Embedding spaces and hyperspaces of polyhedra in 2-manifolds

Tatsuhiko Yagasaki

Department of Mathematics, Kyoto Institute of Technology, Matsugasaki, Sakyo, Kyoto 606-8585, Japan

Received 22 November 1999; received in revised form 5 February 2000

Abstract

Suppose $M$ is a 2-manifold and $X$ is a compact polyhedron. Let $\mathcal{E}(X, M)$ denote the space of embeddings of $X$ into $M$ with the compact-open topology and let $\mathcal{K}(X, M)$ denote the hyperspace of copies of $X$ embedded in $M$ with the Fréchet topology. In this paper we show that the natural map $\pi : \mathcal{E}(X, M) \to \mathcal{K}(X, M)$, $\pi(f) = f(X)$, is a principal bundle with fiber $\mathcal{H}(X)$, the homeomorphism group of $X$. As a corollary it follows that the space $\mathcal{K}(X, M)$ is an ANR.

AMS classification: 57N05; 57N20; 57N35

Keywords: 2-manifolds; Hyperspaces; Embeddings; Homeomorphism groups; Infinite-dimensional manifolds

1. Introduction

This paper is a continuation of study of homeomorphism groups of 2-manifolds and related topics [6–8]. In this paper we investigate relations between embedding spaces and hyperspaces of polyhedra in 2-manifolds.

Suppose $Z = (Z, d)$ is a metric space and $X$ is a compact subset of $Z$. Let $\mathcal{H}(X)$ denote the group of homeomorphisms of $X$ onto itself and let $\mathcal{E}(X, Z)$ denote the space of embeddings $f : X \to Z$. These spaces are equipped with the compact-open topology which is induced by the sup-metric $d(f, g) = \sup_{x \in X} d(f(x), g(x))$. Next consider the hyperspace $\mathcal{K}(X, Z)$ consisting of the subsets of $Z$ homeomorphic ($\cong$) to $X$. We provide this hyperspace with the Fréchet topology. This topology is induced from the metric $\rho$ defined as follows: $\rho(Y_1, Y_2) = \inf\{d(h, id_{Y_1}) \mid h : Y_1 \to Y_2$ is a homeomorphism\} ($Y_1, Y_2 \in \mathcal{K}(X, Z)$). These spaces are related by the natural map

$$\pi : \mathcal{E}(X, Z) \to \mathcal{K}(X, Z), \quad \pi(f) = f(X).$$

E-mail address: yagasaki@ipc.kit.ac.jp (T. Yagasaki).
The group $\mathcal{H}(X)$ acts on $\mathcal{E}(X, Z)$ by right composition and this action preserves the fibers of $\pi$.

Sakai [4] studied the case where $X$ is an arc $I$ and $Z$ is a graph $G$ and showed that the map $\pi : \mathcal{E}(I, G) \to K(I, G)$ is a principal bundle with fiber $\mathcal{H}(I)$, where the Fréchet topology coincides with the usual Vietoris topology induced from the Hausdorff metric. In this paper we study a similar problem in the case where $Z$ is a 2-manifold.

**Theorem 1.1.** Suppose $M$ is a 2-manifold and $X$ is a compact polyhedron embedded in $M$. Then there exists an open neighborhood $U$ of $X$ in $K(X, M)$ and a map $F : U \to \mathcal{E}(X, M)$ such that $\pi \circ F = \text{id}_U$ and $F(X) = \text{id}_X$.

Theorem 1.1 implies that each point of $K(X, M)$ has an open neighborhood $U$ with a section $F : U \to \mathcal{E}(X, M)$ of the map $\pi$, which induces an $\mathcal{H}(X)$-equivariant trivialization $\varphi : U \times \mathcal{H}(X) \cong \pi^{-1}(U), \varphi(Y, h) = F(Y)h (\varphi^{-1}(f) = (\pi(f), F(\pi(f))^{-1}f)).$ This means the following

**Corollary 1.1.** Suppose $M$ is a 2-manifold and $X$ is a compact polyhedron. Then the map $\pi : \mathcal{E}(X, M) \to K(X, M)$ is a principal bundle map with fiber $\mathcal{H}(X)$.

Since $\mathcal{E}(X, M)$ is an ANR [6], we have

**Corollary 1.2.** Suppose $M$ is a 2-manifold and let $X$ is a compact polyhedron. Then the space $K(X, M)$ is an ANR.

Professor E. Shchepin suggested that Corollary 1.2 has an application to the “Bundle problem” in the case of 2-manifold fiber [3, Problem 2.4, p. 248], [1]. The author would like to thank him for this comment. In a succeeding paper we will study the homotopy types of the space $K(X, M)$.

Theorem 1.1 means that any copy of $X$ which is sufficiently close to $X$ admits a canonical parametrization by $X$. Since every graph can be decomposed into arc edges and circle edges, we first verify the case of arcs and circles in Section 2. Here we apply the conformal mapping theorem and related results in complex function theory [2,5]. The general cases are treated in Section 3. Throughout the paper, spaces are assumed to be separable and metrizable and maps are continuous. By $C(X, Y)$ we denote the space of maps from $X$ to $Y$ with the compact-open topology. When $(Z, d)$ is a metric space, for $Y \in K(X, Z)$ the notation $N_{\rho}(Y, \varepsilon)$ denotes the $\varepsilon$-neighborhood of $Y$ in $K(X, Z)$ with respect to the Fréchet metric $\rho$, while for a subset $A$ of $Z$ the notation $N(A, \varepsilon)$ denotes the $\varepsilon$-neighborhood of $A$ in $Z$ with respect to $d$.

### 2. Parametrization of arcs and circles

When we work on the complex plane $\mathbb{C}$, we use the following notations: For $z \in \mathbb{C}$ and $r > 0$, 
There exists an open neighborhood $U$ of $\alpha$ in $K(\alpha, D)$ and a map $F : U \to E(\alpha, D)$ such that $F(\beta)(\alpha) = \beta$ ($\beta \in U$) and $F(\alpha) = \text{id}_\alpha$.

In the verification of Proposition 2.1 we can assume that $D = D(2)$ and use the following facts [6, Lemmas 2.2 and 2.3].

**Lemma 2.1.** For each $\beta \in K(\alpha, O(2))$ there exists a unique real number $r = r_\beta$, $0 < r < 2$, and a unique map $h = h_\beta : A(2, r) \to D(2)$ such that $h : \text{Int} A(2, r) \to O(2) \setminus \beta$ is a conformal map and $h(2) = 2$. Furthermore, the map $h$ satisfies the following conditions:

(i) $h$ maps $C(2)$ homeomorphically onto $C(2)$ (hence $h(C(r)) = \beta$), and

(ii) there exists a unique pair of points $\bar{u} = \bar{u}(\beta)$, $\bar{v} = \bar{v}(\beta)$ in $C(r)$ such that $h(\bar{u})$, $h(\bar{v})$ are the end points of $\beta$ and $h$ maps two circular arcs in $C(r)$ with end points $\bar{u}$, $\bar{v}$ homeomorphically onto $\beta$.

For $0 < r < 2$ let $\varphi_r : A(1, 2) \to A(r, 2)$ denote the radial map defined by $\varphi_r(x) = ((2 - r)(|x| - 1) + r)x/|x|$. Using $h_\beta$ and $r_\beta$ in Lemma 2.1, we define $\psi_\beta \equiv h_\beta \varphi_{r_\beta} \in C(A(1, 2), D(2))$. Let $B(\beta) = (\varphi_{r_\beta})^{-1}([\bar{u}(\beta), \bar{v}(\beta)]) \subset C(1)$.

**Lemma 2.2.** The function $K(\alpha, O(2)) \to \mathbb{R} \times C(A(1, 2), D(2)) : \beta \mapsto (r_\beta, \psi_\beta)$ is continuous.

A sequence of compact subsets $X_n$ in a metric space $(Z, d)$ is said to be uniformly locally connected [2, §2.2, p. 22] if for each $\varepsilon > 0$ there exists a $\delta > 0$ such that for any $n \geq 1$ and any $x$, $y \in X_n$ with $d(x, y) < \delta$ there exists a connected subset $A$ of $X_n$ such that $x$, $y \in A$ and $\text{diam} A < \varepsilon$. Note that if $X$ is a locally connected compactum and a sequence $Y_n (n \geq 1)$ converges to $Y$ in $K(X, Z)$, then $Y_n (n \geq 1)$ is uniformly locally connected. Therefore, Lemma 2.2 can be verified by the same argument as in [6, Lemma 2.3].

**Proof of Proposition 2.1.** First we show that the function $K(\alpha, O(2)) \ni \beta \mapsto B(\beta) \in K(\{\pm 1\}, C(1))$ is continuous. Let $\beta \in K(\alpha, O(2))$ and $\varepsilon > 0$ be given. Let $B(\beta) = [u_1, u_2]$ and choose disjoint open neighborhoods $U_i$ of $\psi_\beta(u_i)$ ($i = 1, 2$) in $O(2)$ and disjoint closed
neighborhoods $A_i$ of $u_i$ in $C(1)$ such that $A_i \subset N(u_i, \varepsilon)$ and $\psi_\beta(A_i) \subset U_i$ ($i = 1, 2$). Set $K = C(1) \setminus (A_1 \cup A_2)$ and take $\delta > 0$ such that $N(\psi_\beta(A_i), \delta) \subset U_i$ ($i = 1, 2$) and $N(\psi_\beta(K), \delta) \cap N(\gamma(u_i), \delta) = \emptyset$ ($i = 1, 2$). Since $\gamma \mapsto \psi_\gamma$ is continuous, there exists a neighborhood $U$ of $\beta$ in $K(\alpha, O(2))$ such that for each $\gamma \in U$ (i) $\psi_\gamma (K) \subset N(\psi_\beta(K), \delta)$ and $\psi_\gamma (A_i) \subset U_i$ ($i = 1, 2$), and (ii) implies that one end point of $\gamma$ is in $N(\psi_\beta(u_1), \delta)$ and another is in $N(\psi_\beta(u_2), \delta)$. Hence by the definition of $B(\gamma)$ it follows that $B(\gamma) \subset A_1 \cup A_2$ and $B(\gamma)$ meets both $A_1$ and $A_2$, thus $B(\gamma) \subset N_{\rho}(B(\beta), \varepsilon)$.

We fix a labeling $B(\alpha) = \{u(\alpha), v(\alpha)\}$ and choose small neighborhoods $U$ and $V$ of $u(\alpha)$ and $v(\alpha)$, respectively, in $C(1)$. If we choose a small neighborhood $U_{\beta}$ of $\beta$ in $K(\alpha, O(2))$, then for each $\beta \in U_{\beta}$ we have a unique labeling $B(\beta) = \{u(\beta), v(\beta)\}$ such that $u(\beta) \in U$ and $v(\beta) \in V$. It follows that the functions $u, v : U \rightarrow C(1)$ are continuous. As indicated in Fig. 1, we choose a boundary point $z_0$ of $U$, and for each $\beta \in U_{\beta}$ we represent as $u(\beta) = z_0 e^{i\theta_\beta}$, $v(\beta) = z_0 e^{i\tau_\beta}$ and define a path $c_\beta$ in $C(1)$ by $c_\beta(t) = z_0 e^{i(1-t)\theta_\beta + t\tau_\beta)}$ ($0 \leq t \leq 1$). Then the function $U_{\beta} \ni \beta \mapsto f_\beta = \psi_\beta c_\beta \in E([0, 1), O(2))$ is continuous. The required map $F : U \rightarrow E(\alpha, O(2))$ is defined by $F(\beta) = f_\beta(f_\alpha)^{-1}$. □

2.2. Circles in an annulus

First we introduce some notations: Suppose $(Z, d)$ is a metric space and $K \subset X$ are compact subsets of $Z$. Let $K((X, K), Z) = \{(Y, L) \mid L \subset Y \subset Z \text{ and } (Y, L) \cong (X, K)\}$. The Fréchet topology is induced by the metric $\rho((Y_1, L_1), (Y_2, L_2)) = \inf\{d(h, id) \mid h : (Y_1, L_1) \cong (Y_2, L_2)\}$. The symbol $N_\rho((X, K), \varepsilon)$ denotes the $\varepsilon$-neighborhood with respect to this metric.

Suppose $A$ is an annulus and $\alpha$ is a center (essential) circle in $\text{Int} A$.

Proposition 2.2.

(1) There exists an open neighborhood $U$ of $\alpha$ in $K(\alpha, A)$ and a map $F : U \rightarrow E(\alpha, A)$ such that $F(\beta)(\alpha) = \beta$ ($\beta \in U$) and $F(\alpha) = id_{\alpha}$.
(2) For any distinguished point $x_0$ in $\alpha$ there exists an open neighborhood $U$ of $(\alpha, x_0)$ in $K((\alpha, x_0), A)$ and a map $F : U \to E(\alpha, A)$ such that $F(\beta, y_0)(\alpha, x_0) = (\beta, y_0)$ ($(\beta, y_0) \in U$) and $F(\alpha, x_0) = \text{id}_\alpha$.

Since each Jordan curve in $C$ has an annulus neighborhood, we can work on $C$. We will use the following facts:

**Lemma 2.3.**

1. Suppose $C$ is a Jordan curve in $C$, $D$ is the disk bounded by $C$ and $w$ is a point in $\text{Int} \ C$ (= Int $D$). Then there exists a unique homeomorphism $f = f(C, w) : D(1) \cong D$ such that (i) $f$ maps $O(1)$ conformally onto $\text{Int} \ C$ and (ii) $f(0) = w$ and $f'(0) > 0$.
2. Suppose $(C, w)$ and $(C_n, w_n)$ ($n \geq 1$) are pairs as in (1). If (i) the sequence $C_n$ is uniformly locally connected and converges to $C$ in the Hausdorff metric, and (ii) $w_n \to w$ ($n \to \infty$), then the maps $f(C_n, w_n)$ converges uniformly to $f(C, w)$ in $C(D(1), \mathbb{C})$.

**Lemma 2.4.**

1. Suppose $C$ is a Jordan curve in $C$, $x \in C$, $D$ is the disk bounded by $C$ and $w$ is a point in $\text{Int} \ C$. Then there exists a unique homeomorphism $f = f(C, w, x) : D(1) \cong D$ such that (i) $f$ maps $O(1)$ conformally onto $\text{Int} \ C$ and (ii) $f(0) = w$ and $f(1) = x$.
2. Suppose $(C, w, x)$, $(C_n, w_n, x_n)$ ($n \geq 1$) are triples as in (1). If (i) the sequence $C_n$ is uniformly locally connected and converges to $C$ in the Hausdorff metric, and (ii) $w_n \to w$, $x_n \to x$ ($n \to \infty$), then the maps $f(C_n, w_n, x_n)$ converges uniformly to $f(C, w, x)$ in $C(D(1), \mathbb{C})$.

As for references of these lemmas, we refer to [2, Riemann Mapping Theorem (§1.2, p. 4), Theorem 2.6, Corollary 2.7 and its proof, and §1.2.3 (The Möbius transformations)] for the statements 2.3(1) and 2.4(1), and to [2, Proposition 2.3, Corollary 2.4, Theorem 2.11 and Proof of Theorem 2.1] for 2.3(2) and 2.4(2). Also see [5, Chapter 3].

**Proof of Proposition 2.2.** (1) Let $\alpha$ be any simple closed curve in $C$. Choose any $w \in \text{Int} \alpha$ and $\varepsilon > 0$ with $\varepsilon < d(\alpha, w)$. Then for each $\beta \in U \equiv N_\varepsilon(\alpha, w) \subset K(\alpha, \mathbb{C})$ we have $w \in \text{Int} \beta$, hence by Lemma 2.3(1) we can define $f_\beta = f(\beta, w)|_{C(1)} \in E(C(1), \mathbb{C})$. The function $\beta \mapsto f_\beta$ is continuous by Lemma 2.3(2). The map $F : U \to E(\alpha, \mathbb{C})$, $F(\beta) = f_\beta(f_\alpha)^{-1}$, satisfies the required conditions.

The statement (2) can be verified similarly by using Lemma 2.4. $\square$

### 2.3. Circles in a Möbius band

Suppose $\mathbb{M}$ is a Möbius band and $\alpha$ is a center circle in $\text{Int} \mathbb{M}$.

**Proposition 2.3.**

1. There exists an open neighborhood $U$ of $\alpha$ in $K(\alpha, \mathbb{M})$ and a map $F : U \to E(\alpha, \mathbb{M})$ such that $F(\beta)(\alpha) = \beta$ ($\beta \in U$) and $F(\alpha) = \text{id}_\alpha$. 
(2) For any distinguished point \(x_0\) in \(\alpha\), there exists an open neighborhood \(U\) of \((\alpha, x_0)\) in \(K((\alpha, x_0), \mathbb{M})\) and a map \(F : U \to E(\alpha, \mathbb{M})\) such that \(F(\beta, y_0)(\alpha, x_0) = (\beta, y_0)\) for any \((\beta, y_0) \in U\) and \(F(\alpha, x_0) = \text{id}_{\alpha}\).

**Proof.** (1) Let \(\varphi : A \to \mathbb{M}\) denote the double covering from an annulus \(A\) onto \(\mathbb{M}\). Since \(\alpha\) is a center circle of \(\text{Int}\, \mathbb{M}\), \(\tilde{\alpha} = \varphi^{-1}(\alpha)\) is a center circle of \(\text{Int}\, A\). Take \(\varepsilon > 0\) such that any \(\varepsilon\)-close maps to \(\mathbb{M}\) are homotopic. Then for each \(\beta \in N_\varepsilon(\alpha, \varepsilon) \subset K(\alpha, \mathbb{M})\), it follows that \(\tilde{\beta} = \varphi^{-1}(\beta)\) is a circle and \(\varphi|_{\tilde{\beta}} : \tilde{\beta} \to \beta\) is a double covering.

We show that the function \(U \equiv N_\varepsilon(\alpha, \varepsilon) \ni \beta \mapsto \tilde{\beta} \in \mathbb{K}(\tilde{\alpha}, A)\) is (uniformly) continuous. Let \(\nu > 0\) be given. Since \(\varphi\) is a finite covering, there exists a \(\mu > 0\) such that if \(K\) is a connected subset of \(A\) and \(\text{diam}\, \varphi(K) < \mu\), then \(\text{diam}\, K < \nu\). Take \(\lambda > 0\) such that \(\lambda\)-close maps to \(\mathbb{M}\) are \(\mu\)-homotopic in \(\mathbb{M}\). Suppose \(\beta, \gamma \in U\) and \(\rho(\beta, \gamma) < \lambda\). Then there exist a \(\lambda\)-homeomorphism \(f : \beta \to \gamma\) and a \(\mu\)-homotopy \(f_t : \beta \to \mathbb{M}\) from \(\text{id}_\beta\) to \(f\). We can lift \(f_t\) to a homotopy \(\tilde{f}_t : \tilde{\beta} \to A\) from \(\text{id}_{\tilde{\beta}}\) to a lift \(\tilde{f}\) of \(f\). By the choice of \(\mu\), \(\tilde{f}_t\) is a \(\nu\)-homotopy and \(\tilde{f} : \tilde{\beta} \cong \tilde{\gamma}\) is a \(\nu\)-homeomorphism so that \(\rho(\tilde{\beta}, \tilde{\gamma}) < \nu\) as required.

By Proposition 2.2(1) we have a neighborhood \(\mathcal{V}\) of \(\tilde{\alpha}\) in \(\mathbb{K}(\tilde{\alpha}, A)\) and a map \(\tilde{g} : \mathcal{V} \to E(C(1), A)\) such that \(\tilde{g}(\gamma)(C(1)) = \gamma\). We may assume that \(\tilde{\beta} \in \mathcal{V}\) for each \(\beta \in U\). For each \(\beta \in U\) let \(z_{\beta}\) denote the unique point of \(C(1) \setminus \{1\}\) such that \(\varphi(\tilde{g}(\tilde{\beta})(z_{\beta})) = \varphi(\tilde{g}(\tilde{\beta})(1))\).

The function \(U \ni \beta \mapsto z_{\beta} \in C(1)\) is continuous. To see this, let \(\beta \in U\) and \(\varepsilon > 0\) be given. Let \(K = C(1) \setminus N(z_{\beta}, \varepsilon)\). We can find open subsets \(U_1, V_1, V_2\) of \(A\) such that \(\tilde{g}(\tilde{\beta})(K) \subset U_1, \quad \tilde{g}(\tilde{\beta})(1) \in V_1, \quad \tilde{g}(\tilde{\beta})(z_{\beta}) \in V_2, \quad \varphi(V_1) = \varphi(V_2), \quad V_1 \cap V_2 = \emptyset\) and \(U \cap V_2 = \emptyset\). Since \(\gamma \mapsto \tilde{g}(\tilde{\gamma})\) is continuous, there exists a neighborhood \(\mathcal{V}\) of \(\beta\) in \(U\) such that \(\tilde{g}(\tilde{\gamma})(K) \subset U\) and \(\tilde{g}(\tilde{\gamma})(1) \in V_1\) for each \(\gamma \in \mathcal{V}\). For each \(\gamma \in \mathcal{V}\), the definition of \(z_{\gamma}\) implies that \(\tilde{g}(\tilde{\gamma})(z_{\gamma}) \in V_2\) and \(z_{\gamma} \notin K\), hence \(z_{\gamma} \in N(z_{\beta}, \varepsilon)\).
Represent as \( z_{\beta} = e^{i\theta_{\beta}} \) \((0 < \theta_{\beta} < 2\pi)\) and define a map \( g: \mathcal{U} \to \mathcal{E}(C(1), \mathbb{M}) \) by 
\[
g(\beta)(e^{2\pi i t}) = \varphi(g(\beta)(e^{i\theta_{\beta}})).
\]
Finally the desired map \( F \) can be defined by \( F(\beta) = g(\beta)g(\alpha)^{-1} \) (cf. Fig. 2).

The statement (2) can be verified by a similar argument. \( \square \)

3. Proof of Theorem 1.1

A finite graph \( G \) is a compact 1-dimensional polyhedron. Let \( R(G) \) denote the set of points of \( G \) which have a neighborhood homeomorphic to \( \mathbb{R} \), and set \( V(G) = G \setminus R(G) \). Each point of \( V(G) \) is called a vertex of \( G \) and the closure of each component of \( R(G) \) in \( G \) is called an edge of \( G \). Therefore, an edge \( e \) is an arc or a simple closed curve: in the former case the end points of \( e \) are vertices and in the latter case \( e \) contains at most one vertex. By \( E(G) \) we denote the set of edges of \( G \). Note that \( V(G) \) and \( E(G) \) are topological invariants of \( G \).

**Proof of Theorem 1.1.** If \( X \subset Z \) satisfies the condition in the conclusion of Theorem 1.1 (replacing \( M \) by \( Z \)) and \( X \subset Y \subset Z \), then \( X \subset Y \) also satisfies the same condition. Thus, attaching the open collar along \( \partial M \), we may assume that \( \partial M = \emptyset \). Fix a metric \( d \) in \( M \).

(i) First we consider the case where \( X \) is a graph. Let \( V(X) = \{v_1(X), \ldots, v_l(X)\} \) and \( E(X) = \{e_1(X), \ldots, e_m(X)\} \). If we choose \( \delta > 0 \) to be sufficiently small, then for each \( Y \in N_{\delta}(X, \delta) \) we can write uniquely as \( V(Y) = \{v_1(Y), \ldots, v_l(Y)\} \) and \( E(Y) = \{e_1(Y), \ldots, e_m(Y)\} \) so that \( V(Y) \cap N_{\delta}(v_1(X), \delta) = \{v_1(Y)\} \) \((1 \leq i \leq \ell)\) and \( E(Y) \cap N_{\delta}(e_j(X), \delta) = \{e_j(Y)\} \) \((1 \leq j \leq m)\). Note that if \( h: Z \to Y \) is a \( \delta \)-homeomorphism then \( v_1(Y) = h(v_1(X)) \) and \( e_j(Y) = h(e_j(X)) \). Thus we have the maps \( N_{\delta}(X, \delta) \ni Y \mapsto v_1(Y) \in M, N_{\delta}(X, \delta) \ni Y \mapsto e_j(Y) \in K(e_j(X), M) \).

For each edge \( e_j(X), 1 \leq j \leq m \), we apply Propositions 2.1–2.3:

(i) If \( e_j(X) \) is an arc, then there exists \( 0 < \delta_j < \delta \) and a map \( f_j: N_{\delta_j}(e_j(X), \delta_j) \to \mathcal{E}(e_j(X), M) \) as in Propositions 2.1. We may assume that \( \text{Im} f_j \subset \text{Int}(d_{e_j(X)}(X)) \) (in the sup-metric). Note that, if \( e_j(X) \) has the end vertices \( v_1(X) \) and \( v_k(X) \), then for each \( Y \in N_{\delta_j}(X, \delta) \), \( e_j(Y) \) has the end vertices \( v_i(Y) \) and \( v_k(Y) \), and if \( e_j(Y) \in N_{\delta_j}(e_j(X), \delta_j) \) then \( f_j : e_j(Y) : e_j(X) \to e_j(Y) \) is a \( \delta \)-homeomorphism, hence it maps \( v_1(X) \) onto \( v_i(Y) \) and \( v_k(X) \) onto \( v_k(Y) \).

(ii) If \( e_j(X) \) is a circle, then it has an annulus or Möbius band neighborhood in \( M \).

(a) If \( e_j(X) \) does not contain any vertex of \( X \), then there exists \( 0 < \delta_j < \delta \) and a map \( f_j: N_{\delta_j}(e_j(X), \delta_j) \to \mathcal{E}(e_j(X), M) \) as in Propositions 2.2(1) or 2.3(1).

(b) If \( e_j(X) \) is a circle which contains a (unique) vertex \( v_k(X) \) of \( X \), then there exists \( 0 < \delta_j < \delta \) and a map \( f_j: N_{\delta_j}((e_j(X), v_k(X)), \delta_j) \to \mathcal{E}(e_j(X), M) \) as in Propositions 2.2(2) or 2.3(2).

Take \( \epsilon > 0 \) with \( \epsilon < \delta, \delta_j \) \((1 \leq j \leq m)\) and for each \( Y \in \mathcal{U} = N_{\delta}(X, \epsilon) \) define a homeomorphism \( F(Y): X \to Y \) by \( F(Y)(v_i(X)) = v_i(Y) \) \((1 \leq i \leq \ell)\). \( F(Y)|_{e_j(X)} = f_j(e_j(Y)) \) in the case (i), (iii) and \( F(Y)|_{e_j(X)} = f_j(e_j(Y), v_k(Y)) \) in the case (ii).

(2) When \( \dim X = 2 \), consider a 1-dimensional subpolyhedron \( X_1 = \{x \in X \mid x \) does not have any neighborhood homeomorphic to \( \mathbb{R}^2 \}, \) the non-2-manifold set of \( X \). Note that
for $Y, Y' \in \mathcal{K}(X, M)$ any homeomorphism $h : Y \cong Y'$ maps $Y_1$ onto $Y'_1$, thus $\rho(Y_1, Y'_1) \leq \rho(Y, Y')$, and the function $\mathcal{K}(X, M) \to \mathcal{K}(X_1, M) : Y \mapsto Y_1$ is continuous.

By [6, Theorem 1.1] there exists a neighborhood $V$ of $\text{Id}_X$ in $\mathcal{E}(X_1, M)$ and a map $\phi : V \to \mathcal{H}(M)$ such that $\phi(g)|X_1 = g$ and $\phi(\text{Id}_{X_1}) = \text{Id}_M$. By the case (1) there exists $\varepsilon > 0$ such that $\phi(Y) = Y$ for each $Y \in N_\varepsilon(X_1, \varepsilon)$ and $f(X_1) = \text{Id}_{X_1}$. We may assume that $\text{Im} f \subset V$. Therefore, if $Y \in N_\varepsilon(X_1, \varepsilon)$ then $Y \in N_\varepsilon(X_1, \varepsilon)$ and we obtain $\phi_Y \equiv \phi(f(Y)) \in \mathcal{H}(M)$.

Next we note that the 2-manifold $X \setminus X_1$ is a finite union of connected components $U_i (i = 1, \ldots, n)$ since it is a finite union of open simplices. Choose a point $x_i \in U_i$ ($i = 1, \ldots, n$). Since each $U_i$ is open in the 2-manifold $M$, there exists a $\mu > 0$ such that $x_i \in g(U_i)$ if $1 \leq i \leq n$ and $g : U_i \to M$ is an embedding which is $\mu$-close to $\text{Id}_{U_i}$. Since $Y \mapsto \phi_Y$ is continuous and $\phi_X = \text{Id}_M$, if we choose $\varepsilon > 0$ to be sufficiently small, then $\varepsilon < \mu$ and $\phi_Y|X$ is $\mu$-close to $\text{Id}_X$ for each $Y \in N_\varepsilon(X, \varepsilon)$.

Moreover there exists an $\varepsilon$-homeomorphism $h : X \to Y$. Then $Y \setminus Y_1 = h(X \setminus X_1) = \bigcup_i h(U_i)$ and $x_i \in h(U_i)$. Note that $\phi_Y(U_i)$’s and $h(U_i)$’s are connected components of $M \setminus Y_1$ since they are connected and clopen in $M \setminus Y_1$. Thus $h(U_i) = \phi_Y(U_i)$ ($i = 1, \ldots, n$) and $\phi_Y(X) = Y$.

It suffices to define $F(Y) = \phi_Y|X$. This completes the proof. $\blacksquare$

References