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# On the minimal eccentric connectivity indices of graphs☆

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ABSTRACT

Let $G$ be a simple connected graph. The eccentric connectivity index $\xi^c(G)$ of $G$ is defined as $\xi^c(G) = \sum_{v\in V(G)} d(v)ec_G(v)$, where the eccentricity $ec_G(v)$ is the largest distance between $v$ and any other vertex $u$ of $G$. In this paper, we obtain lower bounds on $\xi^c(G)$ in terms of the number of edges among $n$-vertex connected graphs with given diameter. Over connected graphs on $n$ vertices with $m$ edges and diameter at least $s$, and connected graphs on $n$ vertices with diameter at least $d$, we provide lower bounds on $\xi^c(G)$ and characterize the extremal graphs, respectively.



## 1. Introduction

Let $G$ be a simple connected graph with vertex set $V(G)$ and edge set $E(G)$. For vertices $u, v \in V(G)$, the distance $d(u, v)$ is defined as the length of the shortest path connecting $u$ and $v$ in $G$. A $u - v$ path of length $d(u, v)$ is called a $u - v$ geodesic. The eccentricity $ec_G(u)$ is the number $\max_{v\in V(G)} d(u, v)$, that is, $ec_G(u)$ is the largest distance between $u$ and all other vertices of $G$. The radius $\mathrm{rad}(G)$ and diameter $\mathrm{diam}(G)$ are, respectively, the minimum and maximum eccentricity among vertices of $G$. A vertex $v$ is a central vertex if $ec_G(v) = \mathrm{rad}(G)$, an edge $uv$ is called a central edge if $u$ and $v$ are all central vertices.

The eccentric connectivity index of $G$, denoted by $\xi^c(G)$, is defined as [7]

$$\xi^c(G) = \sum_{u\in V(G)} ec_G(u)d(u).$$

Obviously,

$$\xi^c(G) = \sum_{uv\in E(G)} \omega_G(uv),$$

where $\omega_G(uv)$, the weight of edge $uv$ in $G$, equals $ec_G(u) + ec_G(v)$. Let $\omega_G(H) = \sum_{e\in E(H)} \omega_G(e)$ for any subgraph $H$ of $G$. Then $\omega_G(G) = \xi^c(G)$.

As a novel, distance-cum-adjacency topological descriptor, the eccentric connectivity index provides excellent correlations with regard to both physical and biological properties of chemical substances. The simplicity amalgamated with high correlating ability of this index can be easily exploited in QSPR/QSAR studies (see [2,3,7]). So it is of interest to study the mathematical properties of this invariant.

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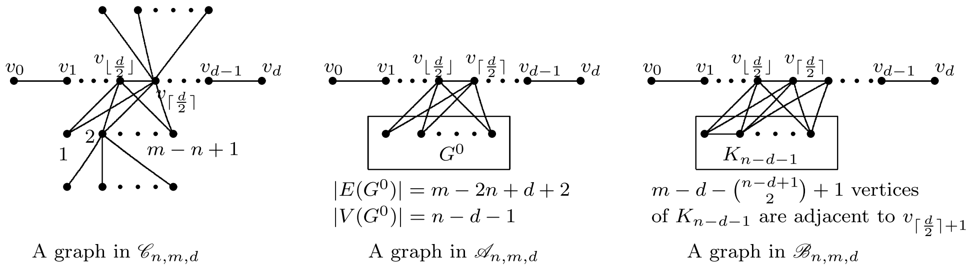


**Fig. 1.** A graph in $\mathscr{C}_{n,m,d}$, $\mathscr{A}_{n,m,d}$ and $\mathscr{B}_{n,m,d}$, respectively.

Recently, Ashrafi et al. [1] gave exact formulas for the eccentric connectivity index of $TUC_4C_8(s)$ nanotube and $TC_4C_8(s)$ nanotorus. Saheli and Ashrafi [6] gave an expression for the eccentric connectivity index of an armchair nanotube. Ilić and Gutman [4] considered the eccentric connectivity index of chemical trees. Morgan et al. [5] provided asymptotically upper bound on the eccentric connectivity index in terms of order and diameter, Zhou and Du [8] and Morgan et al. [5] provided independently lower bound on the eccentric connectivity index in terms of order, and provided upper and lower bounds for the eccentric connectivity index of the trees of given order and diameter. In this paper, we consider the minimal eccentric connectivity index of the simple connected graphs.

## 2. The lower bounds for $\xi^c(G)$ of connected graphs with odd diameter

If $G$ is a connected graph with $n$ vertices, $m$ edges and diameter $d$, then it is easily seen that $n-1 \le m \le 2n-d-2+\binom{n-d}{2}$. Suppose first that $d$ is an odd integer not less than 3. Let $P_{d+1} = v_0v_1 \cdots v_{\lfloor d/2 \rfloor} v_{\lceil d/2 \rceil} \cdots v_d$ be a path of length $d$.

If $2n-d-1 \le m \le n-1+\binom{n-d}{2}$, then let $\mathscr{A}_{n,m,d}$ be the set of graphs with $n$ vertices, $m$ edges, and diameter $d$, obtained from $P_{d+1}$ by joining each vertex of $G^0$ to $v_{\lfloor d/2 \rfloor}$ and $v_{\lceil d/2 \rceil}$, where $G^0$ is a graph with $n-d-1$ vertices and $m-2n+d+2$ edges (see Fig. 1).

**Theorem 1.** *Let $G$ be a connected graph with $n$ vertices, $m$ edges and diameter $d$, where $d$ is odd and $d \ge 3$. If $2n-d-1 \le m \le n-1+\binom{n-d}{2}$, then*

$$\xi^c(G) \ge \frac{(d-1)^2}{2} + m(d+1) \tag{1}$$

*with equality if and only if $G \in \mathscr{A}_{n,m,d}$.*

**Proof.** Let $P_{d+1} = v_0v_1 \cdots v_d$ be a diametrical path of $G$. Note that $d(v_i, v_j) = |i-j|$,

$$ec_G(v_i) \ge \max\{d(v_i, v_0), d(v_i, v_d)\}, \tag{2}$$

and

$$\omega_G(P_{d+1}) \ge \omega_{P_{d+1}}(P_{d+1}) = \left\lfloor \frac{3d^2+1}{2} \right\rfloor. \tag{3}$$

Moreover, equality holds in (3) if and only if the equality holds in (2) for all $i$ with $0 \le i \le d$.

Since $d$ is odd and $\mathrm{rad}(G) \le d \le 2\mathrm{rad}(G)$, we have $\mathrm{rad}(G) \ge \frac{d+1}{2}$ and $ec_G(u) \ge \mathrm{rad}(G) \ge \frac{d+1}{2}$ for any vertex $u$ of $G$. It follows that $\omega_G(e) \ge 2(\frac{d+1}{2}) = d+1$ for any edge $e \in E(G)$, and

$$\xi^c(G) \ge \omega_{P_{d+1}}(P_{d+1}) + (m-d)(d+1) = \frac{(d-1)^2}{2} + m(d+1) \tag{4}$$

with equality if and only if

(i) $\mathrm{rad}(G) = \frac{d+1}{2}$,

(ii) equality holds in (3), and

(iii) the weight of any edge $e$ of $G$ not in $P_{d+1}$ must be $d+1$, that is, the two terminal vertices of $e$ must be central vertices of $G$. Thus, any vertex $u \in V(G) \setminus V(P_{d+1})$ must be a central vertex of $G$.

From the inequality (2), we know that any vertex of $P_{d+1}$ except $v_{\lfloor d/2 \rfloor}$ and $v_{\lceil d/2 \rceil}$ must be not central vertex. It follows that $d(v_0) = d(v_d) = 1$, $d(v_i) = 2$ for $1 \le i < \lfloor d/2 \rfloor$ and $\lceil d/2 \rceil < i \le d-1$.

Since $G$ is connected, for any $u \in V(G) \setminus V(P_{d+1})$, there exists a path connecting $u$ and the diametral path $P_{d+1}$, and then its last edge must be incident to $v_{\lfloor d/2 \rfloor}$ or $v_{\lceil d/2 \rceil}$. Without loss of generality, suppose that $u$ is not incident with $v_{\lfloor d/2 \rfloor}$, then

$ec_G(u) \geq d(u, v_0) > \frac{d}{2} + 1$, contradicting to the fact of (iii) that $u$ is a central vertex of $G$. Thus $u$ must be incident to both $v_{\lfloor d/2 \rfloor}$ and $v_{\lceil d/2 \rceil}$, and consequently $(V(G) \setminus V(P_{d+1})) \cup \{v_{\lfloor d/2 \rfloor}, v_{\lceil d/2 \rceil}\}$ is the set of the central vertices in $G$.

Since $2n - d - 1 \leq m \leq n - 1 + \binom{n-d}{2}$, $G$ must be a graph in $\mathscr{A}_{n,m,d}$. Conversely, if $G \in \mathscr{A}_{n,m,d}$, then the result is clear from the analysis above. □

If $n + \binom{n-d}{2} \leq m \leq 2n - d - 2 + \binom{n-d}{2}$, where $m \geq n$, then let $\mathscr{B}_{n,m,d}$ be the set of graphs with $n$ vertices, $m$ edges, and diameter $d$, obtained from $P_{d+1}$ by joining each vertex of a complete graph $K_{n-d-1}$ to $v_{\lfloor d/2 \rfloor}$ and $v_{\lceil d/2 \rceil}$, and joining $m - d - \binom{n-d+1}{2} + 1$ vertices of $K_{n-d-1}$ to $v_{\lfloor d/2 \rfloor - 1}$ or $v_{\lceil d/2 \rceil + 1}$, but not to both (see Fig. 1).

**Theorem 2.** *Let $G$ be a connected graph with $n$ vertices, $m$ edges and diameter $d$, where $d$ is odd and $d \geq 3$. If $n + \binom{n-d}{2} \leq m \leq 2n - d - 2 + \binom{n-d}{2}$, then*

$$\xi^c(G) \geq m(d+2) - \frac{3d - 3 + n(n - 2d + 1)}{2} \tag{5}$$

*with equality if and only if $G \in \mathscr{B}_{n,m,d}$.*

**Proof.** If $\mathrm{rad}(G) > \frac{d+1}{2}$, then for any edge $e \in E(G)$, $\omega_G(e) \geq 2(\frac{d+1}{2} + 1) = d + 3$, and

$$\begin{aligned}\xi^c(G) &\geq \omega_{P_{d+1}}(P_{d+1}) + (m - d)(d + 3) \\ &= \frac{3d^2 + 1}{2} + m(d+2) + m - d^2 - 3d \\ &> m(d+2) - \frac{3d - 3 + n(n - 2d + 1)}{2}.\end{aligned}$$

Now suppose that $\mathrm{rad}(G) = \frac{d+1}{2}$. Let $P_{d+1} = v_0 v_1 \cdots v_d$ be a diametral path of $G$. Then there are at most $n - d - 1 + 2$ central vertices and $\binom{n-d+1}{2}$ central edges including the edge $v_{\lfloor d/2 \rfloor} v_{\lceil d/2 \rceil}$. Since the weight of non-central edge is at least $d + 2$, we have

$$\begin{aligned}\xi^c(G) &\geq \omega_{P_{d+1}}(P_{d+1}) + \left[\binom{n-d+1}{2} - 1\right](d+1) + \left[m - d - \binom{n-d+1}{2} + 1\right](d+2) \\ &= m(d+2) - \frac{3d - 3 + n(n - 2d + 1)}{2}\end{aligned}$$

with equality if and only if there are just $n - d + 1$ central vertices including $v_{\lfloor d/2 \rfloor}$ and $v_{\lceil d/2 \rceil}$, and the subgraph induced by the central vertices is a clique $K_{n-d+1}$; And for any $e = uv \in E(G) \setminus \{E(K_{n-d+1}) \cup E(P_{d+1})\}$, there is a central vertex in $\{u, v\}$, say $u$, and $ec_G(v)$ must be equal to $\frac{d+1}{2} + 1$. Then $u$ is a vertex of $K_{n-d+1}$ and $v$ is a vertex of $P_{d+1}$. Since $ec_G(v_{\lfloor d/2 \rfloor - 1}) = ec_G(v_{\lceil d/2 \rceil + 1}) = \frac{d+1}{2} + 1$ and $ec_G(v_i) > \frac{d+1}{2} + 1$ for $i \neq \lfloor d/2 \rfloor - 1, \lfloor d/2 \rfloor, \lceil d/2 \rceil, \lceil d/2 \rceil + 1$, $v$ must be a vertex of $\{v_{\lfloor d/2 \rfloor - 1}, v_{\lceil d/2 \rceil + 1}\}$. Thus $G \in \mathscr{B}_{n,m,d}$. □

If $n - 1 \leq m \leq 2n - d - 2$, then let $\mathscr{C}_{n,m,d}$ be the set of graphs with $n$ vertices, $m$ edges, and diameter $d$, obtained by attaching $2n - m - d - 2$ pendent vertices to $G'$ such that each is adjacent to some central vertex of $G'$, where $G'$ is the graph obtained from $P_{d+1}$ by joining $m - n + 1$ isolated vertices to both $v_{\lfloor d/2 \rfloor}$ and $v_{\lceil d/2 \rceil}$ (see Fig. 1).

**Theorem 3.** *Let $G$ be a connected graph with $n$ vertices, $m$ edges and diameter $d$, where $d$ is odd and $d \geq 3$. If $n - 1 \leq m \leq 2n - d - 2$, then*

$$\xi^c(G) \geq \frac{d^2 - 4d - 3}{2} + md + 2n \tag{6}$$

*with equality if and only if $G \in \mathscr{C}_{n,m,d}$.*

**Proof.** We prove the result by induction on $n$. It is clear for $n = d + 1$. Suppose that $n \geq d + 2$. Let $P_{d+1} = v_0 v_1 \cdots v_d$ be a diametral path of $G$. If $n = d + 2$, then $d + 1 \leq m \leq d + 2$. If $m = d + 1$, then $G$ is a tree obtained from $P_{d+1}$ by joining an isolated vertex $u$ to $v_i$ for some $1 \leq i \leq d - 1$, and thus

$$\xi^c(G) = \omega_G(P_{d+1}) + \omega_G(uv_i) \geq \frac{3d^2 + 1}{2} + (d + 2) = \frac{d^2 - 4d - 3}{2} + md + 2n$$

with equality if and only if $i = \lfloor d/2 \rfloor$ or $\lceil d/2 \rceil$, i.e., $G \in \mathscr{C}_{d+2,d+1,d}$. If $m = d+2$, then $G$ is an unicyclic graph obtained from $P_{d+1}$ by joining an isolated vertex to both $v_i$ and $v_j$, where $0 \le i < j \le d$ and $j - i \le 2$, and thus

$$\xi^c(G) = \omega_G(P_{d+1}) + \omega_G(uv_i) + \omega_G(uv_j) \ge \frac{3d^2+1}{2} + 2(d+1) = \frac{d^2-4d-3}{2} + md + 2n$$

with equality if and only if $\{i,j\} = \{\lfloor d/2 \rfloor, \lceil d/2 \rceil\}$, i.e., $G \in \mathscr{C}_{d+2,d+2,d}$.

Now suppose that $n \ge d+3$ and the result is true for smaller values. If $\mathrm{rad}(G) \ge \frac{d+1}{2} + 1$, then

$$\xi^c(G) = \omega_G(P_{d+1}) + (m-d)(d+3) \ge \frac{3d^2+1}{2} + (m-d)(d+3) > \frac{d^2-4d-3}{2} + md + 2n.$$

In the following we suppose that $\mathrm{rad}(G) = \frac{d+1}{2}$.

Let $v_i \in V(P_{d+1})$ $(0 \le i \le \lfloor \frac{d}{2} \rfloor)$ be the first vertex adjacent to a vertex not in $P_{d+1}$, that is, if $d(v_0) \ge 2$, then $v_i = v_0$. Else, $d(v_0) = 1$, $d(v_1) = \cdots = d(v_{i-1}) = 2$, and $d(v_i) \ge 3$ for $1 \le i \le \lfloor \frac{d}{2} \rfloor$.

*Case* 1. $G - v_i$ has a component, say $F$, such that $F$ contains no vertex of $P_{d+1}$. Let $uv_i$ be an edge of $G$ for $u \in V(F)$, we obtain a new graph $G'$ by identifying $u$ and $v_i$ and deleting loop and parallel edges. Then $G'$ is a connected graph of diameter $d$ with $n(G') = n-1$, $m(G') = m-x$, where $x \ge 1$. Clearly, for any $w \in V(G')$, $e \in E(G')$, $ec_{G'}(w) \le ec_G(w)$, thus $\omega_{G'}(e) \le \omega_G(e)$.

If $x = 1$, then $u$ and $v_i$ have no common neighbor vertex, and $ec_G(u) \ge \frac{d+1}{2} + 1$. If $m - 1 = 2(n-1) - d - 1$, that is, $m = 2n - d - 2$, by Lemma 1 we have

$$\begin{aligned}\xi^c(G) &= \sum_{e \in E(G')} \omega_G(e) + \omega_G(uv_i) \ge \xi^c(G') + d + 2 \\ &\ge \frac{(d-1)^2}{2} + (m-1)(d+1) + d + 2 \\ &= \frac{d^2-4d-1}{2} + md + 2n \\ &> \frac{d^2-4d-3}{2} + md + 2n.\end{aligned}$$

If $m - 1 \le 2(n-1) - d - 2$, by the induction hypothesis we have

$$\begin{aligned}\xi^c(G) &= \sum_{e \in E(G')} \omega_G(e) + \omega_G(uv_i) \ge \xi^c(G') + d + 2 \\ &\ge \frac{d^2-4d-3}{2} + (m-1)d + 2(n-1) + d + 2 \\ &\ge \frac{d^2-4d-3}{2} + md + 2n\end{aligned}$$

with equality if and only if $ec_G(u) = \frac{d+1}{2} + 1$, $i = \lfloor \frac{d}{2} \rfloor$, $G' \in \mathscr{C}_{n-1,m-1,d}$, and for any $w \in V(G')$, $ec_{G'}(w) = ec_G(w)$. Thus $uv_i$ is a pendent vertex. Note that $i = \lfloor \frac{d}{2} \rfloor$ and $G' \in \mathscr{C}_{n-1,m-1,d}$, then $G \in \mathscr{C}_{n,m,d}$.

If $x \ge 2$, then $m - x \le 2(n-1) - d - 2$, and for $ec_G(u) \ge \max\{d(u,v_0), d(u,v_d)\}$, we have

$$\begin{aligned}\xi^c(G) &= \sum_{e \in E(G')} \omega_G(e) + \sum_{e \notin E(G')} \omega_G(e) \ge \xi^c(G') + x(d+2) \\ &\ge \frac{d^2-4d-3}{2} + (m-x)d + 2(n-1) + x(d+2) \\ &> \frac{d^2-4d-3}{2} + md + 2n.\end{aligned}$$

*Case* 2. Any component of $G - v_i$ contains some vertex of the fixed diametral path $P_{d+1}$. Let $u \notin V(P_{d+1})$ be a vertex adjacent to $v_i$. We identify $u$ and $v_{i+1}$, delete loop and parallel edges, and obtain the new graph $H$ with $n(H) = n-1$ and $m(H) = m - y (y \ge 1)$. Clearly, for any vertex $s$, $ec_H(s) \le ec_G(s)$, and we have the following.

**Claim 1.** *The fixed diametral path $P_{d+1}$ of $G$ is also a diametral path of $H$, that is, $d_H(v_s, v_t) = t - s$ for all $0 \le s \le t \le d$. Otherwise, there exist two integers $k$, $l (k > l)$ such that $d_H(v_l, v_k) < k - l$, then $l = i+1$, and $d_G(v_i, v_k) < k - i$, a contradiction.*

*Subcase* 2.1.1. Let $i \le \lfloor \frac{d}{2} \rfloor - 1$ and $y \ge 2$. Then $ec_G(v_i) \ge \frac{d+1}{2} + 1$ and $m - y \le 2(n-1) - d - 2$. By the induction hypothesis we have

$$\xi^c(G) \ge \sum_{e \in E(H)} \omega_G(e) + \sum_{e \notin E(H)} \omega_G(e) \ge \xi^c(H) + (y-1)(d+1) + d + 2$$

$$\geq \frac{d^2-4d-3}{2}+(m-y)d+2(n-1)+yd+y+1$$
$$> \frac{d^2-4d-3}{2}+md+2n.$$

*Subcase* 2.1.2. Suppose that $i \leq \lfloor \frac{d}{2} \rfloor - 1$ and $y = 1$. If $m-1 = 2(n-1)-d-1$, by a similar proof of Case 1, we can prove the result. If $m-1 \leq 2(n-1)-d-2$, by the induction hypothesis we have

$$\xi^c(G) \geq \sum_{e \in E(H)} \omega_G(e) + \omega_G(uv_i) \geq \xi^c(H)+d+2$$
$$\geq \frac{d^2-4d-3}{2}+(m-1)d+2(n-1)+d+2$$
$$\geq \frac{d^2-4d-3}{2}+md+2n.$$

The equalities maybe hold only if $H \in \mathscr{C}_{n-1,m-1,d}$ and $\omega_G(uv_i) = d+2$. But if $H \in \mathscr{C}_{n-1,m-y,d}$, then $ec_G(u) \geq d_G(u, v_d) \geq \lfloor \frac{d+1}{2} \rfloor + 1$. Since $y = 1$, we can show that $\omega_G(uv_i) \geq d+3$. Thus the inequality is strict.

*Case* 2.2.1. Let $i = \lfloor \frac{d}{2} \rfloor$ and $y \geq 2$. Then $m - y \leq 2(n-1)-d-2$. By induction hypothesis we have

$$\xi^c(G) \geq \sum_{e \in E(H)} \omega_G(e) + \sum_{e \notin E(H)} \omega_G(e) \geq \xi^c(H)+y(d+1)$$
$$\geq \frac{d^2-4d-3}{2}+(m-y)d+2(n-1)+yd+y$$
$$\geq \frac{d^2-4d-3}{2}+md+2n$$

with equality if and only if $y = 2$, $ec_G(u) = \frac{d+1}{2}$, $H \in \mathscr{C}_{n-1,m-2,d}$, and for any $w \in V(H)$, $ec_H(w) = ec_G(w)$. Thus $uv_i$ is a pendent vertex, $i = \lfloor \frac{d}{2} \rfloor$ and $G \in \mathscr{C}_{n,m,d}$.

*Case* 2.2.2. By symmetry, we can suppose that $d(v_1) = \cdots = d(v_{\lfloor \frac{d}{2} \rfloor - 1}) = d(v_{\lceil \frac{d}{2} \rceil + 1}) = \cdots = d(v_{d-1}) = 2$, $d(v_0) = d(v_d) = 1$, $d(v_{\lfloor \frac{d}{2} \rfloor}), d(v_{\lceil \frac{d}{2} \rceil}) \geq 3$, and $y = 1$. Thus $i = \lfloor \frac{d}{2} \rfloor$. Clearly, $u$ is not adjacent to $v_{\lceil \frac{d}{2} \rceil}$ for $y = 1$. Furthermore, suppose that $v_{\lfloor \frac{d}{2} \rfloor}$ and $v_{\lceil \frac{d}{2} \rceil}$ have no common neighbor, otherwise, it returns to the Case 2.2.1 by choosing a new vertex to substitute for $u$. From above analysis, we know that $ec_G(u) \geq \lfloor \frac{d}{2} \rfloor + 1$, $H \notin \mathscr{C}_{n-1,m-1,d}$ for $m-1 \leq 2(n-1)-d-2$, and $H \notin \mathscr{A}_{n-1,m-1,d}$ for $m-1 = 2(n-1)-d-1$. By the induction hypothesis and Lemma 1 we have that if $m-1 \leq 2(n-1)-d-2$, then

$$\xi^c(G) \geq \sum_{e \in E(H)} \omega(e) + \omega_G(uv_i) \geq \xi^c(H)+d+2$$
$$> \frac{d^2-4d-3}{2}+(m-1)d+2(n-1)+d+2$$
$$= \frac{d^2-4d-3}{2}+md+2n,$$

and if $m-1 = 2(n-1)-d-1$,

$$\xi^c(G) \geq \sum_{e \in E(H)} \omega(e) + \omega_G(uv_i) \geq \xi^c(H)+d+2$$
$$> \frac{3d^2+1}{2}+(m-1-d)(d+1)+d+2$$
$$= \frac{d^2-4d-2}{2}+md+2n$$
$$> \frac{d^2-4d-3}{2}+md+2n.$$

This completes the proof. □

## 3. The lower bounds for $\xi^c(G)$ of connected graphs with even diameter

Now suppose that $d$ is an even integer. Let $K_{2,a+1}$ denote the complete bipartite graph with bipartite sets $X$, $Y$ ($|X| = 2$, $|Y| = a + 1$).

If $d \geq 6$ and $a = \left\lceil \frac{-3+\sqrt{17+8(m-n)}}{2} \right\rceil$, then let $\mathscr{D}_{n,m,d}$ be the set of graphs obtained from $K_{2,a+1}$ by adding $m - n + 1 - a$ edges and $n - d - a - 1$ pendent vertices (each is adjacent to some vertex of $Y$) to $Y$, and attaching a pendent path of length $\frac{d}{2} - 1$ at each vertex of $X$; If $d = 4$, and $m \notin \{n, n + 2\}$, then let $\mathscr{D}_{n,m,4}$ be the set of graphs obtained from $K_{2,a+1}$ by adding $m - n + 1 - a$ edges to $Y$ such that there exists at least one vertex adjacent with the other vertices (we denote by $S$ the set of vertices which are adjacent with others in $Y$), adding $n - a - 5$ pendent vertices to $Y$ such that each is adjacent to some vertex of $S$, and attaching a pendent vertex at each vertex of $X$.

**Theorem 4.** *Let $G$ be a connected graph with $n$ vertices, $m$ edges, and diameter $d$, where $d$ is even and $d \geq 4$, and when $d = 4$, $m \neq n, n + 2$. Then*

$$\xi^c(G) \geq \frac{d^2 - 2d - 2}{2} + md + n + \left\lceil \frac{-3 + \sqrt{17 + 8(m - n)}}{2} \right\rceil \tag{7}$$

*with equality if and only if $G \in \mathscr{D}_{n,m,d}$.*

**Proof.** If $\mathrm{rad}(G) \geq \frac{d}{2} + 1$, since $m \geq n - 1$, we have

$$\begin{aligned}
\xi^c(G) > \omega_{P_{d+1}}(P_{d+1}) + (m - d)(d + 2) &> \frac{3d^2}{2} + (m - d)(d + 1) \\
&> \frac{3d^2}{2} + (n - d - 1 + a)(d + 1) + (m - n + 1 - a)d \\
&= \frac{d^2 - 2d - 2}{2} + md + n + \left\lceil \frac{-3 + \sqrt{17 + 8(m - n)}}{2} \right\rceil.
\end{aligned}$$

In the following we suppose that $\mathrm{rad}(G) = \frac{d}{2}$. Fix a diametral path $P_{d+1} = v_0 v_1 \cdots v_{\frac{d}{2}} \cdots v_d$.

**Claim 1.** *For any central vertex $u$, $ec_G(u) = \frac{d}{2} = d(u, v_0) = d(u, v_d)$. Otherwise,*

$$d(v_0, v_d) = d \leq d(u, v_0) + d(u, v_d) \leq d - 1.$$

*That is, any central vertex must be in a $v_0 - v_d$ geodesic. Thus for any two central vertices $u$ and $w$, the two edges incident to $u$ in the $v_0 - v_d$ geodesic passing $u$ are different with such edges incident to $w$. It follows that $x$ central vertices must cost at least $2x$ edges. Without loss of generality, suppose that $v_{\frac{d}{2}}$ is a central vertex for the case $\mathrm{rad}(G) = \frac{d}{2}$.*

Suppose that $G$ has $b$ central edges. Let $a = \left\lceil \frac{-3+\sqrt{17+8(m-n)}}{2} \right\rceil$.

**Claim 2.** *$b \leq m - n + 1 - a$. Let $H$ be the graph induced by the $b$ central edges. If $b \geq m - n + 2 - a$, then $m - n + 2 - a > \binom{a}{2}$, and $|V(H)| = c \geq a + 1$. That is, $G$ contains at least $a + 1$ central vertices. Delete the edges of $H$ and the $2(c - 1)$ edges, where two of the $2(c - 1)$ edges are incident to a common vertex of $H$ (except $v_{\frac{d}{2}}$ if $v_{\frac{d}{2}} \in V(H)$) in a $v_0 - v_d$ geodesic, then there are at most $c$ components. Thus $m - 2(c - 1) - b \geq \sum_{i=1}^{c}(n_i - 1) = n - c$, where $n_i$ is the number of vertices of the $n_i$-th component. Thus $a \geq m - n + 2 - b \geq c \geq a + 1$, a contradiction.*

Thus

$$\begin{aligned}
\xi^c(G) &\geq \omega_{P_{d+1}}(P_{d+1}) + (m - d - b)(d + 1) + bd = \frac{3d^2}{2} + m(d + 1) - d(d + 1) - b \\
&\geq \frac{d^2 - 2d - 2}{2} + md + n + \left\lceil \frac{-3 + \sqrt{17 + 8(m - n)}}{2} \right\rceil.
\end{aligned}$$

If equality holds, then $b = m - n + 1 - a$, and the weight of each non-central edge $uv$ that is not in $P_{d+1}$ must be $d + 1$, that is $\omega(uv) = \frac{d}{2} + (\frac{d}{2} + 1)$. Thus all central vertices are adjacent to $v_{\frac{d}{2}-1}$ and $v_{\frac{d}{2}+1}$. We can prove that $a + 1 = m - n + 2 - b \geq c$ with a similar proof of Claim 2. Considering the case $m - n + 1 - a > \binom{a-1}{2}$, it follows that $a + 1 \geq c \geq a$.

If $c = a$, then $n - d - 1 - (a - 1) > m - d - 2(a - 1) - b$, then $G$ is not connected, a contraction; if $c = a + 1$, then we have $m - d - 2a - b = n - d - 1 - a$, there are exact $a + 1$ central vertices including $v_{\frac{d}{2}}$. Let $G^*$ be the graph obtained from $P_{d+1}$ and graph $G^0$ of $a$ vertices and $b$ edges by joining each vertex of $G^0$ with $v_{\frac{d}{2}-1}$ and $v_{\frac{d}{2}+1}$. Since the weight of non-central edge is $d + 1$, $G$ can be viewed as obtained from $G^*$ by adding $n - d - a - 1$ pendent vertices, each is adjacent to a central vertex of $G^*$. Thus $G \in \mathscr{D}_{n,m,d}$. Clearly, the equality holds for $G \in \mathscr{D}_{n,m,d}$. Thus we complete the proof. □

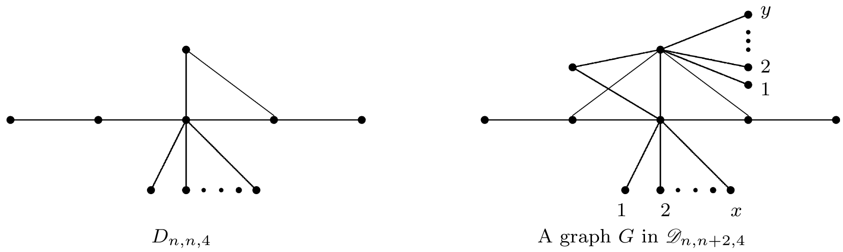


**Fig. 2.** The graph $D_{n,n,4}$ and a graph $G$ in $\mathscr{D}_{n,n+2,4}$.

We define $D_{n,n,4}$ as the graph in Fig. 2, and denote by $\mathscr{D}_{n,n+2,4}$ the set of graphs, each of which is the type as in Fig. 2 for a pair of nonnegative positive integers $x, y$ satisfying $x + y = n - 7$.

**Theorem 5.** *Let G be a connected graph with n vertices, m edges, and diameter 4. If* $m = n \geq 9$, *then*

$$\xi^c(G) \geq 5(n+1)$$

*with equality if and only if* $G \cong D_{n,n,4}$; *if* $m = n + 2$ *and* $n \geq 12$, *then*

$$\xi^c(G) \geq 5n + 13$$

*with equality if and only if and* $G \in \mathscr{D}_{n,n+2,4}$.

**Proof.** The result is clear for $\text{rad}(G) \geq 3$, we now suppose that $\text{rad}(G) = 2$.

If $m = n$, then $G$ contains a unique cycle, by Claim 1 of Theorem 4 we know that $G$ contains at most two central vertices. Suppose that $u, v$ are all central vertices of $G$, and $P_5 = v_0v_1v_2v_3v_4$ is a diametral path. Then one of the vertices $u, v$ is the vertex $v_2$, say $v = v_2$, and $u, v$ are in the unique cycle. Note that $uv_2 \notin E(G)$, otherwise, $ec(u) \geq \max\{d(u, v_0), d(u, v_4)\} = ec(v_2) + 1 = 3$. If $u, v_2$ have at most one common neighbor, then the length of cycle of $G$ at least 6, which is contradiction to the fact that $u, v_2$ are the central vertices of $G$. Thus $u, v_2$ have exact two common neighbors, say $w, r$, then $G$ can be viewed as obtained from $C_4 = uwv_2ru$ by adding some pendent vertices adjacent to $w$ and $r$, thus $\xi^c(G) = 7n - 8 > 5(n+1)$. If $G$ contains a unique central vertex, the result is obvious.

Similarly, if $m = n + 2$, we can verify that $G$ contains at most one central edge. If $G$ has no central edge, then $\omega_G(e) \geq 5$ for any edge $e \in E(G)$, $\omega_G(v_0v_1) = \omega_G(v_3v_4) = 7$, and $\xi(G) \geq 5n + 14 > 5n + 13$. We now suppose that $G$ contains exact one central edges, if there are three central vertices, by a similar proof as above it is easy to prove $\xi^c(G) > 5n + 13$. If $G$ contains exact one central edge and two central vertices, the result is immediate. □

If $d = 2$, then $n - 1 \leq m \leq \frac{n^2-n-2}{2}$. Let $c = \left\lfloor \frac{2n-1-\sqrt{4n^2-4n+1-8m}}{2} \right\rfloor$ and $\mathscr{E}_{n,m,2}$ be the set of graphs obtained from the complete graph $K_c$ by joining every vertex of $K_c$ to every vertex of $E$, where $E$ is a graph with $n - c$ vertices and $m - \binom{c}{2} - c(n-c)$ edges.

**Theorem 6.** *Let G be a connected graph with n vertices, m edges, and diameter 2. Then*

$$\xi^c(G) \geq 4m + \left\lfloor \frac{2n-1-\sqrt{4n^2-4n+1-8m}}{2} \right\rfloor (1-n) \tag{8}$$

*with equality if and only if* $G \in \mathscr{E}_{n,m,2}$.

**Proof.** If $\text{rad}(G) = \text{diam}(G) = 2$, then $\xi^c(G) = 4m$, and the result is obvious. If $\text{rad}(G) = 1$ and $u$ is a central vertex, then $d(u) = n-1$. if there are $x$ central vertices, then $G$ contains $\binom{x}{2}$ central edges, $x(n-x)$ edges of weight 3, and $m - \binom{x}{2} - x(n-x)$ edges of weight 4. Thus

$$\xi^c(G) = 2\binom{x}{2} + 3x(n-x) + 4\left(m - \binom{x}{2} - x(n-x)\right).$$

Note that $m \geq \binom{x}{2} + x(n-x)$, thus $x \leq \left\lfloor \frac{2n-1-\sqrt{4n^2-4n+1-8m}}{2} \right\rfloor = c$, and

$$\begin{aligned}\xi^c(G) &\geq 2\binom{c}{2} + 3c(n-c) + 4\left(m - \binom{c}{2} - c(n-c)\right)\\ &= 4m + c(1-n).\end{aligned}$$

Clearly, the equality holds if and only if $G \in \mathscr{E}_{n,m,2}$. □

**Corollary 1.** *Let $G$ be a connected graph with $n$ vertices and $m$ edges.*

(i) *If $m = \binom{n}{2}$, then $G \cong K_n$, and $\xi^c(K_n) = \binom{n}{2}$;*

(ii) *If $n-1 \le m < \binom{n}{2}$, then $\xi^c(G) \ge 4m + (1-n)\left\lfloor \frac{2n-1-\sqrt{4n^2-4n+1-8m}}{2} \right\rfloor$, and equality holds if and only if $G \in \mathscr{E}_{n,m,2}$.*

**Proof.** Let $G$ be a connected graph of $n$ vertices and $m$ edges. If $\mathrm{diam}(G) \ge 3$, then $\mathrm{rad}(G) \ge 2$, $\omega_G(e) \ge 4$ for any edge $e \in E(G)$. Since $\left\lfloor \frac{2n-1-\sqrt{4n^2-4n+1-8m}}{2} \right\rfloor \ge 1$, we have

$$\xi^c(G) \ge 4m > 4m + (1-n)\left\lfloor \frac{2n-1-\sqrt{4n^2-4n+1-8m}}{2} \right\rfloor.$$

If $\mathrm{diam}(G) = 2$, by Theorem 6, the result is immediately clear. □

## 4. The lower bounds for $\xi^c(G)$ of connected graphs

Set $d = 2k+1$ in the right of Inequality 1, 5, 6 and $d = 2k$ in the right of Inequality 7. We obtain that

$$\phi_1(k) = 2k^2 + (2m-2)k + m + 2n - 3, \quad \text{for } 1 \le k \le \left\lfloor \frac{2n-3-m}{2} \right\rfloor$$

$$\phi_2(k) = 2k^2 + 2mk + 2m \quad \text{for } \max\left\{1, \frac{2n-1-m}{2}\right\} \le k \le \left\lfloor \frac{2n-3-\sqrt{9+8(m-n)}}{4} \right\rfloor$$

$$\phi_3(k) = 3m + k(2m+2n-3) - \frac{n^2-n}{2} \quad \text{for } \left\lceil \frac{2n-3-\sqrt{1+8(m-n)}}{4} \right\rceil \le k \le \left\lfloor \frac{2n-1-\sqrt{17+8(m-n)}}{4} \right\rfloor$$

$$\phi_4(k) = 2k^2 + (2m-2)k + n - 1 + \left\lceil \frac{-3+\sqrt{17+8(m-n)}}{2} \right\rceil \quad \text{for } k \ge 2,$$

and let

$$\phi_5 = 4m + (1-n)\left\lfloor \frac{2n-1-\sqrt{4n^2-4n+1-8m}}{2} \right\rfloor.$$

Then we have the following.

**Lemma 1.** *Let $\phi_1(k), \phi_2(k), \phi_4(k)$ and $\phi_5$ are defined as above.*

(i) *Let $x, y, z$ be three positive integers satisfying that $1 \le x \le \left\lfloor \frac{2n-3-m}{2} \right\rfloor$, $\max\{1, \frac{2n-1-m}{2}\} \le y \le \left\lfloor \frac{2n-3-\sqrt{9+8(m-n)}}{4} \right\rfloor$ and $\left\lceil \frac{2n-3-\sqrt{1+8(m-n)}}{4} \right\rceil \le z \le \left\lfloor \frac{2n-1-\sqrt{17+8(m-n)}}{4} \right\rfloor$. Then*

$$\phi_1(x) < \phi_2(y) < \phi_3(z).$$

(ii) *Let $k$ be a positive integer. Then $\phi_5 < \phi_4(k)$ for $k \ge 2$, and*

$$\phi_4(k) < \phi_1(k) < \phi_4(k+1) \quad \textit{for } 1 \le k \le \left\lfloor \frac{2n-3-m}{2} \right\rfloor,$$

$$\phi_4(k) < \phi_2(k) < \phi_4(k+1) \quad \textit{for } \max\left\{1, \frac{2n-1-m}{2}\right\} \le k \le \left\lfloor \frac{2n-3-\sqrt{9+8(m-n)}}{4} \right\rfloor,$$

$$\phi_4(k) < \phi_3(k) < \phi_4(k+1) \quad \textit{for } \left\lceil \frac{2n-3-\sqrt{1+8(m-n)}}{4} \right\rceil \le k \le \left\lfloor \frac{2n-1-\sqrt{17+8(m-n)}}{4} \right\rfloor.$$

**Proof.** (i) Clearly, $\phi_1(k), \phi_2(k)$ and $\phi_3(k)$ are all monotonically increasing functions in $k$. Then

$$\phi_1(x) \le \phi_1\left(\frac{2n-3-m}{2}\right) = \phi_2\left(\frac{2n-3-m}{2}\right) < \phi_2\left(\frac{2n-1-m}{2}\right) \le \phi_2(y),$$

similarly, we have $\phi_2(y) \le \phi_2\left(\left\lfloor \frac{2n-3-\sqrt{9+8(m-n)}}{4} \right\rfloor\right) < \phi_3\left(\left\lfloor \frac{2n-3-\sqrt{9+8(m-n)}}{4} \right\rfloor + 1\right) \le \phi_3\left(\left\lceil \frac{2n-3-\sqrt{1+8(m-n)}}{4} \right\rceil\right) \le \phi_3(z)$. Then the result (i) follows.

(ii) Note that

$$\phi_1(k)-\phi_4(k)=m+n-2-\left\lceil\frac{-3+\sqrt{17+8(m-n)}}{2}\right\rceil>0,$$

$$\phi_1(k)-\phi_4(k+1)=-\left(4k+m-n+2+\left\lceil\frac{-3+\sqrt{17+8(m-n)}}{2}\right\rceil\right)<0.$$

It follows that $\phi_4(k)<\phi_1(k)<\phi_4(k+1)$ for $1\le k\le\left\lfloor\frac{2n-3-m}{2}\right\rfloor$. Similarly, we can verify that $\phi_4(k)<\phi_2(k)<\phi_4(k+1)$ for $\max\{1,\frac{2n-1-m}{2}\}\le k\le\left\lfloor\frac{2n-3-\sqrt{9+8(m-n)}}{4}\right\rfloor$ and $\phi_5<\phi_4(k)$.

Let $\omega(k)=3m+k(2m+2n-3)-\frac{n^2-n}{2}-2k^2-2mk+2k-n+1-\left\lfloor\frac{-3+\sqrt{17+8(m-n)}}{4}\right\rfloor$. Then

$$\begin{aligned}\omega(k)&=-2k^2+(2n-1)k+3m-\frac{n^2-n}{2}+1-\left\lfloor\frac{-3+\sqrt{17+8(m-n)}}{2}\right\rfloor\\&\ge-2k^2+(2n-1)k+3m-\frac{n^2-n}{2}+2-\frac{\sqrt{17+8(m-n)}}{2}.\end{aligned}$$

Since $\left\lceil\frac{2n-3-\sqrt{1+8(m-n)}}{4}\right\rceil\le k\le\left\lfloor\frac{2n-1-\sqrt{17+8(m-n)}}{4}\right\rfloor<\frac{2n-1}{4}$, $\omega(k)$ is a monotonically increasing function for $\left\lceil\frac{2n-3-\sqrt{1+8(m-n)}}{4}\right\rceil\le k\le\left\lfloor\frac{2n-1-\sqrt{17+8(m-n)}}{4}\right\rfloor$, and we have

$$\begin{aligned}\omega\left(\left\lceil\frac{2n-3-\sqrt{1+8(m-n)}}{4}\right\rceil\right)&\ge\omega\left(\frac{2n-3-\sqrt{1+8(m-n)}}{4}\right)\\&=2m+2+\frac{n-2}{4}-\frac{\sqrt{1+8(m-n)}+\sqrt{17+8(m-n)}}{2}\\&>0.\end{aligned}$$

Then $\phi_3(k)>\phi_4(k)$ for $\left\lceil\frac{2n-3-\sqrt{1+8(m-n)}}{4}\right\rceil\le k\le\left\lfloor\frac{2n-1-\sqrt{17+8(m-n)}}{4}\right\rfloor$. Similarly, we can prove $\phi_3(k)<\phi_4(k+1)$ for $\left\lceil\frac{2n-3-\sqrt{1+8(m-n)}}{4}\right\rceil\le k\le\left\lfloor\frac{2n-1-\sqrt{17+8(m-n)}}{4}\right\rfloor$. Thus we complete the proof. □

**Theorem 7.** *Let G be a connected graph with n vertices and m edges. For a positive integer s of not less than 2, if* $\mathrm{diam}(G)\ge s$, *then* $2\le s\le\frac{2n+1-\sqrt{17+8(m-n)}}{2}$, *and*

(i) *if s is odd, then*

$$\xi^c(G)\ge\begin{cases}\dfrac{s^2-4s-3}{2}+ms+2n & \text{if } 3\le s\le 2n-m-2,\text{ where } 2n\ge m+5\\ \dfrac{(s-1)^2}{2}+m(s+1) & \text{if } \max\{3,2n-m-1\}\le s\le\left\lfloor\dfrac{2n-1-\sqrt{9+8(m-n)}}{2}\right\rfloor\\ m(s+2)-\dfrac{3s-3+n(n-2s+1)}{2} & \text{if } \left\lceil\dfrac{2n-1-\sqrt{1+8(m-n)}}{2}\right\rceil\le s\le\left\lfloor\dfrac{2n+1-\sqrt{17+8(m-n)}}{2}\right\rfloor\end{cases}$$

*with equalities if and only if* $G\in\mathscr{C}_{n,m,s},\mathscr{A}_{n,m,s},\mathscr{B}_{n,m,s}$, *respectively;*

(ii) *if s is even, then*

$$\xi^c(G)\ge\begin{cases}4m+\left\lfloor\dfrac{2n-1-\sqrt{4n^2-4n+1-8m}}{2}\right\rfloor(1-n) & \text{if } s=2\\ \dfrac{s^2-2s-2}{2}+ms+n+\left\lceil\dfrac{-3+\sqrt{17+8(m-n)}}{2}\right\rceil & \text{if } s\ge4,\text{ here } m\ne n,\,n+2\text{ when } s=4\\ 5n+13 & \text{if } s=4,\ m=n+2\ge12\\ 5n+5 & \text{if } s=4,\ m=n\ge9\end{cases}$$

*with equalities if and only if* $G\in\mathscr{E}_{n,m,s},\mathscr{D}_{n,m,s},\mathscr{D}_{n,n+2,4}$, *and* $G\cong D_{n,n,4}$, *respectively.*

**Proof.** Let $G$ be a graph on $n$ vertices with $m$ edges and diameter $d$, where $d\ge s$. Suppose that $s$ is an odd positive integer and $3\le s\le 2n-m-2$. If $d=s$, then the result is clear for Theorem 1.

If $d$ is odd and $d > s$, then by Theorems 1–3, we have

$$\xi^c(G) \geq \begin{cases} \frac{d^2-4d-3}{2} + md + 2n & \text{if } 3 \leq d \leq 2n-m-2, \text{ where } 2n \geq m+5, \\ \frac{(d-1)^2}{2} + m(d+1) & \text{if } \max\{3, 2n-m-1\} \leq d \leq \left\lfloor \frac{2n-1-\sqrt{9+8(m-n)}}{2} \right\rfloor, \\ m(d+2) - \frac{3d-3+n(n-2d+1)}{2} & \text{if } \left\lceil \frac{2n-1-\sqrt{1+8(m-n)}}{2} \right\rceil \\ \quad \leq d \leq \left\lfloor \frac{2n+1-\sqrt{17+8(m-n)}}{2} \right\rfloor, \end{cases}$$

and by Lemma 1, we have $\phi_1(\frac{s-1}{2}) < \phi_1(\frac{d-1}{2})$ for $3 \leq d \leq 2n-m-2$, $\phi_1(\frac{s-1}{2}) < \phi_2(\frac{d-1}{2})$ for $\max\{3, 2n-m-1\} \leq d \leq \left\lfloor \frac{2n-1-\sqrt{9+8(m-n)}}{2} \right\rfloor$, and $\phi_1(\frac{s-1}{2}) < \phi_3(\frac{d-1}{2})$ for $\left\lceil \frac{2n-1-\sqrt{1+8(m-n)}}{2} \right\rceil \leq d \leq \left\lfloor \frac{2n+1-\sqrt{17+8(m-n)}}{2} \right\rfloor$. It follows that $\xi^c(G) > \phi_1(\frac{s-1}{2}) = \frac{s^2-4s-3}{2} + ms + 2n$ for the graph $G$ with odd diameter $d$ and $m$ edges ($d > s$, $3 \leq s \leq 2n-m-2$).

If $d$ is even and $d > s$, then by Theorems 4 and 5, we have that

$$\xi^c(G) \geq \begin{cases} \frac{d^2-2d-2}{2} + md + n + \left\lceil \frac{-3+\sqrt{17+8(m-n)}}{2} \right\rceil & \text{if } d \geq 4, \text{ here } m \neq n, n+2 \text{ when } d = 4, \\ 5n+13 & \text{if } d = 4, m = n+2 \geq 12, \\ 5n+5 & \text{if } d = 4, m = n \geq 9, \end{cases}$$

and since $\phi_4(\frac{d}{2}) > \phi_1(\frac{d}{2}-1) \geq \phi_1(\frac{s-1}{2})$, $5n+13 > 5n+3$ and $5n+5 > 5n-3$. It follows $\xi^c(G) > \phi_1(\frac{s-1}{2}) = \frac{s^2-4s-3}{2} + ms + 2n$ for the graph $G$ with odd diameter $d$ and $m$ edges ($d > s$, $3 \leq s \leq 2n-m-2$). This completes the proof for the case.

Similarly, we can prove the results for other cases, and thus complete the proof. □

We denote by $\Delta(G)$ the maximum vertex degree of $G$. Note that $\mathrm{diam}(G) \leq 2$ for $\Delta(G) = n-1$, and $\mathrm{diam}(G) \geq 3$ for $\Delta(G) \leq n-2$. By Theorem 7, we have the following.

**Corollary 2.** *Let $G$ be a connected graph with $n$ vertices and $m$ edges. If $\Delta(G) \leq n-2$, then*

$$\xi^c(G) \geq 3m + 2n - 3$$

*with equality if and only if $G \in \mathscr{C}_{n,m,3}$.*

**Theorem 8.** *Let $G$ be a connected graph on $n$ vertices with diameter at least $d$.*

(i) *Let $d$ be an odd positive integer. If $d = 1$, then $G \cong K_n$, and $\xi^c(K_n) = \binom{n}{2}$; If $d \geq 3$, then*

$$\xi^c(G) \geq \frac{d^2-6d-3}{2} + n(d+2)$$

*with equality if and only if $G \in \mathscr{C}_{n,n-1,d}$.*

(ii) *Let $d$ be an even positive integer. If $d = 2$, then $\xi^{(G)} \geq 3(n-1)$, with equality if and only if $G$ is isomorphic to the star $S_n$; If $d \geq 4$, then*

$$\xi^c(G) \geq \frac{d^2-4d-2}{2} + n(d+1)$$

*with equality if and only if $G \in \mathscr{D}_{n,n-1,d}$.*

**Proof.** (i) The result is obvious for $d = 1$. Suppose $d \geq 3$, and let

$$f(d, m) = \frac{d^2-4d-3}{2} + md + 2n,$$

$$g(d, m) = \frac{(d-1)^2}{2} + m(d+1),$$

$$h(d, m) = m(d+2) - \frac{3d-3+n(n-2d+1)}{2}.$$

Since they are all increasing functions on $m$, and $g(d, 2n-d-1) = f(d, 2n-d-2)$, $h\left(d, n-1+\binom{n-d}{2}\right) = g\left(d, n-1+\binom{n-d}{2}\right)$, by Theorem 7 we have that for any connected graph $G$ on $n$ vertices with diameter at least $d$, $\xi^c(G) \geq f(d, n-1) = \frac{d^2-6d-3}{2} + n(d+2)$, with equality if and only if $G \in \mathscr{C}_{n,n-1,d}$.

(ii) Let

$$\varphi(m) = 4m + \left\lfloor \frac{2n - 1 - \sqrt{4n^2 - 4n + 1 - 8m}}{2} \right\rfloor (1 - n),$$

$$\psi(d, m) = \frac{d^2 - 2d - 2}{2} + md + n + \left\lceil \frac{-3 + \sqrt{17 + 8(m - n)}}{2} \right\rceil.$$

Since $\varphi(m+1) - \varphi(m) = 4 + (1-n)\left\{\left\lfloor \frac{2n-1-\sqrt{4n^2-4n+9-8m}}{2} \right\rfloor - \left\lfloor \frac{2n-1-\sqrt{4n^2-4n+1-8m}}{2} \right\rfloor\right\} > 0$, $\varphi(m)$ is an increasing function for $m \geq n - 1$. Clearly, $\psi(d, m)$ is also an increasing function for $m$. Combining these cases and Theorem 7, we can complete the proof. □

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