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An Elementary Proof of a Theorem of Erdös on the Sum of Divisors Function

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Let $\sigma(n)$ be the sum of the positive divisors of the positive integer n. We give an elementary proof of the following theorem due to P. Erdös: *If $g(x)$ is the number of positive integers m such that $\sigma(m) \leqslant x$, then there is a positive constant c such that $g(x) = cx + o(x)$.*

In addition we derive

$$c = \prod_p \{1-1/p)(1 + (1/p + 1) + (1/p^2 + p + 1) + (1/p^3 + p^2 + p + 1) + \cdots)\}.$$

1. INTRODUCTION

In an earlier paper an elementary proof of the following theorem of P. Erdös is given: If φ is the Euler function and $f(x)$ is the number of positive integers m such that $\varphi(m) \leqslant x$, then there is a positive constant c_1 such that $f(x) = c_1x + o(x)$. It is the purpose of this paper to give an elementary proof of a related theorem also due to P. Erdös: If σ is the sum of divisors function and $g(x)$ is the number of positive integers m such that $\sigma(m) \leqslant x$, then there is a positive constant c_2 such that $g(x) = c_2x + o(x)$. Erdös' proofs use analytic results of I.J. Schoenberg, and the above results can also be obtained from the Wiener–Ikehara Theorem. Our proof given here for σ is more difficult than our proof for φ because the multiplicity of the prime divisors of an integer m must be taken into account—something that can be ignored in the case of φ.

2. NOTATION

Let $A = \{a_i\}_{i=1}^{\infty}$ be a sequence of positive real numbers $\geqslant 1$. For a positive integer j, define $\#(A, j)$ to be the number of integers i such that $a_i \leqslant j$ (i.e., the number of elements of A *counting multiplicity* which are $\leqslant j$). We denote the i-th prime by p_i and p will always be a prime.

3. The Main Result

We begin by remarking that a well-known formula for σ is

$$\sigma(n) = n \prod_{p^e \| n} (1 + (1/p) + \cdots + (1/p^e)).$$

We define

$$\sigma_{r,k}(n) = n \prod_{\substack{p^e \| n \\ p \leqslant p_r}} (1 + (1/p) + \cdots + (1/p^{\min(e,k)})).$$

We let $\#(\sigma_{r,k}, n)$ denote the number of positive integers m such that $\sigma_{r,k}(m) \leqslant n$ and we define $\Delta(\sigma_{r,k}) = \lim_{n\to\infty} \#(\sigma_{r,k}, n)/n$. As a first step we show $\Delta(\sigma_{r,k})$ exists and calculate its value.

Corresponding to each r-tuple $(t_1, t_2, \ldots, t_r)$ with $0 \leqslant t_j \leqslant k+1$, let $A = A(t_1, \ldots, t_r)$ be the set of those integers m such that $p_j^{t_j} \mid m$ $(j = 1, 2, \ldots, r)$ but $p_j^{t_j+1} \nmid m$ if $t_j < k+1$ $(j = 1, 2, \ldots, r)$. For each $m \in A$ we have

$$\sigma_{r,k}(m) = m \prod_{j=1}^{r} (1 + (1/p_j) + (1/p_j^2) + \cdots + (1/p_j^{t_j - 1 + \operatorname{sgn}(k+1-t_j)}))$$

where as usual

$$\operatorname{sgn} x = \begin{Bmatrix} -1 & \text{if } x < 0 \\ 0 & \text{if } x = 0 \\ 1 & \text{if } x > 0 \end{Bmatrix}.$$

An integer $m \in A$ satisfies $\sigma_{r,k}(m) \leqslant n$ if and only if it satisfies

$$m \leqslant n \prod_{j=1}^{r} (1 + (1/p_j) + \cdots + (1/p_j^{t_j - 1 + \operatorname{sgn}(k+1-t_j)}))^{-1}.$$

An easy counting argument then shows that the number of $m \in A$ with $\sigma_{r,k}(m) \leqslant n$ is

$$\left\{ \prod_{j=1}^{r} (p_j - \operatorname{sgn}(k+1-t_j))/p_j^{t_j+1} \right\}$$

$$\times \left\{ n \prod_{j=1}^{r} (1 + (1/p_j) + \cdots + (1/p_j^{t_j - 1 + \operatorname{sgn}(k+1-t_j)}))^{-1} + \theta p_1^{k+1} \cdot \cdots \cdot p_r^{k+1} \right\}$$

for some θ satisfying $|\theta| \leqslant 1$.

Now

$$((p_j - 1)/p_j) + ((p_j - 1)/p_j^2) + \cdots + ((p_j - 1)/p_j^{k+1}) + (p_j/p_j^{k+2}) = 1$$

and so, summing over all r-tuples $(t_1, \ldots, r_r)$ gives

$$\#(\sigma_{r,k}, n) = n\, \Delta(\sigma_{r,k}) + \theta' p_1^{k+1} \cdot \cdots \cdot p_r^{k+1} \qquad (*)$$

where $|\theta'| \leqslant 1$ and

$$\Delta(\sigma_{r,k}) = \sum_{t_1, t_2, \ldots, t_r = 0}^{k+1} \prod_{j=1}^{r} (p_j - \operatorname{sgn}(k + 1 - t_j))/p_j^{t_j+1}$$

$$\times (1 + (1/p_j) + \cdots + (1/p_j^{t_j - 1 + \operatorname{sgn}(k+1-t_j)}))^{-1}$$

$$= \prod_{j=1}^{r} \sum_{t_1, t_2, \ldots, t_r = 0}^{k+1} (1 - p_j^{-1} \operatorname{sgn}(k + 1 - t_j))/(p_j^{t_j} + p_j^{t_j - 1} + \cdots + p_j^{1 - \operatorname{sgn}(k+1-t_j)}).$$

We note that

$$\lim_{r,k \to \infty} \Delta(\sigma_{r,k}) = \prod_p \{(1 - 1/p)(1 + (1/(p + 1))$$

$$+ (1(/p^2 + p + 1)) + (1/(p^3 + p^2 + p + 1)) + \cdots)\}.$$

Next notice that for any positive integer m, if we have $r_1 \geqslant r$ and $k_1 \geqslant k$, then $\sigma_{r_1,k_1}(m) \geqslant \sigma_{r,k}(m)$. Thus, for any positive integer x we have $\#(\sigma_{r,k}, x) \geqslant \#(\sigma_{r_1,k_1}, x)$. So, $\Delta(\sigma_{r_1,k_1}) \leqslant \Delta(\sigma_{r,k})$. In addition $\sigma(m) \geqslant \sigma_{r,k}(m)$ for all positive integers r, k, and m and so

$$\lim_{n\to\infty} \sup\, (\#(\sigma, n)/n) \leqslant \lim_{n\to\infty} \sup\, (\#(\sigma_{r,k}, n)/n)$$

$$= \lim_{n\to\infty} (\#\sigma_{r,k}, n)/n) = \Delta(\sigma_{r,k}).$$

Thus $\lim_{n\to\infty} \sup(\#(\sigma, n)/n) \leqslant \lim_{r,k\to\infty} \Delta(\sigma_{r,k})$.

We now prove that $\lim_{n\to\infty} \inf(\#(\sigma, n)/n) \geqslant \lim_{r,k\to\infty} \Delta(\sigma_{r,k})$. For any positive integers r, k, and m we have

$$\sigma(m) = \sigma_{r,k}(m) \prod_{\substack{p^e \| m \\ p > p_r}} \left(1 + \frac{1}{p} + \cdots + \frac{1}{p^e}\right) \prod_{\substack{p^e \| m \\ p \leqslant p_r \\ e > k}} \frac{(1 + (1/p) + \cdots + (1/p^e))}{(1 + (1/p) + \cdots + (1/p^k))}.$$

So if $\sigma_{r,k}(m) \leqslant y$, we have

$$\sigma(m) \leqslant y \prod_{\substack{p^e \| m \\ p > p_r}} \left(1 + \frac{1}{p} + \cdots + \frac{1}{p^e}\right) \prod_{\substack{p^e \| m \\ p \leqslant p_r \\ e > k}} \left(\frac{1 + (1/p) + \cdots + (1/p^e)}{1 + (1/p) + \cdots + (1/p^k)}\right).$$

So if $\sigma_{r-1,k}(m) \leqslant nT_{m,r-1,k}$, where

$$T_{m,r-1,k} = \prod_{\substack{p^e \| m \\ p > p_{r-1}}} \frac{1}{\left(1 + \frac{1}{p} + \cdots + \frac{1}{p^e}\right)} \prod_{\substack{p^e \| m \\ p \leqslant p_{r-1} \\ e > k}} \left(\frac{1 + (1/p) + \cdots + (1/p_k)}{1 + (1/p) + \cdots + (1/p^e)}\right),$$

then $\sigma(m) \leqslant n$. Now suppose that for $n > 1$ we have $(p_1 p_2 \cdots p_r)^{k+1} \leqslant n < (p_1 p_2 \cdots p_r)^{k+2}$. If for some integer m, $\sigma_{r-1,k}(m) \leqslant nT_{m,r-1,k}$ then m has fewer than $r(k+2)$ distinct prime divisors and so

$$1 \geqslant T_{m,r-1,k} \geqslant S_{r-1,}k ,$$

where

$$S_{r-1,k} = \prod_{p \leqslant p_{r-1}} (1 + (1/p) + \cdots + (1/p^k))/(1 + (1/p) + (1/p^2) + \cdots)$$

$$\times \prod_{i=r}^{r(k+2)-1} (1/(1 + (1/p_i) + (1/p_i^2) + \cdots)).$$

Thus, $\#(\sigma, n) \geqslant \#(\sigma_{r-1,k}, nS_{r-1,k})$.

Now for r and k large with $\log r/\log k$ also large (e.g.,

$$r = [(\log n)(\log\log n)^{-2}] \quad \text{and} \quad k + 1 = [\log n/(\log p_1 + \cdots + \log p_r)])$$

we have

$$\prod_{i=r}^{r(k+2)-1} (1/(1 + (1/p_i) + (1/p_i^2) + \cdots)) = \prod_{i=r}^{r(k+2)-1} ((p_i - 1)/p_i)$$

which is close to 1 by Mertens' Theorem and Tchebychef's Theorem [5; pp. 351, 9]. Also

$$1 \geqslant \prod_{p \leqslant p_{r-1}} ((1 + (1/p) + \cdots + (1/p_k))/(1 + (1/p) + (1/p^2) + \cdots))$$

$$= \prod_{p \leqslant p_{r-1}} (p^{k+1} - 1)/p^{k+1} \geqslant 1/\zeta(k+1)$$

which is close to 1. Thus for r and k large with $\log r/\log k$ also large we have that $S_{r-1,k}$ is close to 1.

Now from (*), where we replace r by $r - 1$ and n by $nS_{r-1,k}$, we have that $\#(\sigma_{r-1,k}, nS_{r-1,k}) = nS_{r-1,k}\,\Delta(\sigma_{r-1},k) + \theta' p_1^{k+1} \cdot \cdots \cdot p_{r-1}^{k+1}$, where $|\theta'| \leqslant 1$. Since $n \geqslant (p_1 \cdot \cdots \cdot p_r)^{k+1}$, we see that

$$\#(\sigma, n)/n \geqslant \#(\sigma_{r-1,k}, nS_{r-1,k})/n \geqslant S_{r-1,k}\,\Delta(\sigma_{r-1,k}) - (1/p_r^{k+1}).$$

Thus, by letting $n \to \infty$ and by choosing r's and k's as above we see that $\liminf_{n\to\infty}(\#(\sigma, n)/n) \geqslant \lim_{r,k\to\infty} \Delta(\sigma_{r,k}))$. We have now shown that

$$\begin{aligned}\Delta(\sigma) &= \lim_{n\to\infty} (\#(\sigma, n)/n) = \lim_{r,k\to\infty} \Delta(\sigma_{r,k}) \\ &= \prod_p \{(1 - 1/p)(1 + (1/p + 1) + (1/p^2 + p + 1) \\ &\quad + (1/(p^3 + p^2 + p + 1) + \cdots)\}.\end{aligned}$$

But this last product is greater than

$$\prod_p \{(1 - 1/p)(1 + (1/p + 1) + (1/(p + 1)^2 + \cdots)\} = \prod_p (1 - (1/p^2)) > 3/5$$

and so $\Delta(\sigma)$ is positive (in fact, greater than 3/5) and we are done.

Finally, it is worth observing that the theorem we prove contains only the weakest error term. Bateman [1], using analytic techniques, has considerably strengthened the error term for both the φ-function and the σ-function.

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