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# Inverse eigenvalue problems of tridiagonal symmetric matrices and tridiagonal bisymmetric matrices<sup>☆</sup>

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## Abstract

The problem of generating a matrix  $A$  with specified eigenpairs, where  $A$  is a tridiagonal symmetric matrix, is presented. A general expression of such a matrix is provided, and the set of such matrices is denoted by  $S_E$ . Moreover, the corresponding least-squares problem under spectral constraint is considered when the set  $S_E$  is empty, and the corresponding solution set is denoted by  $S_L$ . The best approximation problem associated with  $S_E(S_L)$  is discussed, that is: to find the nearest matrix  $\hat{A}$  in  $S_E(S_L)$  to a given matrix. The existence and uniqueness of the best approximation are proved and the expression of this nearest matrix is provided. At the same time, we also discuss similar problems when  $A$  is a tridiagonal bisymmetric matrix.

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**Keywords:** Inverse eigenvalue problem; Tridiagonal symmetric matrices; Tridiagonal bisymmetric matrices; Kronecker product; Moore–Penrose generalized inverse

## 1. Introduction

The notation used in this paper can be summarized as follows: denote the set of all  $m \times n$  real matrices by  $R^{m \times n}$ , and the set of all  $n \times n$  real symmetric matrices by  $SR^{n \times n}$ . For  $A \in R^{m \times n}$ ,  $A^T$  and  $A^+$  denote the transpose and the Moore–Penrose generalized inverse of  $A$ , respectively. The identity matrix of order  $n$  is denoted by  $I_n$ , and let  $S_n = (e_n, e_{n-1}, \dots, e_1)$ , where  $e_i$  is the  $i$ th column of  $I_n$ . It is easy to see that  $S_n^T = S_n^{-1} = S_n$ , that is to say, the matrix  $S_n$  is a symmetric orthogonal matrix. We define the inner product:  $\langle A, B \rangle = \text{tr}(B^T A)$  for all  $A, B \in R^{m \times n}$ ; then  $R^{m \times n}$  is a Hilbert inner product space and the norm of a matrix generated by this inner product is the Frobenius norm  $\| \cdot \|$ .  $\|x\|_2$  represents the 2-norm of a vector  $x$ . For  $A = (a_{ij}) \in R^{m \times n}$ ,  $B = (b_{ij}) \in R^{p \times q}$ , the symbol  $A \otimes B = (a_{ij} B) \in R^{mp \times nq}$  stands for the Kronecker product of  $A$  and  $B$ .

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**Definition 1.** An  $n \times n$  matrix  $A$  is called a tridiagonal matrix if

$$A = \begin{pmatrix} a_1 & c_1 & & & \\ b_1 & a_2 & c_2 & & \\ & \ddots & \ddots & \ddots & \\ & & b_{n-2} & a_{n-1} & c_{n-1} \\ & & & b_{n-1} & a_n \end{pmatrix}. \quad (1)$$

If  $b_i = c_i$  ( $i = 1, 2, \dots, n-1$ ),  $A$  is a tridiagonal symmetric matrix. Denote by  $R_3^{n \times n}$  and  $SR_3^{n \times n}$  the set of all  $n \times n$  tridiagonal matrices and tridiagonal symmetric matrices, respectively. If  $b_i = c_i > 0$ , then the tridiagonal symmetric matrix  $A$  is a Jacobi matrix.

**Definition 2.**  $A \in R_3^{n \times n}$  is called a tridiagonal bisymmetric matrix if  $A^T = A$  and  $(S_n A)^T = S_n A$ . Denote by  $BSR_3^{n \times n}$  the set of all  $n \times n$  tridiagonal bisymmetric matrices.

**Definition 3.**  $A \in R^{n \times n}$  is called a skew-symmetric matrix if  $(S_n A)^T = S_n A$ , and  $B \in R^{n \times n}$  is called a skew-anti-symmetric matrix if  $(S_n B)^T = -S_n B$ . Thus,

$$(A, B) = \text{tr}(B^T A) = \text{tr}[(S_n B)^T (S_n A)] = (S_n A, S_n B) = 0,$$

that is, skew-symmetric and skew-anti-symmetric matrices are mutually orthogonal.

The inverse problem of constructing the tridiagonal symmetric matrix and Jacobi matrix from spectral data has been investigated by Hochstadt [1], De Boor and Golub [2]. However, we should point out that the eigenvectors provide also very useful data in control theory [3,4], vibration theory [5,6], and structure design [7]. This kind of problem is called the inverse eigenvalue problem under spectral restriction [8], and can be described as follows:

**Problem 1.** Given a set  $\mathcal{L} \subseteq R^{n \times n}$ , and the eigenpairs  $X = (x_1, x_2, \dots, x_m) \in R^{n \times m}$ ,  $\Lambda = \text{diag}(\lambda_1, \lambda_2, \dots, \lambda_m) \in R^{m \times m}$  ( $1 \leq m < n$ ), find  $A \in \mathcal{L}$  such that

$$AX = X\Lambda.$$

The prototype of this problem initially arose in the design of Hopfield neural networks [9], and many important results on the discussions of the inverse eigenvalue problem associated with several kinds of different sets  $\mathcal{L}$  have been obtained in [10–18] by using the SVD (Singular Value Decomposition) and Moore–Penrose generalized inverse. They provided some solvability conditions for the problem and derived an expression for the general solution. However, the eigenvalues and eigenvectors data is frequently derived from scientific experiments, and the solution set of **Problem 1** may be empty. Hence, we need to study the corresponding least-squares problem, which can be described as follows:

**Problem 2.** Given a set  $\mathcal{L} \subseteq R^{n \times n}$ , and the eigenpairs  $X = (x_1, x_2, \dots, x_m) \in R^{n \times m}$ ,  $\Lambda = \text{diag}(\lambda_1, \lambda_2, \dots, \lambda_m) \in R^{m \times m}$  ( $1 \leq m < n$ ), find  $A \in \mathcal{L}$  such that

$$\|AX - X\Lambda\| = \min_{B \in \mathcal{L}} \|BX - X\Lambda\|.$$

The least-squares problem under spectral restriction associated with several kinds of different sets  $\mathcal{L}$ , for instance, symmetric matrices, positive semi-definite matrices, bisymmetric nonnegative definite matrices and so on (see [19–21]), has been considered.

The best approximation problem occurs frequently in experimental design, see for instance [22]. A preliminary estimation  $A^*$  of the unknown matrix  $A$  can be obtained from experiments, but it may not satisfy the structural requirement (for example, tridiagonal symmetric matrices) and/or spectral requirement. The best estimation of  $A$  is the matrix  $\hat{A}$  in  $\mathcal{L}$  that satisfies both requirements of  $A$  and is the best approximation of  $A^*$  [8,23,24]. So the best approximation problem associated with **Problems 1** and **2** is described as follows:

**Problem 3.** Given  $A^* \in R^{n \times n}$ , find  $\hat{A} \in S_E(S_L)$  such that

$$\|\hat{A} - A^*\| = \min_{A \in S_E(S_L)} \|A - A^*\|,$$

where  $S_E$  and  $S_L$  denote the solution sets of [Problems 1](#) and [2](#), respectively.

In this paper, we will solve the three problems for two sets  $\mathcal{L}$  defined in [Definitions 1](#) and [2](#), i.e.,  $\mathcal{L} = SR_3^{n \times n}$  and  $\mathcal{L} = BSR_3^{n \times n}$ . The inverse eigenvalue problem of tridiagonal symmetric matrices is also called the best approximation problem of tridiagonal symmetric matrices under spectral restriction [11]. To facilitate discussion, we still denote by  $S_E$  and  $S_L$  the solution sets of [Problem 1](#) and [Problem 2](#) in [Problem 3](#) when  $\mathcal{L} = SR_3^{n \times n}$ , and denote by  $S'_E$  and  $S'_L$  the solution sets of [Problems 1](#) and [2](#) in [Problem 3](#) when  $\mathcal{L} = BSR_3^{n \times n}$ . Without loss of generality, we may assume the given matrix  $A^* \in SR_3^{n \times n}$  when  $\mathcal{L} = SR_3^{n \times n}$  and  $A^* \in BSR_3^{n \times n}$  when  $\mathcal{L} = BSR_3^{n \times n}$  in [Problem 3](#), respectively. In fact, for any  $A^* = (a_{ij}^*) \in R^{n \times n}$ , let

$$A_1^* = \begin{pmatrix} a_{11}^* & a_{12}^* & & & \\ a_{21}^* & a_{22}^* & a_{23}^* & & \\ & \ddots & \ddots & \ddots & \\ & & a_{(n-1)(n-2)}^* & a_{(n-1)(n-1)}^* & a_{(n-1)n}^* \\ & & & a_{n(n-1)}^* & a_{nn}^* \end{pmatrix} \in R_3^{n \times n}, \quad A_2^* = A^* - A_1^*.$$

Then  $A_1^*$  is the tridiagonal part of  $A^*$ . Thus, for any  $A \in SR_3^{n \times n}$ , we have

$$\|A - A^*\|^2 = \|A - A_1^* - A_2^*\|^2 = \|A - A_1^*\|^2 + \|A_2^*\|^2.$$

Due to the mutual orthogonality of symmetric matrices and anti-symmetric matrices in  $R^{n \times n}$ , we have

$$\|A - A_1^*\|^2 = \|A - B^* - C^*\|^2 = \|A - B^*\|^2 + \|C^*\|^2,$$

where the matrices  $B^*$  and  $C^*$  denote the symmetric part and the anti-symmetric part of the tridiagonal matrix  $A_1^*$ , respectively. Combining the above two equations, we have

$$\|A - A^*\| = \min \Leftrightarrow \|A - A_1^*\| = \min \Leftrightarrow \|A - B^*\| = \min.$$

Furthermore, since  $BSR_3^{n \times n} \subseteq SR_3^{n \times n}$ ,  $\forall D \in BSR_3^{n \times n}$ , we still have

$$\begin{aligned} \|D - A^*\|^2 &= \|D - A_1^* - A_2^*\|^2 = \|D - A_1^*\|^2 + \|A_2^*\|^2, \\ \|D - A_1^*\|^2 &= \|D - B^* - C^*\|^2 = \|D - B^*\|^2 + \|C^*\|^2. \end{aligned}$$

Let

$$B_1^* = \frac{1}{2}(B^* + S_n B^* S_n), \quad B_2^* = \frac{1}{2}(B^* - S_n B^* S_n).$$

Obviously,  $B_1^* \in BSR_3^{n \times n}$ ,  $(S_n B_2^*)^T = -S_n B_2^*$ . Due to the mutual orthogonality of skew-symmetric and skew-anti-symmetric matrices, we have

$$\|D - B^*\|^2 = \|D - B_1^* - B_2^*\|^2 = \|D - B_1^*\|^2 + \|B_2^*\|^2.$$

So we have

$$\|D - A^*\| = \min \Leftrightarrow \|D - A_1^*\| = \min \Leftrightarrow \|D - B^*\| = \min \Leftrightarrow \|D - B_1^*\| = \min.$$

The paper is organized as follows: in [Sections 2](#) and [3](#), we discuss the structure of the basis matrices [25] for  $SR_3^{n \times n}$  and  $BSR_3^{n \times n}$ , and provide the general expressions of these solutions of [Problems 1–3](#) by using the Moore–Penrose generalized inverse, respectively; in [Section 4](#), we propose two direct algorithms to compute the solutions of [Problem 3](#) and report our experimental results.

## 2. The solutions of Problems 1–3 for the case $\mathcal{L} = SR_3^{n \times n}$

First, we discuss the structure of the basis matrix for  $SR_3^{n \times n}$ . For  $A = (a_{ij})_{m \times n}$ , denote by  $\text{vec}(A)$  the following vector containing all the entries of matrix  $A$ :

$$\text{vec}(A) = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} \in R^{mn},$$

where  $a_j = (a_{1j}, a_{2j}, \dots, a_{mj})^T (j = 1, 2, \dots, n)$  denotes the  $j$ th column of matrix  $A$ . For  $A \in R_3^{n \times n}$  having the form (1), let

$$A_1 = (a_1, \sqrt{2}b_1), \quad A_2 = (a_2, \sqrt{2}b_2), \dots, A_{n-1} = (a_{n-1}, \sqrt{2}b_{n-1}), \quad A_n = a_n,$$

and denote by  $\text{vec}_{S_3}(A)$  the following vector:

$$\text{vec}_{S_3}(A) = (A_1, A_2, \dots, A_{n-1}, A_n)^T \in R^{2n-1}. \quad (2)$$

**Definition 4** ([25]). Denote by  $R^p$  the real vector space of finite dimension  $p > 0$ , and by  $\Gamma$  a given subspace of  $R^p$  with dimension  $s \leq p$ . Let  $d_1, d_2, \dots, d_s$  be a set of basis vectors for  $\Gamma$ , then the  $p \times s$  matrix

$$K = (d_1, d_2, \dots, d_s)$$

is called a basis matrix for  $\Gamma$ . Obviously, the basis matrix is not unique. To facilitate discussion, we may assume that the basis matrix  $K$  is standard column orthogonal, i.e.,  $K^T K = I_s$ .

**Lemma 1.** Suppose  $A \in R_3^{n \times n}$ . Then  $A \in SR_3^{n \times n}$  if and only if

$$\text{vec}(A) = K_{S_3} \text{vec}_{S_3}(A), \quad (3)$$

where  $\text{vec}_{S_3}(A)$  is defined by (2), and the basis matrix  $K_{S_3}$  of  $SR_3^{n \times n}$  is of the following form:

$$K_{S_3} = \begin{pmatrix} e_1 & \frac{1}{\sqrt{2}}e_2 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}}e_1 & e_2 & \frac{1}{\sqrt{2}}e_3 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{\sqrt{2}}e_2 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & e_{n-1} & \frac{1}{\sqrt{2}}e_n & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & \frac{1}{\sqrt{2}}e_{n-1} & e_n \end{pmatrix} \in R^{n^2 \times (2n-1)}. \quad (4)$$

**Proof.** The matrix  $A \in SR_3^{n \times n}$  can be expressed as

$$\begin{aligned} A &= \begin{pmatrix} a_1 & b_1 & & & \\ b_1 & a_2 & b_2 & & \\ & \ddots & \ddots & \ddots & \\ & & b_{n-2} & a_{n-1} & b_{n-1} \\ & & & b_{n-1} & a_n \end{pmatrix} \\ &= a_1(e_1, 0, \dots, 0, 0) + \sqrt{2}b_1\left(\frac{1}{\sqrt{2}}e_2, \frac{1}{\sqrt{2}}e_1, \dots, 0, 0\right) + a_2(0, e_2, \dots, 0, 0) + \sqrt{2}b_2\left(0, \frac{1}{\sqrt{2}}e_3, \dots, 0, 0\right) \end{aligned}$$

$$+ \cdots + a_{n-1}(0, 0, \dots, e_{n-1}, 0) + \sqrt{2}b_{n-1} \left( 0, 0, \dots, \frac{1}{\sqrt{2}}e_n, \frac{1}{\sqrt{2}}e_{n-1} \right) + a_n(0, 0, \dots, 0, e_n).$$

It then follows that

$$\begin{aligned} \text{vec}(A) &= \begin{pmatrix} e_1 & \frac{1}{\sqrt{2}}e_2 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}}e_1 & e_2 & \frac{1}{\sqrt{2}}e_3 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{\sqrt{2}}e_2 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & e_{n-1} & \frac{1}{\sqrt{2}}e_n & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & \frac{1}{\sqrt{2}}e_{n-1} & e_n \end{pmatrix} \begin{pmatrix} a_1 \\ \sqrt{2}b_1 \\ a_2 \\ \sqrt{2}b_2 \\ \vdots \\ \sqrt{2}b_{n-1} \\ a_n \end{pmatrix} \\ &= K_{S_3} \text{vec}_{S_3}(A). \end{aligned}$$

Conversely, if the matrix  $A \in R_3^{n \times n}$  satisfies (3), then it is easy to see that  $A \in SR_3^{n \times n}$ . The proof is completed.  $\square$

From Lemma 1, we immediately have the following conclusion.

**Corollary 1.** For the diagonal matrix  $\Lambda = \text{diag}(\lambda_1, \lambda_2, \dots, \lambda_m) \in R^{m \times m}$ , one gets

$$\text{vec}(\Lambda) = K_0 \text{vec}_{S_3}(\Lambda),$$

where

$$K_0 = \begin{pmatrix} e_1 & & & \\ & e_2 & & \\ & & \ddots & \\ & & & e_m \end{pmatrix} \in R^{m^2 \times m} \quad \text{and} \quad \text{vec}_{S_3}(\Lambda) = \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_m \end{pmatrix} \in R^m. \quad (5)$$

The following Lemmas 2–4 are well known results, see, for instance, Ben-Israel and Greville [26].

**Lemma 2.** Given  $A \in R^{m \times n}$  and  $b \in R^n$ , then the system of equations  $Ax = b$  has a solution  $x \in R^n$  if and only if  $AA^+b = b$ . In that case it has the general solution

$$x = A^+b + (I - A^+A)y, \quad (6)$$

where  $y \in R^n$  is an arbitrary vector.

**Lemma 3.** The least-squares solutions of the system of equations  $Ax = b$ , with  $A \in R^{m \times n}$  and  $b \in R^n$ , are of the form (6).

**Lemma 4.** For any matrices  $A$ ,  $B$  and  $C$  with suitable size, one gets

$$\text{vec}(ABC) = (C^T \otimes A) \text{vec}(B).$$

Based on the above discussions, we now study Problems 1–3 for the case  $\mathcal{L} = SR_3^{n \times n}$ .

**Theorem 1.** Suppose the matrices  $X$  and  $\Lambda$  are given in Problem 1,  $K_{S_3}$  and  $K_0$  are the basis matrices in Lemma 1 and Corollary 1. The vectors  $\text{vec}_{S_3}(A)$  and  $\text{vec}_{S_3}(\Lambda)$  are defined by (2) and (5), respectively. Let  $P_1 = (X^T \otimes I)K_{S_3}$  and  $P_2 = (I \otimes X)K_0$ . Then the solution set  $S_E$  of Problem 1 is nonempty if and only if

$$P_1 P_1^+ P_2 \text{vec}_{S_3}(\Lambda) = P_2 \text{vec}_{S_3}(\Lambda). \quad (7)$$

When condition (7) is satisfied,  $S_E$  can be expressed as

$$S_E = \{A | \text{vec}(A) = K_{S_3} P_1^+ P_2 \text{vec}_{S_3}(A) + K_{S_3} (I - P_1^+ P_1) z\}, \quad (8)$$

where the vector  $z \in R^{(2n-1)}$  is arbitrary.

**Proof.** If the solution set  $S_E$  is nonempty, then Problem 1 has a solution  $A \in SR_3^{n \times n}$ . From Lemma 4, we have

$$(X^T \otimes I) \text{vec}(A) = (I \otimes X) \text{vec}(A), \quad (9)$$

which is, in view of (2), Lemma 1 and Corollary 1, equivalent to

$$(X^T \otimes I) K_{S_3} \text{vec}_{S_3}(A) = (I \otimes X) K_0 \text{vec}_{S_3}(A). \quad (10)$$

It then follows from Lemma 2 that (7) holds, and the set  $S_E$  can be expressed by (8).

Conversely, if (7) holds, we have from Lemma 2 that Eq. (10) has a solution which possesses the explicit expression

$$\text{vec}_{S_3}(A) = P_1^+ P_2 \text{vec}_{S_3}(A) + (I - P_1^+ P_1) z, \quad \forall z \in R^{2n-1}, \quad (11)$$

which implies that  $\text{vec}(A) = K_{S_3} \text{vec}_{S_3}(A)$  is the solution of Eq. (9). From Lemma 1, we know that  $A \in SR_3^{n \times n}$  and the solution set  $S_E$  is nonempty. The proof is completed.  $\square$

Moreover, if  $\text{rank}(P_1) = 2n - 1$ , then the solution of Problem 1 is unique and the solution has the following form

$$\text{vec}(A) = K_{S_3} P_1^+ P_2 \text{vec}_{S_3}(A).$$

**Theorem 2.** If the notation and conditions are the same as in Theorem 1, then the solution set  $S_L$  of Problem 2 can be expressed as

$$S_L = \{A | \text{vec}(A) = K_{S_3} P_1^+ P_2 \text{vec}_{S_3}(A) + K_{S_3} (I - P_1^+ P_1) z\}, \quad (12)$$

where the vector  $z \in R^{(2n-1)}$  is arbitrary.

**Proof.** By Lemmas 1 and 4,

$$\begin{aligned} \|AX - XA\| &= \|\text{vec}(AX) - \text{vec}(XA)\|_2 \\ &= \|(X^T \otimes I) K_{S_3} \text{vec}_{S_3}(A) - (I \otimes X) K_0 \text{vec}_{S_3}(A)\|_2 \\ &= \|P_1 \text{vec}_{S_3}(A) - P_2 \text{vec}_{S_3}(A)\|_2. \end{aligned}$$

From Lemmas 1 and 3, it follows that (12) holds, and this proves the theorem.  $\square$

Next we investigate Problem 3 and assume that the solution set of Problem 1 is nonempty. It is easy to verify that  $S_E(S_L)$  is a closed convex set. Therefore there exists a unique solution of Problem 3. Then we have the following theorems for the solution to Problem 3 over  $S_E(S_L)$ .

**Theorem 3.** If the notation and conditions are the same as in Theorem 1, and  $A^* = (a_{ij}^*) \in SR_3^{n \times n}$ , then the unique solution  $\hat{A} \in S_E$  for Problem 3 can be expressed as

$$\text{vec}(\hat{A}) = K_{S_3} \text{vec}_{S_3}(A^*) + K_{S_3} P_1^+ (P_2 \text{vec}_{S_3}(A) - P_1 \text{vec}_{S_3}(A^*)). \quad (13)$$

**Proof.** Since the basis matrix  $K_{S_3}$  defined in (4) is standard column orthogonal, in view of the orthogonal invariance of the 2-norm, we have

$$\begin{aligned} \min_{A \in S_E} \|A - A^*\| &= \min_{A \in S_E} \|\text{vec}(A) - \text{vec}(A^*)\|_2 \\ &= \min_{A \in S_E} \|K_{S_3} \text{vec}_{S_3}(A) - K_{S_3} \text{vec}_{S_3}(A^*)\|_2 \\ &= \min_{A \in S_E} \|\text{vec}_{S_3}(A) - \text{vec}_{S_3}(A^*)\|_2. \end{aligned}$$

By substituting (11) into the above equation, we know that Problem 3 is equivalent to the following least-squares problem:

$$\min_{z \in R^{2n-1}} \|(I - P_1^+ P_1)z - (\text{vec}_{S_3}(A^*) - P_1^+ P_2 \text{vec}_{S_3}(\Lambda))\|_2. \quad (14)$$

From [Lemma 3](#), we know that the solution of the least-squares problem (14) can be expressed as

$$\hat{z} = (I - P_1^+ P_1)^+ (\text{vec}_{S_3}(A^*) - P_1^+ P_2 \text{vec}_{S_3}(\Lambda)) + (I - (I - P_1^+ P_1)^+ (I - P_1^+ P_1))y, \quad (15)$$

where the vector  $y \in R^{2n-1}$  is arbitrary. Since  $I - P_1^+ P_1$  is a projection matrix, i.e.,  $(I - P_1^+ P_1)(I - P_1^+ P_1) = (I - P_1^+ P_1)$ , it is easy to verify that

$$(I - P_1^+ P_1)(I - P_1^+ P_1)^+ = (I - P_1^+ P_1)^+ = I - P_1^+ P_1.$$

Hence, the unique solution  $\hat{A} \in S_E$  for **Problem 3** is given by

$$\begin{aligned} \text{vec}(\widehat{A}) &= K_{S_3} \text{vec}_{S_3}(\widehat{A}) \\ &= K_{S_3}(I - P_1^+ P_1) \widehat{z} + K_{S_3} P_1^+ P_2 \text{vec}_{S_3}(\Lambda) \\ &= K_{S_3} \text{vec}_{S_3}(A^*) + K_{S_3} P_1^+ (P_2 \text{vec}_{S_3}(\Lambda) - P_1 \text{vec}_{S_3}(A^*)), \end{aligned}$$

and this proves the assertion.  $\square$

Similar to the proof of [Theorem 3](#), we have the following conclusion.

**Theorem 4.** *If the notation and conditions are the same as in Theorem 1, and  $A^* = (a_{ij}^*) \in SR_3^{n \times n}$ , then the unique solution  $\hat{A} \in S_L$  for Problem 3 can be expressed as*

$$\text{vec}(\hat{A}) = K_{S_3} \text{vec}_{S_3}(A^*) + K_{S_3} P_1^+ (P_2 \text{vec}_{S_3}(\Lambda) - P_1 \text{vec}_{S_3}(A^*)). \quad (16)$$

In [Theorems 3 and 4](#), if  $\text{rank}(I - P_1^+ P_1) = 2n - 1$ , then  $P_1 = 0$ , and we can get  $\hat{A} = A^*$ . If  $\text{rank}(P_1) = 2n - 1$ , then  $S_E(S_L) = \{A | \text{vec}(A) = K_{S_3} P_1^+ P_2 \text{vec}_{S_3}(\Lambda)\}$ , and we can get  $\text{vec}(\hat{A}) = K_{S_3} P_1^+ P_2 \text{vec}_{S_3}(\Lambda)$ .

### 3. The solutions of Problems 1–3 for the case $\mathcal{L} = BSR_3^{n \times n}$

At first, we discuss the structure of  $BSR_3^{n \times n}$ . From [Definition 2](#), it is easy to see that

**Lemma 5.** (1) When  $n = 2k$ ,  $A$  is called an  $n \times n$  tridiagonal bisymmetric matrix if

$$A = \begin{pmatrix} a_1 & b_1 & & & & & & & & \\ b_1 & a_2 & b_2 & & & & & & & \\ & \ddots & \ddots & \ddots & & & & & & \\ & & b_{k-1} & a_k & b_k & & & & & \\ & & & b_k & a_k & b_{k-1} & & & & \\ & & & b_{k-1} & a_{k-1} & b_{k-2} & & & & \\ & & & & \ddots & \ddots & \ddots & & & \\ & & & & & b_2 & a_2 & b_1 & & \\ & & & & & & b_1 & a_1 & & \end{pmatrix}. \quad (17)$$

(2) When  $n = 2k + 1$ ,  $A$  is called an  $n \times n$  tridiagonal bisymmetric matrix if

$$A = \begin{pmatrix} a_1 & b_1 & & & & & & & \\ b_1 & a_2 & b_2 & & & & & & \\ & \ddots & \ddots & \ddots & & & & & \\ & & b_{k-1} & a_k & b_k & & & & \\ & & & b_k & a_{k+1} & b_k & & & \\ & & & & b_k & a_k & b_{k-1} & & \\ & & & & & \ddots & \ddots & \ddots & \\ & & & & & & b_2 & a_2 & b_1 \\ & & & & & & & b_1 & a_1 \end{pmatrix}. \quad (18)$$

For  $A \in BSR_3^{n \times n}$  having the form (17) or (18), let

$$A_1 = (\sqrt{2}a_1, 2b_1), \quad A_2 = (\sqrt{2}a_2, 2b_2), \dots, A_k = (\sqrt{2}a_k, 2b_k), \quad A_{k+1} = a_{k+1},$$

and denote by  $\text{vec}_{B_3}(A)$  the following vectors:

$$\begin{aligned} \text{vec}_{B_3}(A) &= (A_1, A_2, \dots, A_{k-1}, A_k)^T \in R^{2k}, \quad \text{when } n = 2k, \\ \text{vec}_{B_3}(A) &= (A_1, A_2, \dots, A_k, A_{k+1})^T \in R^{2k+1}, \quad \text{when } n = 2k + 1. \end{aligned} \quad (19)$$

**Lemma 6.** Suppose  $A \in R_3^{n \times n}$ . Then  $A \in BSR_3^{n \times n}$  if and only if

$$\text{vec}(A) = K_{B_3} \text{vec}_{B_3}(A), \quad (20)$$

where  $\text{vec}_{B_3}(A)$  is defined by (19), and the basis matrix  $K_{B_3} \in R^{n^2 \times n}$  is of the following form:

(1) when  $n = 2k$ ,

$$K_{B_3} = \begin{pmatrix} \frac{1}{\sqrt{2}}e_1 & \frac{1}{2}e_2 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2}e_1 & \frac{1}{\sqrt{2}}e_2 & \frac{1}{2}e_3 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2}e_2 & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & \frac{1}{\sqrt{2}}e_{k-1} & \frac{1}{2}e_k & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & \frac{1}{2}e_{k-1} & \frac{1}{\sqrt{2}}e_k & \frac{1}{\sqrt{2}}e_{k+1} \\ 0 & 0 & 0 & 0 & \cdots & 0 & \frac{1}{2}e_{k+2} & \frac{1}{\sqrt{2}}e_{k+1} & \frac{1}{\sqrt{2}}e_k \\ 0 & 0 & 0 & 0 & \cdots & \frac{1}{\sqrt{2}}e_{k+2} & \frac{1}{2}e_{k+1} & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \frac{1}{2}e_{2k-1} & \cdots & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2}e_{2k} & \frac{1}{\sqrt{2}}e_{2k-1} & \frac{1}{2}e_{2k-2} & \cdots & 0 & 0 & 0 & 0 \\ \frac{1}{\sqrt{2}}e_{2k} & \frac{1}{2}e_{2k-1} & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \end{pmatrix}, \quad (21)$$



(2) when  $n = 2k + 1$ ,

$$K_{B_3} = \begin{pmatrix} \frac{1}{\sqrt{2}}e_1 & \frac{1}{2}e_2 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2}e_1 & \frac{1}{\sqrt{2}}e_2 & \frac{1}{2}e_3 & \cdots & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2}e_2 & \cdots & 0 & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & \frac{1}{\sqrt{2}}e_{k-1} & \frac{1}{2}e_k & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & \frac{1}{2}e_{k-1} & \frac{1}{\sqrt{2}}e_k & \frac{1}{2}e_{k+1} & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \frac{1}{2}e_k + \frac{1}{2}e_{k+2} & e_{k+1} \\ 0 & 0 & 0 & 0 & \cdots & 0 & \frac{1}{2}e_{k+3} & \frac{1}{\sqrt{2}}e_{k+2} & \frac{1}{2}e_{k+1} & 0 \\ 0 & 0 & 0 & 0 & \cdots & \frac{1}{\sqrt{2}}e_{k+3} & \frac{1}{2}e_{k+2} & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \frac{1}{2}e_{2k} & \cdots & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2}e_{2k+1} & \frac{1}{\sqrt{2}}e_{2k} & \frac{1}{2}e_{2k-1} & \cdots & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{\sqrt{2}}e_{2k+1} & \frac{1}{2}e_{2k} & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \quad (22)$$

**Proof.** We consider only the case when  $n = 2k + 1$ , with a similar argument applicable to the other case. For the matrix  $A \in BSR_3^{n \times n}$ , from (18), we have

$$A = a_1(e_1, 0, \dots, 0, e_n) + b_1(e_2, e_1, \dots, e_n, e_{n-1}) + \cdots + a_k(0, 0, \dots, e_k, 0, e_{k+2}, \dots, 0, 0) \\ + b_k(0, 0, \dots, e_{k+1}, e_k + e_{k+2}, e_{k+1}, \dots, 0) + a_{k+1}(0, 0, \dots, 0, e_{k+1}, 0, \dots, 0).$$

It then follows that  $\text{vec}(A) = K_{B_3} \text{vec}_{B_3}(A)$ .

Conversely, if  $\forall A \in R_3^{n \times n}$  and  $\text{vec}(A) = K_{B_3} \text{vec}_{B_3}(A)$ , it is easy to see that  $A \in BSR_3^{n \times n}$ . The proof is completed.  $\square$

Similar to the proofs of Theorems 1–4, we have the following conclusions by Lemmas 5 and 6.

**Theorem 5.** Suppose the matrices  $X$  and  $\Lambda$  are given in Problem 1,  $K_{B_3}$  and  $K_0$  are the basis matrices in Lemma 6 and Corollary 1. The vectors  $\text{vec}_{B_3}(A)$  and  $\text{vec}_{S_3}(\Lambda)$  are defined by (19) and (5), respectively. Let  $Q_1 = (X^T \otimes I)K_{B_3}$  and  $Q_2 = (I \otimes X)K_0$ . Then the solution set  $S'_E$  of Problem 1 is nonempty if and only if

$$Q_1 Q_1^+ Q_2 \text{vec}_{S_3}(\Lambda) = Q_2 \text{vec}_{S_3}(\Lambda). \quad (23)$$

When condition (23) is satisfied,  $S'_E$  can be expressed as

$$S'_E = \{A | \text{vec}(A) = K_{B_3} Q_1^+ Q_2 \text{vec}_{S_3}(\Lambda) + K_{B_3} (I - Q_1^+ Q_1) z\}, \quad (24)$$

where the vector  $z \in R^n$  is arbitrary.

Moreover, if  $\text{rank}(Q_1) = n$ , then the solution of Problem 1 is unique and the solution has the following form

$$\text{vec}(A) = K_{B_3} Q_1^+ Q_2 \text{vec}_{S_3}(\Lambda).$$

**Theorem 6.** If the notation and conditions are the same as in Theorem 5, then the solution set  $S'_L$  of Problem 2 can be expressed as

$$S'_L = \{A | \text{vec}(A) = K_{B_3} Q_1^+ Q_2 \text{vec}_{S_3}(A) + K_{B_3} (I - Q_1^+ Q_1) z\}, \quad (25)$$

where the vector  $z \in R^n$  is arbitrary.

Assume that the solution set of Problem 1 is nonempty. It is easy to verify that  $S'_E(S'_L)$  is a closed convex set. Therefore there exists a unique solution of Problem 3.

**Theorem 7.** If the notation and conditions are the same as in Theorem 5, and  $A^* = (a_{ij}^*) \in BSR_3^{n \times n}$ , then the unique solution  $\hat{A} \in S'_E$  for Problem 3 can be expressed as

$$\text{vec}(\hat{A}) = K_{B_3} \text{vec}_{B_3}(A^*) + K_{B_3} Q_1^+ (Q_2 \text{vec}_{S_3}(A) - Q_1 \text{vec}_{B_3}(A^*)). \quad (26)$$

**Theorem 8.** If the notation and conditions are the same as in Theorem 5, and  $A^* = (a_{ij}^*) \in BSR_3^{n \times n}$ , then the unique solution  $\hat{A} \in S'_L$  for Problem 3 can be expressed as

$$\text{vec}(\hat{A}) = K_{B_3} \text{vec}_{B_3}(A^*) + K_{B_3} Q_1^+ (Q_2 \text{vec}_{S_3}(A) - Q_1 \text{vec}_{B_3}(A^*)). \quad (27)$$

In Theorems 7 and 8, if  $\text{rank}(I - Q_1^+ Q_1) = n$ , then  $Q_1 = 0$ , and we can get  $\hat{A} = A^*$ . If  $\text{rank}(Q_1) = n$ , then  $S'_E(S'_L) = \{A | \text{vec}(A) = K_{B_3} Q_1^+ Q_2 \text{vec}_{S_3}(A)\}$ , and we can get  $\text{vec}(\hat{A}) = K_{B_3} Q_1^+ Q_2 \text{vec}_{S_3}(A)$ .

#### 4. Numerical solution for Problem 3

Based on the discussions in Sections 2 and 3, we can get  $\hat{A}$  according to (13) or (16) when  $\mathcal{L} = SR_3^{n \times n}$  and get  $\hat{A}$  according to (26) or (27) when  $\mathcal{L} = BSR_3^{n \times n}$ . It is easy to see that if (7) holds, then  $\hat{A} \in S_E$ , otherwise  $\hat{A} \in S_L$ . If (23) holds, then  $\hat{A} \in S'_E$ , otherwise  $\hat{A} \in S'_L$ . Now we establish the following direct algorithms for finding the solution  $\hat{A}$  of Problem 3.

When  $\mathcal{L} = SR_3^{n \times n}$ , we have the following algorithm for solving Problem 3.

- Algorithm 1.** (1) Input matrices  $X$ ,  $A$  and  $A^* (A^* \in SR_3^{n \times n})$ .  
 (2) Compute  $K_{S_3}$  and  $K_0$  according to (4) and (5), respectively.  
 (3) Compute  $P_1 = (X^T \otimes I) K_{S_3}$ ,  $P_2 = (I \otimes X) K_0$  and  $P_1^+$ .  
 (4) If  $\text{rank}(I - P_1^+ P_1) = 2n - 1$ , we have  $\hat{A} = A^*$ , stop. Otherwise go to (5).  
 (5) Compute  $\text{vec}(\hat{A})$  according to (16), and we can get  $\hat{A}$ .

When  $\mathcal{L} = BSR_3^{n \times n}$ , we have the following algorithm for solving Problem 3.

- Algorithm 2.** (1) Input matrices  $X$ ,  $A$  and  $A^* (A^* \in BSR_3^{n \times n})$ .  
 (2) Compute  $K_{B_3}$  and  $K_0$  according to (5), (21) and (22), respectively.  
 (3) Compute  $Q_1 = (X^T \otimes I) K_{B_3}$ ,  $Q_2 = (I \otimes X) K_0$  and  $Q_1^+$ .  
 (4) If  $\text{rank}(I - Q_1^+ Q_1) = n$ , we have  $\hat{A} = A^*$ , stop. Otherwise go to (5).  
 (5) Compute  $\text{vec}(\hat{A})$  according to (27), and we can get  $\hat{A}$ .

**Example 1.** Based on the discussions in Section 2 and Algorithm 1, if  $\text{rank}(P_1) = 2n - 1$  and (7) hold, then  $\hat{A}$  can be computed by (13) or  $\text{vec}(\hat{A}) = K_{S_3} P_1^+ P_2 \text{vec}_{S_3}(A)$ , which are denoted by  $\hat{A}_1$  or  $\hat{A}_2$ , respectively. Let  $G = \text{hadamard}(n)$ ,  $A = 1/2(G_1 + G_1^T)$ , where  $G_1$  is the tridiagonal part of  $G$ . Denote  $n = n_0 \times 10^l$  by the scientific notation. Let  $c = 1:n$ ,  $r = n:(2n - 1)$ ,  $G^* = n_0 \times \text{hankel}(c, r)$ ,  $A^*$  is the tridiagonal part of  $G^*$ . It is easy to see that  $A, A^* \in SR_3^{n \times n}$ . Assume that the eigenpair of  $A$  is  $\lambda_i, x_i$  ( $i = 1, 2, \dots, n$ ), we take  $X = (x_1, x_2, \dots, x_n)$ ,  $A = (\lambda_1, \lambda_2, \dots, \lambda_n)$ , obviously, (7) holds. We have tested Algorithm 1 using the software Matlab 6.5, and our numerical results are listed in Table 1.

Table 1  
Numerical results for Example 1

$n$	$\text{rank}(P_1)$	$\ A\ $	$\ A^*\ $	$\ \hat{A}_1 - A\ $	$\ \hat{A}_2 - A\ $	$\ \hat{A}_1 - A^*\ $	$\ \hat{A}_2 - A^*\ $
20	39	7.6158	174.4133	1.0668e-013	7.6420e-015	174.6253	174.6253
40	79	10.8625	999.2792	1.1329e-012	1.7904e-014	999.0185	999.0185
64	127	13.7840	3.2512e+003	8.5456e-012	3.4967e-014	3.2512e+003	3.2512e+003
128	255	19.5448	1.8464e+003	1.8713e-011	1.9132e-013	1.8465e+003	1.8465e+003
160	319	21.8632	3.2281e+003	6.1364e-012	6.3475e-014	3.2281e+003	3.2281e+003

**Example 2.** Let

$$X = \begin{pmatrix} -0.7508 & 0 & -0.1437 & 0 & 0.6037 & 0 \\ 0.6298 & 0 & 0.0653 & 0 & 0.6873 & 0 \\ 0.1166 & 0 & -0.2760 & 0 & 0.3738 & 0 \\ -0.1243 & 0 & 0.9481 & 0 & 0.1530 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -0.1243 & 0 & 0.9481 & 0 & 0.1530 \\ 0 & 0.1166 & 0 & -0.2760 & 0 & 0.3738 \\ 0 & 0.6298 & 0 & 0.0653 & 0 & 0.6873 \\ 0 & -0.7578 & 0 & -0.1437 & 0 & 0.6037 \end{pmatrix},$$

$$A = \begin{pmatrix} 3.7339 & 0 & 0 & 0 & 0 & 0 \\ 0 & 3.7339 & 0 & 0 & 0 & 0 \\ 0 & 0 & 3.8604 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3.8604 & 0 & 0 \\ 0 & 0 & 0 & 0 & 4.3955 & 0 \\ 0 & 0 & 0 & 0 & 0 & 4.3955 \end{pmatrix},$$

$$A^* = \begin{pmatrix} 0.5556 & 0.4340 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0.4340 & 0.3125 & 0.2753 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.2753 & 0.2381 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.2383 & 0.2042 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.2042 & 5 & 0.2042 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.2042 & 0.2383 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0.2383 & 0.2753 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0.2753 & 0.3125 & 0.4340 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.4340 & 0.5556 \end{pmatrix},$$

where  $\text{rank}(X) = 6$ ,  $\text{rank}(Q_1) = 8$ , and (23) holds. By Algorithm 2, we have a unique  $\hat{A} \in S'_E$  as follows:

$$\hat{A} = \begin{pmatrix} 4.0131 & 0.3359 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0.3359 & 4.1574 & -0.1047 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -0.1047 & 4.5079 & 0.1957 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.1957 & 3.9174 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 5 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 3.9174 & 0.1957 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.1957 & 4.5079 & -0.1047 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -0.1047 & 4.1574 & 0.3359 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.3359 & 4.0131 \end{pmatrix}.$$

We can compute  $\|\hat{A}X - XA\| = 4.7748 \times 10^{-5}$ ,  $\|\hat{A} - A^*\| = 10.8602$ .

From the above two examples, we can clearly see that Algorithms 1 and 2 are feasible for solving Problem 3.

## 5. Comments

In this paper, we have applied the Kronecker product, Moore–Penrose generalized inverse, and the basis matrix for  $\mathcal{L}$  to investigate Problems 1–3 for  $\mathcal{L}$ , when  $\mathcal{L} = SR_3^{n \times n}$  or  $\mathcal{L} = BSR_3^{n \times n}$ . Moreover, a direct method for computing the best approximation has been established. The algorithms for finding  $\hat{A} \in S_E(S_L)$  or  $\hat{A} \in S'_E(S'_L)$  have been described in detail, and two examples have been used to show their feasibility.

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