Note

On the combinatorics of an origami model

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1. Introduction

Place a cube at each vertex of a dodecahedron so that adjacent cubes intersect at their appropriate vertices, and also, require maximum symmetry for the resulting compound. This configuration – the “Ball-Of-Cubes” or “Dodecahedron-of-Cubes” – shown in Fig. 1, was introduced by internationally known origami artist David Mitchell [3]. The geometry of this model and those of similar configurations were examined in [2].

The Ball-Of-Cubes model was designed so that at an intersection – representing an edge of the dodecahedron – a vertex of one of the two adjacent cubes is inverted where a vertex of the other cube is inserted. There will be 30 inverted corners to be distributed among the 20 cubes that make up the compound. A possible – and the most balanced – arrangement is to have 10 cubes with each having two inverted corners, and 10 cubes with each having one inverted corner.

To assemble a model, we add the cubes, one-by-one, to the partially assembled compound. At each step, we may have several choices due to the highly symmetric nature of the model. A cube might be placed at different vertices with the dimple(s) possibly occurring at different edges of the dodecahedron. Can we ever get stuck? Is it possible that for a partially assembled compound, at one of the un-built locations, we would need a cube with, say two dimples, but all the remaining cubes that have not been added to the compound have one dimple?

Our aim is to investigate this problem in a graph theoretical context. We will present a condition which guarantees that the assembly process can always be completed, and that also appears to be quite natural.

2. Graphs

Graphs are a convenient tool to describe the combinatorial aspects of the Ball-Of-Cubes configuration. The vertices and edges of the graph correspond to the vertices and edges of the dodecahedron, respectively, so that if \((u, v)\) is a directed edge of the graph, then a corner of the cube at \(u\) is inserted into an inverted corner of the cube at \(v\), see Fig. 2.

In general, we can formalize the assembly process as follows. For an undirected graph \(G = (V, E)\) and degrees \((d_v)\), we may be given a collection of in-degree/out-degree pairs \(((i_1, o_1), (i_2, o_2), \ldots, (i_n, o_n))\) with \(\sum_{p=1}^{n} i_p = \sum_{p=1}^{n} o_p(= |E|)\)

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A partial assembly of the graph $G$ means we have a one-to-one mapping $f$ of some subset $U$ of vertices $V$ to $\{1, 2, \ldots, n\}$, so that for a vertex $v \in U$, if $f(v) = s$, then $d_v = i_s + o_s$. In addition, the edges adjacent to vertex $v$ are assigned a direction so that the in-degree/out-degree pair of $v$ will be $(i_s, o_s)$.

Hence, for a given graph and a given in-degree/out-degree sequence, we may think of a partial assembly as the mapping $f$ together with the orientation of edges incident to $U$.

A full assembly is a partial assembly with $U = V$. A partial assembly $A_1$ is an extension of a partial assembly $A_2$ if every vertex that belongs to $A_2$ also belongs to $A_1$, and every edge in $A_2$ has the same orientation in $A_1$ as in $A_2$. The assembly process is a sequence of partial assemblies $A_0, A_1, \ldots, A_n$ such that $A_0$ is trivial with $U$ being the empty set, $A_n$ is a full assembly, and for each $i = 1, 2, \ldots, n$, the partial assembly $A_i$ is an extension of $A_{i-1}$ by a single vertex.

3. Main result

Is there some condition under which every partial assembly can be extended to a full assembly? We have the following theorem.

**Theorem.** Assume that $G = (V, E)$ is an undirected graph with all vertices having degree less than or equal to 3. For an assembly process, assume a balanced in-degree/out-degree sequence, i.e., $|i_p - o_p| \leq 1$ for all $p = 1, 2, \ldots, n$. Then, a partial assembly of $G$ can always be extended to a full assembly provided that the vertices not in the partial assembly form a connected subgraph of $G$.

**Proof.** Suppose $B$ is a partial assembly of $G$ such that $H$, the subgraph of $G$ formed by the vertices not in $B$, is connected. Let $v$ be a vertex in $H$ such that removing $v$ from $H$ results in a still connected subgraph of $G$. We distinguish three cases based on $d_G(v)$, the degree of $v$ in $G$, and subcases based on $d_H(v)$, the degree of $v$ in $H$.

Assume $d_G(v) = 1$. If $d_H(v) = 1$ then we can freely assign any available $(0, 1)$ or $(1, 0)$ in-degree/out-degree pair to $v$ and assign an orientation to the only edge adjacent to $v$ accordingly. By our assumptions, for any $k = 1, 2, 3$, the sequence
that going into vertex v, x ≥ have 0
and the two inequalities to get together yield H 2 represents the two edges that belong to B in the (0, 1) if the only edge adjacent to v has an orientation towards v, and (0, 1) if that edge is oriented away from v. Otherwise we could assemble G from an in-degree/out-degree sequence that is obtained from ((i 1, o 1), (i 2, o 2), . . . , (i n, o n)) by changing one of the pairs from (0, 1) to (1, 0) or vice versa. But then the condition \( \sum_{p=1}^{n} i_p = \sum_{p=1}^{n} o_p \) would hold for the altered sequence and not for the original one, a contradiction.

Assume \( d_C(v) = 2 \). If \( d_B(v) = 1 \) or 2 then we can assign any available (1, 1) in-degree/out-degree pair to v and assign appropriate orientations to the one or two edges that have no orientation yet and are adjacent to v. Note that a balanced in-degree/out-degree pair with total degrees 2 must be of type (1, 1). Thus, our assumptions on the in-degree/out-degree pairs \(((i_1, o_1), (i_2, o_2), \ldots, (i_n, o_n))\) guarantee the availability of such an in-degree/out-degree pair. If \( d_H(v) = 0 \) then H = \{v\} and both edges adjacent to vertex v have orientation that must be compatible with in-degree/out-degree pair (1, 1). Otherwise, as in the previous case, the condition \( \sum_{p=1}^{n} i_p = \sum_{p=1}^{n} o_p \) would be violated.

Assume \( d_C(v) = 3 \). If \( d_B(v) = 2 \) or 3 then any available degree assignment, (2, 1) or (1, 2) can be used to extend B by vertex v. Again, the existence of such an in-degree/out-degree pair is guaranteed by the assumptions on the in-degree/out-degree pairs \(((i_1, o_1), (i_2, o_2), \ldots, (i_n, o_n))\).

Suppose the degree of v in H is 1. Then, two of the edges adjacent to v belong to B, and are assigned directions. If these directions differ, i.e., one edge is going into v and the other out of v, then any available degree pair, (2, 1) or (1, 2) can be assigned to v while preserving the directions on these edges. Hence, B can be extended by adding v to it.

Assume the directions are the same, say, both edges go into v (see Fig. 3). Then, if a degree pair (2, 1) is available, B can again be extended using v. What if there is no available in-degree/out-degree pair (2, 1) ? Let us count the edges of \( G \) by introducing the following notations:

For non-negative integers k and l, such that 0 ≤ k + l ≤ 3, let \( n_{kl} \) denote the number of pairs in the in-degree/out-degree sequence that are equal to \( (k, l) \). We consider balanced in-degree/out-degree sequences, thus, the interesting \( n_{kl} \) non-zero cases are \( (k, l) = (0, 1), (1, 0), (1, 1), (1, 2), \) and \( (2, 1) \). Define also \( x_{kl} \) to be the number of vertices in B with in-degree/out-degree pair \( (k, l) \). First note that

\[
|E| \geq \sum_{(k,l)} k \cdot x_{kl} + 2 + \left[ \left( \sum_{(k,l)} n_{kl} - \sum_{(k,l)} x_{kl} \right) - 1 \right].
\]

In the expression on the right-hand side, the first summation represents the edges that go into a vertex in B, the term 2 represents the two edges that belong to B and go into v. The third term in brackets represents the number of edges in H which needs to be at least the number of vertices in H minus 1 for H to be connected.

On the other hand,

\[
\sum_{(k,l)} k \cdot x_{kl} + 2 + \left[ \left( \sum_{(k,l)} n_{kl} - \sum_{(k,l)} x_{kl} \right) - 1 \right] = 1 + \sum_{(k,l)} |n_{kl} + (k-1)x_{kl}| \geq 1 + \sum_{(k,l)} k \cdot n_{kl} = 1 + |E|,
\]

and the two inequalities together yield \( |E| \geq 1 + |E| \), a contradiction. To see why the second inequality is true, note that it is implied by the fact that the condition \( \sum_{p=1}^{n} i_p = \sum_{p=1}^{n} o_p \) is now equivalent to \( |E| = \sum_{(k,l)} k \cdot n_{kl} = \sum_{(k,l)} l \cdot n_{kl} \), and by the collection of inequalities

\[ n_{kl} + (k-1) \cdot x_{kl} \geq k \cdot n_{kl} \Leftrightarrow (k-1) \cdot x_{kl} \geq (k-1) \cdot n_{kl}. \]

To see the latter consider the cases \( k = 0, 1, \) and 2 separately. If \( k = 0 \) then we have \( n_{kl} \geq x_{kl} \), and if \( k = 1 \) then we have \( 0 \geq 0 \), both clearly true. If \( k = 2 \), then we must have \( l = 1 \), and thus we obtain \( x_{21} \geq n_{21} \). But this is true since the non-availability of the required pair \( (2, 1) \) is equivalent to \( x_{21} = n_{21} \).

When both edges go out of v, we can repeat the calculations above by considering edges that go out of, instead of edges that go into vertex v and vertices in B.
In the remaining case, when the degree of \( v \) in \( H \) is 0, we can argue as before: since \( H \) is connected \( H \) must consist of \( v \) only and the last available in-degree/out-degree pair must be the one we need for the equation \( \sum_{p=1}^{n} i_p = \sum_{p=1}^{n} o_p \) to be true.

Since the subgraph \( H - v \) is connected, the extension can be repeated until we reach a full assembly. \( \square \)

The class of graphs in the Theorem, although contains our initial example of the Ball-Of-Cubes, is an admittedly limited one. It would be desirable to have a more general result, and the reader is encouraged to find one. However, every component of the Theorem in its present form appears to be essential.

First we note that connectivity of the un-built portion of the graph is required. One simple example would be when, in the dodecahedron, a vertex \( v \) is isolated with the first three vertices in the assembly process in such a way that all edges adjacent to \( v \) are directed, say towards \( v \), forcing an unbalanced in-degree/out-degree assignment for \( v \). Less trivial examples can easily be found as well.

The balanced degree requirement cannot be dropped either. Consider the following graph with in-degree/out-degree numbers ((2, 1), (2, 1), (2, 1), (3, 0), (0, 3), (0, 3)). The partial assembly shown in Fig. 4 cannot be extended to a complete configuration.

The next example in Fig. 5 demonstrates that the degree 3 requirement is also essential. We assume balanced assembly, i.e., all in-degree/out-degree pairs are (2, 2). The partial assembly in the diagram cannot be completed to a full configuration.

A modest extension of the Theorem can be obtained by allowing the insertion of individual edges (assigning directions to them) during the assembly process; the proof of the Theorem goes through with no change.

Our approach to modeling the Ball-Of-Cubes is appropriate due to (1) at every vertex of the dodecahedron there are three edges; and (2) the compound has maximum symmetry. If any of these two conditions is violated then graphs with in-degree/out-degree sequences alone are not sufficient to analyze the assembly combinatorics of a polyhedral compound with rigid geometry.

4. Additional considerations

Besides the extensions of partial assemblies further questions arise. First, we may ask if a particular assignment of degree pairs \((i_p, o_p)\) to the vertices of \( G \) is feasible, that is, whether there exists an orientation of the edges of \( G \) that would result in the same in-degree/out-degree values as the assignment given by the pairs \((i_p, o_p)\). Second, we may ask whether a sequence of in-degree/out-degree pairs is feasible, i.e., whether these pairs can be assigned to the vertices of \( G \) so that this assignment is feasible.

Note that in case of the Theorem, the feasibility of the balanced sequence of degree pairs is guaranteed by the Theorem itself. In a way, the Theorem can be interpreted as a statement about the feasibility of in-degree/out-degree sequences together with an algorithm to obtain in-degree/out-degree assignments that are feasible.

For the examples in Figures 4 and 5, the in-degree/out-degree sequences are feasible, however, the (partial) degree pair assignments shown in the diagrams are not.

Frank and Gyárfás [1] characterized in-degree/out-degree assignments that are feasible. One way to state their results is this: An in-degree/out-degree assignment is feasible if and only if for every subset \( U \) of the vertices \( V \) we have

\[
|\{\text{Edges between } U \text{ and } V - U\}| \geq \sum_{v \in U} |i_v - o_v|.
\]

where \( i_v \) and \( o_v \) denote the in-degree and out-degree for each vertex \( v \), respectively.
In the dodecahedron (Fig. 2), take two pentagons joined by an edge so that $U$ would be the collection of the corresponding 8 vertices. A degree assignment where $(i_v, o_v) = (2, 1)$ for each $v$ in $U$ would not be possible as the above inequality would become $6 \geq 8$.

Our Theorem implies that in a graph, where each vertex has degree 3 or less, every balanced in-degree/out-degree sequence, with $(i_1 + o_1, i_2 + o_2, \ldots, i_n + o_n)$ being the same (ignoring the order of the entries) as $(d_v | v \in V)$, and $\sum_{p=1}^{n} i_p = \sum_{p=1}^{n} o_p$, is feasible. The degree requirements again are essential. Extend the above dodecahedron example by attaching a new vertex of degree 1 to each $v \in U$ and consider the balanced in-degree/out-degree sequence that consists of 8 copies of $(1, 0)$ (one for each new vertex); 8 copies of $(2, 2)$ (one for each $v \in U$); 2 copies of $(2, 1)$; and 10 copies of $(1, 2)$. If this sequence was feasible then the pairs $(2, 2)$ would have to be assigned to vertices in $U$ and by removing the new vertices we would find that the assignment in our original example would be feasible. Hence, the assumption that all degrees are less than or equal to 3 cannot be dropped. A sequence with all pairs being $(3, 0)$ or $(0, 3)$ would not be feasible for the dodecahedron (or for the graph in Fig. 4), thus, the assumption that the in-degree/out-degree pairs are balanced is also essential.

References