**Corollary 4.4.** A map $\phi : T_n(\mathbb{R}) \to T_n(\mathbb{R})$ is an algebra anti-isomorphism if and only if there exists an invertible matrix $P \in T_n(\mathbb{R})$ such that

$$\phi(X) = PX\delta P^{-1} \quad \forall X \in T_n(\mathbb{R}).$$

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