A BSDE approach to stochastic differential games with incomplete information

Christine Grün

Laboratoire de Mathematiques de Brest UMR 6205, Université de Bretagne Occidentale, 6 avenue Le Gorgeu CS 93837, 29238 BREST cedex 3, France

Abstract

We consider a two-player zero-sum stochastic differential game in which one of the players has a private information on the game. Both players observe each other, so that the non-informed player can try to guess his missing information. Our aim is to quantify the amount of information the informed player has to reveal in order to play optimally: to do so, we show that the value function of this zero-sum game can be rewritten as a minimization problem over some martingale measures with a payoff given by the solution of a backward stochastic differential equation.
© 2012 Elsevier B.V. All rights reserved.

1. Introduction

In this paper we consider a two player zero-sum game, where the underlying dynamics are given by a diffusion with controlled drift but uncontrolled (non-degenerate) volatility. The game can take place in I different scenarios for the running cost and the terminal outcome as in a classical stochastic differential game. Before the game starts one scenario is picked with the probability $p = (p_i)_{i \in \{1,...,I\}} \in \Delta(I)$. The information is transmitted only to Player 1. So at the beginning he knows in which scenario he is playing, while Player 2 only knows the probability

E-mail address: Christine.Gruen@univ-brest.fr.

p. It is assumed that both players observe the actions of the other one, so Player 2 might infer from the actions of his opponent in which scenario the game is actually played.

It has been proved by Cardaliaguet and Rainer in [7] that this game has a value. To investigate the game under the perspective of information transmission we establish an alternative representation of this value. We achieve this by directly modeling the amount of information the informed player reveals during the game. To that end we enlarge the canonical Wiener space to a space which carries besides a Brownian motion, càdlàg martingales with values in $\Delta(I)$. These martingales can be interpreted as possible beliefs of the uninformed player, i.e. the probability in which scenario the game is played in according to his information at time t.

The very same ansatz has been used in the case of deterministic differential games by Cardaliaguet and Rainer in [6], while the original idea of the so called a posteriori martingale can already be found in the classical work of Aumann and Maschler (see [2]). Bearing in mind the ideas of Hamadène and Lepeltier [13] we show that the value of our game can be represented by minimizing the solution of a backward stochastic differential equation (BSDE) with respect to possible beliefs of the uninformed player.

A cornerstone in the investigation of stochastic differential games has been laid by Fleming and Souganidis in [12] who extend the results of Evans and Souganidis [11] to a stochastic framework. Therein it is shown that under Isaacs condition the value function of a stochastic differential game is given as the unique viscosity solution of a Hamilton–Jacobi–Isaacs (HJI) equation.

The theory of BSDE, which was originally developed by Peng [17] for stochastic control theory, has been introduced to stochastic differential games by Hamadène and Lepeltier [13] and Hamadène et al. [14]. The former results have been extended to cost functionals defined by controlled BSDEs in [3], where the admissible control processes are allowed to depend on events occurring before the beginning of the game.

The study of games with incomplete information has its starting point in the pioneering work of Aumann and Maschler (see [2] and references given therein). The extension to stochastic differential games has been given in [7]. The proof is accomplished introducing the notion of dual viscosity solutions to the HJI equation of a usual stochastic differential game, where the probability p just appears as an additional parameter. A different unique characterization via the viscosity solution of the HJI equation with an obstacle in the form of a convexity constraint in p is given in [5]. We use this latter characterization in order to prove our main representation result.

The outline of the paper is as follows. In Section 2 we describe the game and restate the results of [7,5] which build the basis for our investigation. In Section 3 we give our main theorem and derive the optimal behavior for the informed player under some smoothness condition. The whole Section 4 is devoted to the proof of the main theorem, while in the Appendix we summarize extensions to classical BSDE results, which are needed in our case.

2. Setup

2.1. Formal description of the game

Let $\mathcal{C}([0, T]; \mathbb{R}^d)$ be the set of continuous functions from $\mathbb{R}$ to $\mathbb{R}^d$, which are constant on $(-\infty, 0]$ and on $[T, +\infty)$. We denote by $B_s(\omega_B) = \omega_B(s)$ the coordinate mapping on $\mathcal{C}([0, T]; \mathbb{R}^d)$ and define $\mathcal{H} = (\mathcal{H}_s)$ as the filtration generated by $s \mapsto B_s$. We denote $\Omega_t = \{\omega \in \mathcal{C}([t, T]; \mathbb{R}^d)\}$ and $\mathcal{H}_{t,s}$ the σ-algebra generated by paths up to time s in Ω_t. Furthermore we provide $\mathcal{C}([0, T]; \mathbb{R}^d)$ with the Wiener measure $\mathbb{P}^0$ on $(\mathcal{H}_s)$ and we consider the respective filtrations augmented by $\mathbb{P}^0$ nullsets without changing the notation.

In the following we investigate a two-player zero-sum differential game starting at a time $t \geq 0$ with terminal time T. The dynamics are given by a controlled diffusion on $(\mathcal{C}([t, T]; \mathbb{R}^d), (\mathcal{H}_{t,s})_{s \in [t,T]}, \mathcal{H}, \mathbb{P}^0)$, i.e. for $t \in [0, T], x \in \mathbb{R}^d$

$$dX_s^{t,x,u,v} = b(s, X_s^{t,x,u,v}, u_s, v_s)ds + \sigma(s, X_s^{t,x,u,v})dB_s \qquad X_t^{t,x} = x. \tag{1}$$

We assume that the controls of the players u, v can only take their values in some set U, V respectively, where U, V are compact subsets of some finite dimensional spaces.

Let $\Delta(I)$ denote the simplex of $\mathbb{R}^I$. The objective to optimize is characterized by

(i) running costs: $(l_i)_{i \in \{1,...,I\}} : [0, T] \times \mathbb{R}^d \times U \times V \to \mathbb{R}$
(ii) terminal payoffs: $(g_i)_{i \in \{1,...,I\}} : \mathbb{R}^d \to \mathbb{R}$,

which are chosen with probability $p \in \Delta(I)$ before the game starts. Player 1 chooses his control to minimize, Player 2 chooses his control to maximize the expected payoff. We assume both players observe their opponents control. However Player 1 knows which payoff he optimizes, Player 2 just knows the respective probabilities p_i for scenario $i \in \{1, \ldots, I\}$.

The following will be the standing assumption throughout the paper.

Assumption (H).

(i) $b : [0, T] \times \mathbb{R}^d \times U \times V \to \mathbb{R}^d$ is bounded and continuous in all its variables and Lipschitz continuous with respect to (t, x) uniformly in (u, v).
(ii) For $1 \leq k, l \leq d$ the function $\sigma_{k,l} : [0, T] \times \mathbb{R}^d \to \mathbb{R}$ is bounded and Lipschitz continuous with respect to (t, x). For any $(t, x) \in [0, T] \times \mathbb{R}^d$ the matrix $\sigma^*(t, x)$ is non-singular and $(\sigma^*(t, x))^{-1}$ is bounded and Lipschitz continuous with respect to (t, x).
(iii) $(l_i)_{i \in I} : [0, T] \times \mathbb{R}^d \times U \times V \to \mathbb{R}$ is bounded and continuous in all its variables and Lipschitz continuous with respect to (t, x) uniformly in (u, v). $(g_i)_{i \in I} : \mathbb{R}^d \to \mathbb{R}$ is bounded and uniformly Lipschitz continuous.
(iv) Isaacs condition: for all $(t, x, \xi, p) \in [0, T] \times \mathbb{R}^d \times \mathbb{R}^d \times \Delta(I)$

$$\inf_{u \in U} \sup_{v \in V} \left\{ \langle b(t, x, u, v), \xi \rangle + \sum_{i=1}^{I} p_i l_i(t, x, u, v) \right\}$$

$$= \sup_{v \in V} \inf_{u \in U} \left\{ \langle b(t, x, u, v), \xi \rangle + \sum_{i=1}^{I} p_i l_i(t, x, u, v) \right\} =: H(t, x, \xi, p). \tag{2}$$

By [Assumption (H)](#) the Hamiltonian H is Lipschitz in (ξ, p) uniformly in (t, x) and Lipschitz in (t, x) with Lipschitz constant $c(1 + |\xi|)$, i.e. it holds for all $t, t' \in [0, T], x, x' \in \mathbb{R}^d, \xi, \xi' \in \mathbb{R}^d, p, p' \in \Delta(I)$

$$|H(t, x, \xi, p)| \leq c(1 + |\xi|) \tag{3}$$

and

$$|H(t, x, \xi, p) - H(t', x', \xi', p')| \leq c(1 + |\xi|)(|x - x'| + |t - t'|)$$

$$+ c|\xi - \xi'| + c|p - p'|. \tag{4}$$

2.2. *Strategies and value function*

Definition 2.1. For any $t \in [0, T[$ an admissible control $u = (u_s)_{s \in [t,T]}$ for Player 1 is a progressively measurable process with respect to the filtration $(\mathcal{H}_{t,s})_{s \in [t,T]}$ with values in U. The set of admissible controls for Player 1 is denoted by $\mathcal{U}(t)$.

The definition for admissible controls $v = (v_s)_{s \in [t,T]}$ for Player 2 is similar. The set of admissible controls for Player 2 is denoted by $\mathcal{V}(t)$.

In differential games with complete information as in [12] it is sufficient, that one player chooses at the beginning an admissible control and the other one chooses the optimal reaction to it. In our case the uniformed player tries to infer from the actions of his opponent in which scenario the game is played and adapts his behavior to his beliefs. Thus a permanent interaction has to be allowed. To this end it is necessary to restrict admissible strategies to have a small delay in time.

Definition 2.2. A strategy for Player 1 at time $t \in [0, T[$ is a map $\alpha : [t, T] \times \mathcal{C}([t, T]; \mathbb{R}^d) \times L^0([t, T]; V) \rightarrow U$ which is nonanticipative with delay, i.e. there is $\delta > 0$ such that for all $s \in [t, T]$ for any $f, f' \in \mathcal{C}([t, T]; \mathbb{R}^d)$ and $g, g' \in L^0([t, T]; V)$ it holds: $f = f'$ and $g = g'$ a.e. on $[t, s] \Rightarrow \alpha(\cdot, f, g) = \alpha(\cdot, f', g')$ a.e. on $[t, s + \delta]$. The set of strategies for Player 1 is denoted by $\mathcal{A}(t)$.

The definition of strategies $\beta : [t, T] \times \mathcal{C}([t, T]; \mathbb{R}^d) \times L^0([t, T]; U) \rightarrow V$ for Player 2 is similar. The set of strategies for Player 2 is denoted by $\mathcal{B}(t)$.

Next we restate Lemma 5.1 of [7].

Lemma 2.3. *One can associate to each pair of strategies $(\alpha, \beta) \in \mathcal{A}(t) \times \mathcal{B}(t)$ a unique couple of admissible controls $(u, v) \in \mathcal{U}(t) \times \mathcal{V}(t)$, such that for all $\omega \in \mathcal{C}([t, T]; \mathbb{R}^d)$*

$$\alpha(s, \omega, v(\omega)) = u_s(\omega) \quad and \quad \beta(s, \omega, u(\omega)) = v_s(\omega).$$

The proof is done via a fixed point argument using the delay property of the strategies.

Furthermore it is crucial that the players are allowed to choose their strategies with a certain additional randomness. Intuitively this can be explained by the incentive of the players to hide their information. Thus for the evaluation of a game with incomplete information we introduce random strategies. To this end let $\mathcal{I}$ denote a set of probability spaces which is non trivial and stable by finite product.

Definition 2.4. A random strategy for Player 1 at time $t \in [0, T[$ is a pair $((\Omega_\alpha, \mathcal{G}_\alpha, \mathbb{P}_\alpha), \alpha)$, where $(\Omega_\alpha, \mathcal{G}_\alpha, \mathbb{P}_\alpha)$ is a probability space in $\mathcal{I}$ and $\alpha : [t, T] \times \Omega_\alpha \times \mathcal{C}([t, T]; \mathbb{R}^d) \times L^0([t, T]; V) \rightarrow U$ satisfies

(i) α is a measurable function, where Ω_α is equipped with the σ-field $\mathcal{G}_\alpha$,
(ii) there exists $\delta > 0$ such that for all $s \in [t, T]$ and for any $f, f' \in \mathcal{C}([t, T]; \mathbb{R}^d)$ and $g, g' \in L^0([t, T]; V)$ it holds:

$$f = f' \quad and \quad g = g' \quad \text{a.e. on } [t, s] \Rightarrow \alpha(\cdot, f, g) = \alpha(\cdot, f', g') \text{ a.e. on } [t, s + \delta]$$
for any $\omega \in \Omega_\alpha$.

The set of random strategies for Player 1 is denoted by $\mathcal{A}^r(t)$.

The definition of random strategies $((\Omega_\beta, \mathcal{G}_\beta, \mathbb{P}_\beta), \beta)$, where $\beta : [t, T] \times \Omega_\beta \times \mathcal{C}([t, T]; \mathbb{R}^d) \times L^0([t, T]; U) \to V$ for Player 2 is similar. The set of random strategies for Player 2 is denoted by $\mathcal{B}^r(t)$.

Remark 2.5. Again one can associate to each couple of random strategies $(\alpha, \beta) \in \mathcal{A}^r(t) \times \mathcal{B}^r(t)$ for any $(\omega_\alpha, \omega_\beta) \in \Omega_\alpha \times \Omega_\beta$ a unique couple of admissible strategies $(u^{\omega_\alpha, \omega_\beta}, v^{\omega_\alpha, \omega_\beta}) \in \mathcal{U}(t) \times \mathcal{V}(t)$, such that for all $\omega \in \mathcal{C}([t, T]; \mathbb{R}^d)$, $s \in [t, T]$

$$\alpha(s, \omega_\alpha, \omega, v^{\omega_\alpha, \omega_\beta}(\omega)) = u_s^{\omega_\alpha, \omega_\beta}(\omega) \quad \text{and} \quad \beta(s, \omega_\beta, \omega, u^{\omega_\alpha, \omega_\beta}(\omega)) = v_s^{\omega_\alpha, \omega_\beta}(\omega).$$

Furthermore $(\omega_\alpha, \omega_\beta) \to (u^{\omega_\alpha, \omega_\beta}, v^{\omega_\alpha, \omega_\beta})$ is a measurable map, from $\Omega_\alpha \times \Omega_\beta$ equipped with the σ-field $\mathcal{G}_\alpha \otimes \mathcal{G}_\beta$ to $\mathcal{V}(t) \times \mathcal{U}(t)$ equipped with the Borel σ-field associated to the L^1-distance.

For any $(t, x, p) \in [0, T[\times \mathbb{R}^d \times \Delta(I), \bar{\alpha} \in (\mathcal{A}^r(t))^I, \beta \in \mathcal{B}^r(t)$ we set

$$J(t, x, p, \bar{\alpha}, \beta) = \sum_{i=1}^{I} p_i \, \mathbb{E}_{\bar{\alpha}_i, \beta} \left[\int_0^T l_i(s, X_s^{t, x, \bar{\alpha}_i, \beta}, (\bar{\alpha}_i)_s, \beta_s) ds + g_i(X_T^{t, x, \bar{\alpha}_i, \beta}) \right], \quad (5)$$

where (5) should be understood in the following way. As in Remark 2.5 we associate to $\bar{\alpha}_i, \beta$ for any $(\omega_{\bar{\alpha}_i}, \omega_\beta) \in \Omega_{\bar{\alpha}_i} \times \Omega_\beta$ the couple of controls $(u^{\omega_{\bar{\alpha}_i}, \omega_\beta}, v^{\omega_{\bar{\alpha}_i}, \omega_\beta})$. The process $X^{t, x, \bar{\alpha}_i, \beta}$ is then defined for any $(\omega_{\bar{\alpha}_i}, \omega_\beta)$ as solution to (1) with the associated controls. Furthermore $\mathbb{E}_{\bar{\alpha}_i, \beta}$ is the expectation on $\Omega_{\bar{\alpha}_i} \times \Omega_\beta \times \mathcal{C}([t, T]; \mathbb{R}^d)$ with respect to the probability $\mathbb{P}_{\bar{\alpha}_i} \otimes \mathbb{P}_\beta \otimes \mathbb{P}^0$, where $\mathbb{P}^0$ denotes the Wiener measure on $\mathcal{C}([t, T]; \mathbb{R}^d)$.

Under Assumption (H) the existence of the value of the game is proved in a more general setting in [7].

Theorem 2.6. *For any $(t, x, p) \in [0, T[\times \mathbb{R}^d \times \Delta(I)$ the value of the game with incomplete information $V(t, x, p)$ is given by*

$$V(t, x, p) = \inf_{\bar{\alpha} \in (\mathcal{A}^r(t))^I} \sup_{\beta \in \mathcal{B}^r(t)} J(t, x, p, \bar{\alpha}, \beta)$$

$$= \sup_{\beta \in \mathcal{B}^r(t)} \inf_{\bar{\alpha} \in (\mathcal{A}^r(t))^I} J(t, x, p, \bar{\alpha}, \beta). \quad (6)$$

Remark 2.7. It is well known (e.g. [7, Lemma 3.1]) that it suffices for the uninformed player to use admissible (non-random) strategies if he plays first. Intuitively this is due to the fact that he has no information to hide. So we can use in (6) the easier expression

$$V(t, x, p) = \inf_{\bar{\alpha} \in (\mathcal{A}^r(t))^I} \sup_{\beta \in \mathcal{B}(t)} J(t, x, p, \bar{\alpha}, \beta). \quad (7)$$

The existence and uniqueness of the value function $V : [0, T] \times \mathbb{R}^d \times \Delta(I) \to \mathbb{R}$ is first given [7] using the concept of dual viscosity solutions to HJI equations. Starting from this a characterization of the value function as solution of an obstacle problem is given in [5].

Theorem 2.8. *The function $V : [0, T[\times \mathbb{R}^d \times \Delta(I) \to \mathbb{R}$ is the unique viscosity solution to*

$$\min \left\{ \frac{\partial w}{\partial t} + \frac{1}{2} \text{tr}(\sigma \sigma^*(t, x) D_x^2 w) + H(t, x, D_x w, p), \lambda_{\min} \left(p, \frac{\partial^2 w}{\partial p^2} \right) \right\} = 0 \quad (8)$$

with terminal condition $w(T, x, p) = \sum_i p_i g_i(x)$, where for all $p \in \Delta(I)$, $A \in \mathcal{S}^I$ (set of symmetric $I \times I$ matrices)

$$\lambda_{\min}(p, A) := \min_{z \in T_{\Delta(I)(p)} \setminus \{0\}} \frac{\langle Az, z \rangle}{|z|^2}$$

and $T_{\Delta(I)(p)}$ denotes the tangent cone to $\Delta(I)$ at p, i.e. $T_{\Delta(I)(p)} = \overline{\cup_{\lambda > 0} (\Delta(I) - p)/\lambda}$.

Remark 2.9. Note that unlike the standard definition of viscosity solutions (see e.g. [8]) the subsolution property to (8) is required only on the interior of $\Delta(I)$ while the supersolution property to (8) is required on the whole domain $\Delta(I)$ (see [5,6]). This is due to the fact that we actually consider viscosity solutions with a state constraint, namely $p \in \Delta(I) \subsetneq \mathbb{R}^I$. For a concise investigation of such problems we refer to [4].

We do not go into detail about the rather technical proof of Theorem 2.7 in [5]. However there is an easy intuitive explanation of the convexity constraint, which we give in the following remark.

Remark 2.10. Let $(t, x) \in [0, T] \times \mathbb{R}^d$ be fixed. For any $p_0 \in \Delta(I)$ let $\lambda \in (0, 1)$, $p_1, p_2 \in \Delta(I)$, such that $p_0 = (1 - \lambda)p_1 + \lambda p_2$.

We consider the game in two steps. First the initial distribution for the game with incomplete information p_1, p_2 is picked with probability $(1 - \lambda), \lambda$. If the outcome is transmitted only to Player 1, the value of this game is $V(t, x, (1 - \lambda)p_1 + \lambda p_2) = V(t, x, p_0)$.

On the other hand we consider the game in which both players are told the outcome of the pick of the initial distribution p_1, p_2. The expected outcome of this game is $(1 - \lambda)V(t, x, p_1) + \lambda V(t, x, p_2)$.

In the first game the informed player knows more, hence, if we make the rather reasonable assumption that the value of information is positive, we have $V(t, x, p_0) \leq (1 - \lambda)V(t, x, p_1) + \lambda V(t, x, p_2)$.

3. Alternative representation of the value function

3.1. Enlargement of the canonical space

In the following we establish a representation of the value function by enlarging the canonical Wiener space to a space which will carry besides a Brownian motion a new dynamic. We use this additional dynamic to model the incorporation of the private information into the game. More precisely we model the probability in which scenario the game is played in according to the information of the uniformed Player 2.

To that end let us denote by $\mathcal{D}([0, T]; \Delta(I))$ the set of càdlàg functions from $\mathbb{R}$ to $\Delta(I)$, which are constant on $(-\infty, 0)$ and on $[T, +\infty)$. We denote by $\mathbf{p}_s(\omega_p) = \omega_p(s)$ the coordinate mapping on $\mathcal{D}([0, T]; \Delta(I))$ and by $\mathcal{G} = (\mathcal{G}_s)$ the filtration generated by $s \mapsto \mathbf{p}_s$. Furthermore we recall that $\mathcal{C}([0, T]; \mathbb{R}^d)$ denotes the set of continuous functions from $\mathbb{R}$ to $\mathbb{R}^d$, which are constant on $(-\infty, 0]$ and on $[T, +\infty)$. We denote by $B_s(\omega_B) = \omega_B(s)$ the coordinate mapping on $\mathcal{C}([0, T]; \mathbb{R}^d)$ and by $\mathcal{H} = (\mathcal{H}_s)$ the filtration generated by $s \mapsto B_s$. We equip the product space $\Omega := \mathcal{D}([0, T]; \Delta(I)) \times \mathcal{C}([0, T]; \mathbb{R}^d)$ with the filtration $\mathcal{F} = \mathcal{G} \otimes \mathcal{H}$.

For $0 \leq t \leq T$ we denote $\Omega_t = \{\omega \in \mathcal{D}([t, T]; \Delta(I)) \times \mathcal{C}([t, T]; \mathbb{R}^d)\}$ and $\mathcal{F}_{t,s}$ the σ-algebra generated by paths up to time $s \geq t$ in Ω_t. Furthermore we define the space

$$\Omega_{t,s} = \{\omega \in \mathcal{D}([t, s]; \Delta(I)) \times \mathcal{C}([t, s]; \mathbb{R}^d)\}$$

for $0 \leq t \leq s \leq T$. If $r \in]t, T[$ and $\omega \in \Omega_t$ then let

$$\omega_1 = 1_{[-\infty,r[}\omega \qquad \omega_2 = 1_{[r,+\infty[}(\omega - \omega_{r-})$$

and denote $\pi\omega = (\omega_1, \omega_2)$. The map $\pi : \Omega_t \to \Omega_{t,r} \times \Omega_r$ induces the identification $\Omega_t = \Omega_{t,r} \times \Omega_r$ moreover $\omega = \pi^{-1}(\omega_1, \omega_2)$, where the inverse is defined in an evident way.

For any measure $\mathbb{P}$ on Ω, we denote by $\mathbb{E}_{\mathbb{P}}[\cdot]$ the expectation with respect to $\mathbb{P}$. We equip Ω with a certain class of measures.

Definition 3.1. Given $p \in \Delta(I), t \in [0, T]$, we denote by $\mathcal{P}(t, p)$ the set of probability measures $\mathbb{P}$ on Ω such that, under $\mathbb{P}$

(i) $\mathbf{p}$ is a martingale, such that $\mathbf{p}_s = p \ \forall s < t$, $\mathbf{p}_s \in \{e_i, i = 1, \ldots, I\} \ \forall s \geq T \mathbb{P}$-a.s., where e_i denotes the i-th coordinate vector in $\mathbb{R}^I$, and $\mathbf{p}_T$ is independent of $(B_s)_{s \in (-\infty, T]}$,

(ii) $(B_s)_{s \in [0,T]}$ is a Brownian motion.

Comment 3.2. Assumption (ii) is naturally given by the Brownian structure of the game. Assumption (i) is motivated as follows. Before the game starts the information of the uninformed player is just the initial distribution p. The martingale property, implying $\mathbf{p}_t = \mathbb{E}_{\mathbb{P}}[\mathbf{p}_T | \mathcal{F}_t]$, is due to the best guess of the uniformed player about the scenario he is in. Finally, at the end of the game the information is revealed hence $\mathbf{p}_T \in \{e_i, i = 1, \ldots, I\}$ and since the scenario is picked before the game starts the outcome $\mathbf{p}_T$ is independent of the Brownian motion.

3.2. BSDEs for stochastic differential games with incomplete information

The value of a game with incomplete information is studied in [6] in a simpler setting, namely where $X^{t,x,u,v}$ is constant. An alternative representation of the value is given by directly minimizing the expectation of the Hamiltonian over a similar class of martingale measures as in Definition 3.1(i). In our case $X^{t,x,u,v}$ is a diffusion where the drift is controlled by the players, hence the Hamiltonian (2) depends on the first derivative of the value function and a "direct" representation is not possible.

To solve this problem we use the theory of BSDE and extend the result of [13], where the value of an ordinary stochastic differential game is expressed by the solution of a BSDE with the Hamiltonian as driver. In the case of incomplete information we will have additional to the diffusion an extra forward dynamic, namely the beliefs $\mathbf{p}$ of the uninformed player, which are manipulated by the actions of the informed one. He chooses his control – hence indirectly the dynamics of $\mathbf{p}$ – in order to minimize the expected outcome. In this paper we show that we can represent the value function over a direct minimization of the solutions of BSDEs which can be interpreted as the outcome of a stochastic differential game with information completeness and an additional forward dynamic $\mathbf{p}$ (see (10)).

First we introduce the following spaces. For any $p \in \Delta(I), t \in [0, T]$ and fixed $\mathbb{P} \in \mathcal{P}(t, p)$ we denote by $\mathcal{L}_T^2(\mathbb{P})$ the set of a square integrable $\mathcal{F}_T$-measurable random variables. We define by $\mathcal{H}^2(\mathbb{P})$ the space of all predictable processes θ such that $\int_0^{\cdot} \theta_s dB_s$ is a square integrable martingale, i.e. $\mathbb{E}\left[\int_0^T \theta_s^2 ds\right] < \infty$. Furthermore we denote by $\mathcal{M}_0^2(\mathbb{P})$ the space of square integrable martingales null at zero. In the following we shall identify any $N \in \mathcal{M}_0^2(\mathbb{P})$ with its càdlàg modification.

For all $t \in [0, T], x \in \mathbb{R}^d$ we define the process $X^{t,x}$ by

$$X_s^{t,x} = x \quad s < t, \qquad X_s^{t,x} = x + \int_t^s \sigma(r, X_r^{t,x}) dB_r \quad s \geq t. \tag{9}$$

Let $p \in \Delta(I)$. We consider for each $\mathbb{P} \in \mathcal{P}(t, p)$ the BSDE

$$Y_s^{t,x,\mathbb{P}} = \langle \mathbf{p}_T, g(X_T^{t,x}) \rangle + \int_s^T H(r, X_r^{t,x}, Z_r^{t,x,\mathbb{P}}, \mathbf{p}_r) dr$$

$$- \int_s^T \sigma^*(r, X_r^{t,x}) Z_r^{t,x,\mathbb{P}} dB_r - N_T^{\mathbb{P}} + N_s^{\mathbb{P}}, \tag{10}$$

where $N^{\mathbb{P}} \in \mathcal{M}_0^2(\mathbb{P})$ is strongly orthogonal to the Brownian motion B.

Existence and uniqueness results for the BSDE (10) can be found in more generality in [9]. Our case is much simpler, since the driver does not depend on the jump parts. We mention the results which will be relevant for us in the Appendix. Note in particular that as in the standard case one can establish a comparison principle (Theorem A.3), which will be crucial in our further calculations.

Theorem 3.3. *Under the Assumption* (H) *the BSDE* (10) *has a unique solution* $(Y^{t,x,\mathbb{P}}, Z^{t,x,\mathbb{P}}, N^{\mathbb{P}}) \in \mathcal{H}^2(\mathbb{P}) \times \mathcal{H}^2(\mathbb{P}) \times \mathcal{M}_0^2(\mathbb{P})$ *and it holds for any* $s \leq T$

$$Y_s^{t,x,\mathbb{P}} = \mathbb{E}_{\mathbb{P}} \left[\int_s^T H(r, X_r^{t,x}, Z_r^{t,x,\mathbb{P}}, \mathbf{p}_r) dr + \langle \mathbf{p}_T, g(X_T^{t,x}) \rangle \Big| \mathcal{F}_s \right].$$

In particular it holds

$$Y_{t-}^{t,x,\mathbb{P}} = \mathbb{E}_{\mathbb{P}} \left[\int_t^T H(r, X_r^{t,x}, Z_r^{t,x,\mathbb{P}}, \mathbf{p}_r) dr + \langle \mathbf{p}_T, g(X_T^{t,x}) \rangle \Big| \mathcal{F}_{t-} \right]. \tag{11}$$

Fix $t \in [0, T], x \in \mathbb{R}^d, p \in \Delta(I)$. Note that all $\mathbb{P} \in \mathcal{P}(t, p)$ are equal on $\mathcal{F}_{t-}$, i.e. the distribution of $(B_s, \mathbf{p}_s)s \in [0, t[$ is given by $\delta(p) \otimes \mathbb{P}^0$, where $\delta(p)$ is the measure under which $\mathbf{p}$ is constant and equal to p and $\mathbb{P}^0$ is a Wiener measure. So we can identify each $\mathbb{P} \in \mathcal{P}(t, p)$ on $\mathcal{F}_{t-}$ with a common probability measure $\mathbb{Q}$ and define

$$W(t, x, p) = \text{essinf}_{\mathbb{P} \in \mathcal{P}(t,p)} Y_{t-}^{t,x,\mathbb{P}} \quad \mathbb{Q}\text{-a.s.} \tag{12}$$

The aim of this paper is to show the following alternative representation for the value function.

Theorem 3.4. *For any* $(t, x, p) \in [0, T[\times\mathbb{R}^d \times \Delta(I)$ *the value of the game with incomplete information* $V(t, x, p)$ *can be characterized as*

$$V(t, x, p) = \text{essinf}_{\mathbb{P} \in \mathcal{P}(t,p)} Y_{t-}^{t,x,\mathbb{P}}. \tag{13}$$

We give the proof in the Section 4, where we first show that $W(t, x, p)$ is a deterministic function. Then we establish a Dynamic Programming Principle and show that $W(t, x, p)$ is a viscosity solution to (8). Since $V(t, x, p)$ is by Theorem 2.8 uniquely defined as the viscosity solution to (8), the equality is immediate. Before, let us first investigate under smoothness assumptions a possible behavior of an optimal measure and show how the representation is related to the original game.

3.3. A sufficient condition for optimality

Next we give a sufficient condition for a $\mathbb{P} \in \mathcal{P}(t, p)$ to be optimal in (13). We assume $V \in \mathcal{C}^{1,2,2}([t, T) \times \mathbb{R}^d \times \Delta(I); \mathbb{R})$ and set

$$\mathcal{H} = \left\{ (t, x, p) \in [0, T) \times \mathbb{R}^d \times \Delta(I) : \frac{\partial V}{\partial t} \right.$$
$$\left. + \frac{1}{2} \mathrm{tr}(\sigma \sigma^*(t, x) D_x^2 V) + H(t, x, D_x V, p) = 0 \right\}$$

and

$$\mathcal{H}(t, x) = \{ p \in \Delta(I) : (t, x, p) \in \mathcal{H} \}.$$

In the theory of games with incomplete information the set $\mathcal{H}$ is usually called the non-revealing set. This is due to the fact that on $\mathcal{H}$ the value function fulfills the standard HJI equation, hence the informed player is not "actively" using his information because the belief of the uniformed player stays unchanged.

Theorem 3.5. *Let $(t, x, p) \in [0, T) \times \mathbb{R}^d \times \Delta(I)$. We assume $V \in \mathcal{C}^{1,2,2}([t, T) \times \mathbb{R}^d \times \Delta(I); \mathbb{R})$. Let $\bar{\mathbb{P}} \in \mathcal{P}(t, p)$, such that*

(i) $\mathbf{p}_s \in \mathcal{H}(s, X_s^{t,x}) \; \forall s \in [t, T] \bar{\mathbb{P}}$-*a.s.,*
(ii) $\bar{\mathbb{P}}$-*a.s. it holds $\forall s \in [t, T]$*

$$V(s, X_s^{t,x}, \mathbf{p}_s) - V(s, X_s^{t,x}, \mathbf{p}_{s-}) - \left\langle \frac{\partial}{\partial p} V(s, X_s^{t,x}, \mathbf{p}_{s-}), \mathbf{p}_s - \mathbf{p}_{s-} \right\rangle = 0,$$

(iii) $\mathbf{p}$ *is under $\bar{\mathbb{P}} \in \mathcal{P}(t, p)$ a purely discontinuous martingale.*

Then $\bar{\mathbb{P}}$ is optimal for $V(t, x, p)$.

Remark 3.6. The analysis of the deterministic case in [6] indicates that the conditions (i) and (ii) might also be necessary even in the non-smooth case. In fact under certain assumptions the conditions (i)–(iii) of Theorem 3.5 can expected to be necessary and sufficient. (See [6, Example 4.4].)

Proof. By definition $V(T, x, p) = \langle g(x), p \rangle$. Since $V \in \mathcal{C}^{1,2,2}$ and $\mathbf{p}$ is purely discontinuous we have by Itô's formula and the assumptions (i)–(iii)

$$
\begin{aligned}
\langle g(X_T^{t,x}), \mathbf{p}_T \rangle &= V(T, X_T^{t,x}, \mathbf{p}_T) \\
&= V(s, X_s^{t,x}, \mathbf{p}_s) \\
&\quad + \int_s^T \left(\frac{\partial}{\partial t} V(r, X_r^{t,x}, \mathbf{p}_r) + \frac{1}{2} \mathrm{tr}(\sigma \sigma^*(r, X_r^{t,x}) D_x^2 V(s, X_r^{t,x}, \mathbf{p}_r)) \right) dr \\
&\quad + \int_s^T \sigma^*(r, X_r^{t,x}) D_x V(r, X_r^{t,x}, \mathbf{p}_r) dB_r \\
&\quad + \sum_{s \le r \le T} V(r, X_r^{t,x}, \mathbf{p}_r) - V(r, X_r^{t,x}, \mathbf{p}_{r-}) \\
&\quad - \left\langle \frac{\partial}{\partial p} V(r, X_r^{t,x}, \mathbf{p}_{r-}), \mathbf{p}_r - \mathbf{p}_{r-} \right\rangle \\
&= V(s, X_s^{t,x}, \mathbf{p}_s) - \int_s^T H(r, X_r^{t,x}, D_x V(r, X_r^{t,x}, \mathbf{p}_r), \mathbf{p}_r) dr \\
&\quad + \int_s^T \sigma^*(r, X_r^{t,x}) D_x V(r, X_r^{t,x}, \mathbf{p}_r) dB_r.
\end{aligned}
$$

So by comparison (Theorem A.3) $(Y_s^{t,x,\bar{\mathbb{P}}}, Z_s^{t,x,\bar{\mathbb{P}}}, N_s^{t,x,\bar{\mathbb{P}}}) := (V(s, X_s^{t,x}, \mathbf{p}_s), D_x V(s, X_s^{t,x}, \mathbf{p}_s), 0)$ is the unique solution to the BSDE (10).

We have in particular

$$V(t, x, p) = \langle g(X_T^{t,x}), \mathbf{p}_T \rangle - \int_t^T H(s, X_s^{t,x}, Z_s^{t,x,\bar{\mathbb{P}}}, \mathbf{p}_s) ds + \int_t^T \sigma^*(s, X_s^{t,x}) Z_s^{t,x,\bar{\mathbb{P}}} dB_s,$$

hence the result follows from taking conditional expectation and the representation in Theorem 3.3. $\quad\square$

3.4. Optimal information reveal for the informed player

Our aim is to quantify the amount of information the informed player has to reveal in order to play optimally. Note that in the representation we consider as in [13] the original game under a Girsanov transformation. Hence an optimal measure in (13) gives an information structure of the game only up to a Girsanov transformation, which we have to reverse to get back to our original problem.

We assume $V \in \mathcal{C}^{1,2,2}([t, T) \times \mathbb{R}^d \times \Delta(I); \mathbb{R})$. Let $\bar{\mathbb{P}} \in \mathcal{P}(t, p)$, such that the conditions of Theorem 3.5 are fulfilled, hence $Z_s^{t,x,\bar{\mathbb{P}}} = D_x V(s, X_s^{t,x}, \mathbf{p}_s)$.

Thanks to Isaacs condition, Assumption (H)(iv), one can define the function $u^*(t, x, p, \xi)$ as a Borel measurable selection of $\text{argmin}_{u \in U} \{ \max_{v \in V} \langle b(t, x, u, v), \xi \rangle + \sum_{i=1}^I p_i l_i(t, x, u, v) \}$, hence

$$H(t, x, \xi, p) = \max_{v \in V} \left\{ \langle b(t, x, u^*(t, x, p, \xi), v), \xi \rangle \right.$$

$$\left. + \sum_{i=1}^I p_i l_i(t, x, u^*(t, x, p, \xi), v) \right\}. \tag{14}$$

We define the process

$$\bar{u}_s = u^*(s, X_s^{t,x}, D_x V(s, X_s^{t,x}, \mathbf{p}_s), \mathbf{p}_s), \tag{15}$$

where by definition $\bar{u}$ is progressively measurable with respect to the filtration $(\mathcal{F}_{t,s})_{s \in [t,T]}$ with values in U. In the following we will denote the set of such processes the set of relaxed controls $\bar{\mathcal{U}}(t)$ and the set of progressively measurable processes with respect to the filtration $(\mathcal{F}_{t,s})_{s \in [t,T]}$ with values in V the set of relaxed controls $\bar{\mathcal{V}}(t)$.

We consider for each relaxed control $v \in \bar{\mathcal{V}}(t)$ the (F)BSDE

$$X_s^{t,x} = x + \int_t^s \sigma(r, X_r^{t,x}) dB_r$$

$$Y_s^{t,x,\bar{u},v} = \langle \mathbf{p}_T, g(X_T^{t,x}) \rangle + \int_s^T \left(\langle \mathbf{p}_r, l(r, X_r^{t,x}, \bar{u}_r, v_r) \rangle \right.$$

$$+ \langle b(r, X_r^{t,x}, \bar{u}_r, v_r), D_x V(r, X_r^{t,x}, \mathbf{p}_r) \rangle \right) dr$$

$$- \int_s^T \sigma^*(r, X_r^{t,x}) D_x V(r, X_r^{t,x}, \mathbf{p}_r) dB_r - (N_T - N_s). \tag{16}$$

Theorem 3.7. *For any $v \in \bar{\mathcal{V}}(t)$ we have*

$$Y_{t-}^{t,x,\bar{u},v} \leq Y_{t-}^{t,x,\bar{\mathbb{P}}} = V(t, x, p) \quad \bar{\mathbb{P}}\text{-a.s.}, \tag{17}$$

Proof. Since

$$
\begin{aligned}
&H(r, X_r^{t,x}, D_x V(r, X_r^{t,x}, \mathbf{p}_r), \mathbf{p}_r) \\
&\quad = \min_{u \in U} \max_{v \in V} \left\{ \langle b(r, X_r^{t,x}, u, v), D_x V(r, X_r^{t,x}, \mathbf{p}_r) \rangle + \langle \mathbf{p}_r, l(r, X_r^{t,x}, u_r, v) \rangle \right\} \\
&\quad = \max_{v \in V} \left\{ \langle b(r, X_r^{t,x}, \bar{u}_r, v), D_x V(r, X_r^{t,x}, \mathbf{p}_r) \rangle + \langle \mathbf{p}_r, l(r, X_r^{t,x}, \bar{u}_r, v) \rangle \right\} \\
&\quad \geq \langle b(r, X_r^{t,x}, \bar{u}_r, v_r), D_x V(r, X_r^{t,x}, \mathbf{p}_r) \rangle + \langle \mathbf{p}_r, l(r, X_r^{t,x}, \bar{u}_r, v_r) \rangle,
\end{aligned}
$$

(17) follows from the comparison Theorem A.3. $\quad\square$

As in [13] we define now for any $v \in \bar{\mathcal{V}}(t)$ the equivalent measure $\bar{\mathbb{P}}^{\bar{u},v} = (\Gamma_T^{\bar{u},v})\bar{\mathbb{P}}$ with

$$
\Gamma_s^{\bar{u},v} = \mathcal{E} \left(\int_t^s b(r, X_r^{t,x}, \bar{u}_r, v_r) \sigma^*(r, X_r^{t,x})^{-1} dB_r \right)
$$

for $s \geq t$ and $\Gamma_s^{\bar{u},v} = 1$ for $s < t$. By Girsanov (see e.g. Theorem III.3.24 [15]) we have the following lemma.

Lemma 3.8. *For any $p \in \Delta(I), t \in [0, T], v \in \bar{\mathcal{V}}(t)$, it holds*

(i) *$X^{t,x}$ is under $\bar{\mathbb{P}}^{\bar{u},v}$ a solution to*

$$
X_s^{t,x} = x + \int_t^s b(r, X_r^{t,x}, \bar{u}_r, v_r) dr + \int_t^s \sigma(r, X_r^{t,x}) d\bar{B}_r, \tag{18}
$$

where $\bar{B}$ is a $\bar{\mathbb{P}}^{\bar{u},v}$-Brownian motion.

(ii) *$\mathbf{p}$ is a $\bar{\mathbb{P}}^{\bar{u},v}$ martingale, such that $\mathbf{p}_s = p \; \forall s < t, \mathbf{p}_s \in \{e_i, i = 1, \ldots, I\} \; \forall s \geq T \; \mathbb{P}^{\bar{u},v}$-a.s. and $\mathbf{p}_T$ is independent of $(B_s)_{s \in (-\infty, T]}$.*

For any $\beta \in \mathcal{B}(t)$, i.e. $\beta : [t, T] \times \mathcal{C}([t, T]; \mathbb{R}^d) \times L^0([t, T]; U) \to V$ is nonanticipative with delay, we can define the process $\beta(\bar{u})_s = \beta(s, \cdot, \bar{u}_s)$. By definition $\beta(\bar{u})$ is a V-valued process which is progressively measurable with respect to the filtration $\mathcal{F}_{t,s}$ hence $\beta(\bar{u}) \in \bar{\mathcal{V}}(t)$. So we can define for any $\beta \in \mathcal{B}(t)$ the measure $\bar{\mathbb{P}}^{\bar{u},\beta(\bar{u})}$.

To take into account that the informed player knows the scenario, we define now for any scenario $i \in \{1, \ldots, I\}$ and for any $\beta \in \mathcal{B}(t)$ a probability measure $\bar{\mathbb{P}}_i^{\bar{u},\beta(\bar{u})}$ by: for all $A \in \mathcal{F}$ it holds

$$
\bar{\mathbb{P}}_i^{\bar{u},\beta(\bar{u})}[A] = \bar{\mathbb{P}}^{\bar{u},\beta(\bar{u})}[A|\mathbf{p}_T = e_i] = \frac{1}{p_i} \bar{\mathbb{P}}^{\bar{u},\beta(\bar{u})}[A \cap \{\mathbf{p}_T = e_i\}], \quad \text{if } p_i > 0,
$$

and $\bar{\mathbb{P}}_i^{\bar{u},\beta(\bar{u})}[A] = \bar{\mathbb{P}}^{\bar{u},v}[A]$. Note that by Lemma 3.8(ii) $\bar{B}$ is a $\bar{\mathbb{P}}_i^{\bar{u},v}$-Brownian motion, hence $X^{t,x}$ is under $\bar{\mathbb{P}}_i^{\bar{u},v}$ a solution of the SDE (18).

Theorem 3.9. *For any scenario $i = 1, \ldots, I$ and any strategy of the uniformed player $\beta \in \mathcal{B}(t)$ the information transmission $\bar{\mathbb{P}}_i^{\bar{u},\beta(\bar{u})}$ is optimal for the informed player in the sense that for any $\beta \in \mathcal{B}(t)$*

$$
\sum_{i=1}^I p_i \mathbb{E}_{\bar{\mathbb{P}}_i^{\bar{u},\beta(\bar{u})}} \left[\int_t^T l_i(s, X_s^{t,x}, \bar{u}_s, \beta(\bar{u})_s) ds + g_i(X_T^{t,x}) \right] \leq V(t, x, p). \tag{19}
$$

Proof. By definition we have

$$\sum_{i=1}^{I} p_i \mathbb{E}_{\bar{\mathbb{P}}_i^{\bar{u},\beta(\bar{u})}} \left[\int_t^T l_i(s, X_s^{t,x}, (\bar{u})_s, \beta(\bar{u})_s)ds + g_i(X_T^{t,x}) \right]$$

$$= \sum_{i=1}^{I} p_i \mathbb{E}_{\bar{\mathbb{P}}^{\bar{u},\beta(\bar{u})}} \left[\int_t^T l_i(s, X_s^{t,x}, \bar{u}_s, \beta(\bar{u})_s)ds + g_i(X_T^{t,x}) \Big| \mathbf{p}_T = e_i \right].$$

Furthermore

$$\sum_{i=1}^{I} p_i \mathbb{E}_{\bar{\mathbb{P}}^{\bar{u},\beta(\bar{u})}} \left[\int_t^T l_i(s, X_s^{t,x}, \bar{u}_s, \beta(\bar{u})_s)ds + g_i(X_T^{t,x}) \Big| \mathbf{p}_T = e_i \right]$$

$$= \sum_{i=1}^{I} \bar{\mathbb{P}}^{\bar{u},\beta(\bar{u})}[\mathbf{p}_T = e_i] \mathbb{E}_{\bar{\mathbb{P}}^{\bar{u},\beta(\bar{u})}} \left[\int_t^T l_i(s, X_s^{t,x}, \bar{u}_s, \beta(\bar{u})_s)ds + g_i(X_T^{t,x}) \Big| \mathbf{p}_T = e_i \right]$$

$$= \sum_{i=1}^{I} \mathbb{E}_{\bar{\mathbb{P}}^{\bar{u},\beta(\bar{u})}} \left[1_{\{\mathbf{p}_T=e_i\}} \int_t^T l_i(s, X_s^{t,x}, \bar{u}_s, \beta(\bar{u})_s)ds + g_i(X_T^{t,x}) \right]$$

$$= \mathbb{E}_{\bar{\mathbb{P}}^{\bar{u},\beta(\bar{u})}} \left[\left\langle \mathbf{p}_T, \int_t^T l(s, X_s^{t,x}, \bar{u}_s, \beta(\bar{u})_s)ds \right\rangle + \langle \mathbf{p}_T, g(X_T^{t,x}) \rangle \right]$$

$$= \mathbb{E}_{\bar{\mathbb{P}}^{\bar{u},\beta(\bar{u})}} \left[\int_t^T \langle \mathbf{p}_s, l(s, X_s^{t,x}, \bar{u}_s, \beta(\bar{u})_s) \rangle ds + \langle \mathbf{p}_T, g(X_T^{t,x}) \rangle \right],$$

where in the last step we used the product rule for the $\bar{\mathbb{P}}^{\bar{u},\beta(\bar{u})}$-martingale $\mathbf{p}$ and the adapted finite variation process $\int_t^\cdot l(s, X_s^{t,x}, \bar{u}_s, \beta(\bar{u})_s)ds$.

Furthermore we have

$$\mathbb{E}_{\bar{\mathbb{P}}^{\bar{u},\beta(\bar{u})}} \left[\int_t^T \langle \mathbf{p}_s, l(s, X_s^{t,x}, \bar{u}_s, \beta(\bar{u})_s) \rangle ds + \langle \mathbf{p}_T, g(X_T^{t,x}) \rangle \right] = \mathbb{E}_{\bar{\mathbb{P}}^{\bar{u},\beta(\bar{u})}} \left[Y_{t-}^{t,x,\bar{u},\beta(\bar{u})} \right],$$

since by Girsanov $Y_s^{t,x,\bar{u},\beta(\bar{u})}$ is under $\bar{\mathbb{P}}^{\bar{u},\beta(\bar{u})}$ given by

$$Y_s^{t,x,\bar{u},\beta(\bar{u})} = \langle \mathbf{p}_T, g(X_T^{t,x}) \rangle + \int_s^T \langle \mathbf{p}_r, l(r, X_r^{t,x}, \bar{u}_r, \beta(\bar{u})_r) \rangle dr$$

$$- \int_s^T \sigma^*(r, X_r^{t,x}) D_x V(r, X_r^{t,x}, \mathbf{p}_r) d\bar{B}_r - (N_T - N_s). \tag{20}$$

So since by Theorem 3.7 $Y_{t-}^{t,x,\bar{u},\beta(\bar{u})} \le V(t, x, p) \bar{\mathbb{P}}$-a.s. and $\bar{\mathbb{P}}$ is equivalent to $\bar{\mathbb{P}}^{\bar{u},\beta(\bar{u})}$, we have

$$\mathbb{E}_{\bar{\mathbb{P}}^{\bar{u},\beta(\bar{u})}} \left[Y_{t-}^{t,x,\bar{u},\beta(\bar{u})} \right] \le V(t, x, p). \quad \square$$

Remark 3.10. In the simpler case of [6] the representation (13) allowed to derive an optimal random control for the informed player in a direct feedback form. Here however there are significant differences. By the Girsanov transformation we have for each $\beta \in \mathcal{B}(t)$ at each time $s \in [t, T]$ an optimal reaction $\bar{u}_s = u^*(s, X_s^{t,x}, D_x V(s, X_s^{t,x}, \mathbf{p}_s), \mathbf{p}_s)$ of the informed player. It depends on the state of the system, i.e. $X^{t,x}$ under $\bar{\mathbb{P}}_i^{\bar{u},\beta(\bar{u})}$ and the shifted randomization $\mathbf{p}$ under the optimal measure $\bar{\mathbb{P}}_i^{\bar{u},\beta(\bar{u})}$. Since this shift depends on the strategy β of the uninformed player, we do not find a random control but a kind of random strategy for the informed player. Note that this "strategy" – none of the less giving us a recipe how the informed player can generate

the optimal information flow – is in general not of the form required in Definition 2.4. To get a classical random strategy it would be necessary to show a certain structure of the optimal measure $\bar{\mathbb{P}}$. In a subsequent paper we show how this can be established for ϵ-optimal measures leading to ϵ-optimal strategies in the sense of Definition 2.4.

4. Proof of Theorem 3.4

4.1. The function $W(t, x, p)$ and ϵ-optimal strategies

Recall that we defined $W(t, x, p)$ $\mathbb{Q}$-a.s. as $\operatorname{essinf}_{\mathbb{P}\in\mathcal{P}(t,p)} Y_{t-}^{t,x,\mathbb{P}}$, where by definition a random variable ξ is called $\operatorname{essinf}_{\mathbb{P}\in\mathcal{P}(t,p)} Y_{t-}^{t,x,\mathbb{P}}$, if

(i) $\xi \leq Y_{t-}^{t,x,\mathbb{P}}$, $\mathbb{Q}$-a.s., for any $\mathbb{P} \in \mathcal{P}(t, p)$
(ii) if there is another random variable η such that $\eta \leq Y_{t-}^{t,x,\mathbb{P}}$, $\mathbb{Q}$-a.s., for any $\mathbb{P} \in \mathcal{P}(t, p)$, then $\eta \leq \xi$, $\mathbb{Q}$-a.s.

So by its very definition $W(t, x, p)$ is merely a $\mathcal{F}_{t-}$ measurable random field. However we show that it is deterministic and hence a good candidate to represent the deterministic value function $V(t, x, p)$. Our proof is mainly based on the methods in [3].

Proposition 4.1. *For any $t \in [0, T]$, $x \in \mathbb{R}^d$, $p \in \Delta(I)$ it holds*

$$W(t, x, p) = \mathbb{E}_{\mathbb{Q}}[W(t, x, p)] \quad \mathbb{Q}\text{-a.s.} \tag{21}$$

Hence identifying $W(t, x, p)$ with its deterministic version $\mathbb{E}_{\mathbb{Q}}[W(t, x, p)]$ we can consider $W : [0, T] \times \mathbb{R}^d \times \Delta(I) \to \mathbb{R}$ as a deterministic function.

To prove that $W(t, x, p)$ is deterministic it suffices to show that it is independent of the σ-algebra $\sigma(B_s, s \in [0, t])$. Since **p** is on $[0, t[$ $\mathbb{Q}$-a.s. a constant Proposition 4.1 follows with Lemma 4.1 in [3].

To show the independence of $\sigma(B_s, s \in [0, t])$ we will use as in [3] a perturbation of $\mathcal{C}([0, T]; \mathbb{R}^d)$ with certain elements of the Cameron–Martin space. Let H denote the Cameron–Martin space of all absolutely continuous elements $h \in \mathcal{C}([0, T]; \mathbb{R}^d)$, whose Radon–Nikodym derivative $\dot{h}$ belongs to $L^2([0, T]; \mathbb{R}^d)$. Denote $H_t = \{h \in H : h(\cdot) = h(\cdot \wedge t)\}$. For any $h \in H_t$, we define for all $(\omega_p, \omega_B) \in \mathcal{D}([0, T]; \Delta(I)) \times \mathcal{C}([0, T]; \mathbb{R}^d)$ the mapping $\tau_h(\omega_p, \omega_B) := (\omega_p, \omega_B + h)$. Then $\tau_h : \mathcal{D}([0, T]; \Delta(I)) \times \mathcal{C}([0, T]; \mathbb{R}^d) \to \mathcal{D}([0, T]; \Delta(I)) \times \mathcal{C}([0, T]; \mathbb{R}^d)$ is a $\mathcal{F} - \mathcal{F}$ measurable bijection with $[\tau_h]^{-1} = \tau_{-h}$.

Lemma 4.2. *For any $h \in H_t$*

$$W(t, x, p) \circ \tau_h = W(t, x, p). \tag{22}$$

Proof. Obviously $\tau_h, \tau_h^{-1} : \mathcal{D}([0, T]; \Delta(I)) \times \mathcal{C}([0, T]; \mathbb{R}^d) \to \mathcal{D}([0, T]; \Delta(I)) \times \mathcal{C}([0, T]; \mathbb{R}^d)$ is $\mathcal{F}_t - \mathcal{F}_t$ measurable and $(B_s - B_t) \circ \tau_h = (B_s - B_t)$ for all $s \in [t, T]$.

Step 1: Observe that $X_s^{t,x} \circ \tau_h = X_s^{t,x}$ for all $s \in [t, T]$. Then $Y^{t,x,\mathbb{P}} \circ \tau_h$ is the solution to the BSDE

$$(Y^{t,x,\mathbb{P}} \circ \tau_h)_s = \langle \mathbf{p}_T, g(X_T^{t,x}) \rangle + \int_s^T H(r, X_r^{t,x}, (Z^{t,x,\mathbb{P}} \circ \tau_h)_r, \mathbf{p}_r) ds$$

$$- \int_s^T \sigma^*(r, X_r^{t,x})(Z^{t,x,\mathbb{P}} \circ \tau_h)_r dB_r - (N \circ \tau_h)_T + (N \circ \tau_h)_s \tag{23}$$

which is the original BSDE (10) however under the different $\mathbb{P} \circ [\tau_h]^{-1}$ dynamics for **p**.

Furthermore $X_{s\in[t,T]}^{t,x}$ under $\mathbb{P}$ and under $\mathbb{P} \circ [\tau_h]^{-1}$ are by Girsanov $\mathbb{P}$-a.s. equal. So under $\mathbb{P} \circ [\tau_h]^{-1}$ the process $Y^{t,x,\mathbb{P}\circ[\tau_h]^{-1}}$ by Girsanov solves (23). Since the solution of (23) is unique we have in particular

$$Y_{t-}^{t,x,\mathbb{P}} \circ \tau_h = Y_{t-}^{t,x,\mathbb{P}\circ[\tau_h]^{-1}}. \tag{24}$$

Step 2: We claim that

$$\left(\text{essinf}_{\mathbb{P}\in\mathcal{P}(t,p)} Y_{t-}^{t,x,\mathbb{P}}\right) \circ \tau_h = \text{essinf}_{\mathbb{P}\in\mathcal{P}(t,p)} \left(Y_{t-}^{t,x,\mathbb{P}} \circ \tau_h\right) \quad \mathbb{Q}\text{-a.s.} \tag{25}$$

Observe that the law of τ_h is given by

$$\mathbb{P} \circ [\tau_h]^{-1} = \exp\left(\int_0^t \dot{h}_s dB_s - \frac{1}{2}\int_0^t |\dot{h}_s|^2 ds\right) \mathbb{P} \tag{26}$$

for all measures $\mathbb{P}$ on Ω. Define $I(t,x,p) = \text{essinf}_{\mathbb{P}\in\mathcal{P}(t,p)} Y_{t-}^{t,x,\mathbb{P}}$. Then $I(t,x,p) \le Y_{t-}^{t,x,\mathbb{P}}$. Since $\mathbb{Q} \circ [\tau_h]^{-1}$ is equivalent to $\mathbb{Q}$ on $\mathcal{F}_{t-}$, we have $I(t,x,p) \circ \tau_h \le Y_{t-}^{t,x,\mathbb{P}} \circ \tau_h$ $\mathbb{Q}$-a.s.

Furthermore let ξ be a $\mathcal{F}_{t-}$-measurable random variable, such that $\xi \le Y_{t-}^{t,x,\mathbb{P}} \circ \tau_h$ $\mathbb{Q}$-a.s. Then $\xi \circ [\tau_h]^{-1} \le Y_{t-}^{t,x,\mathbb{P}}$ $\mathbb{Q}$-a.s. hence it holds $\xi \circ [\tau_h]^{-1} \le I(t,x,p)$, so $\xi \le I(t,x,p) \circ \tau_h$.

Consequently we have

$$I(t,x,p) \circ \tau_h = \text{essinf}_{\mathbb{P}\in\mathcal{P}(t,p)}(Y_{t-}^{t,x,\mathbb{P}} \circ \tau_h).$$

Step 3: Using (24) and (25) we have $\mathbb{Q}$-a.s.

$$\begin{aligned}
W(t,x,p) \circ \tau_h &= (\text{essinf}_{\mathbb{P}\in\mathcal{P}(t,p)} Y_{t-}^{t,x,\mathbb{P}}) \circ \tau_h \\
&= \text{essinf}_{\mathbb{P}\in\mathcal{P}(t,p)}(Y_{t-}^{t,x,\mathbb{P}} \circ \tau_h) \\
&= \text{essinf}_{\mathbb{P}\in\mathcal{P}(t,p)} Y_{t-}^{t,x,\mathbb{P}\circ[\tau_h]^{-1}}.
\end{aligned}$$

Note that in general $\mathbb{P} \circ [\tau_h]^{-1} \notin \mathcal{P}(t,p)$, since under $\mathbb{P} \circ [\tau_h]^{-1}$ the process B is no longer a Brownian motion on $[0,t]$. We define $\mathbb{P}^h$ on $\Omega = \Omega_{0,t} \times \Omega_t$, such that

$$\mathbb{P}^h = (\delta(p) \otimes \mathbb{P}^0) \otimes (\mathbb{P} \circ [\tau_h]^{-1}|_{\Omega_t}),$$

where $\delta(p)$ is the measure under which $\mathbf{p}$ is constant and equal to p and $\mathbb{P}^0$ is a Wiener measure on $\Omega_{0,t}$.

So by definition $(B_s)_{s\in[t,T]}$ is a Brownian motion under $\mathbb{P}^h$. Also $(\mathbf{p}_s)_{s\in[t,T]}$ is still a martingale under $\mathbb{P}^h$. We can see this immediately, since for all $t \le s \le r \le T$ by (26)

$$\mathbb{E}_{\mathbb{P}^h}[\mathbf{p}_r|\mathcal{F}_s] = \mathbb{E}_{\mathbb{P}\circ[\tau_h]^{-1}}[\mathbf{p}_r|\mathcal{F}_s] = \mathbb{E}_{\mathbb{P}}[\mathbf{p}_r|\mathcal{F}_s].$$

Furthermore the remaining conditions of Definition 3.1 are obviously met. Hence $\mathbb{P}^h \in \mathcal{P}(t,p)$ and, since $Y^{t,x,\mathbb{P}\circ[\tau_h]^{-1}}$ is a solution of a BSDE, we have

$$Y_{t-}^{t,x,\mathbb{P}\circ[\tau_h]^{-1}} = Y_{t-}^{t,x,\mathbb{P}^h}.$$

On the other hand by considering $\mathbb{P} \circ \tau_h$ one can associate to any $\mathbb{P} \in \mathcal{P}(t,p)$ a $\mathbb{P}^{-h} \in \mathcal{P}(t,p)$, such that

$$Y_{t-}^{t,x,\mathbb{P}^{-h}\circ[\tau_h]^{-1}} = Y_{t-}^{t,x,\mathbb{P}}.$$

Hence $\left\{ Y_{t-}^{t,x,\mathbb{P}\circ[\tau_h]^{-1}} : \mathbb{P} \in \mathcal{P}(t,p) \right\} = \left\{ Y_{t-}^{t,x,\mathbb{P}} : \mathbb{P} \in \mathcal{P}(t,p) \right\}$ and

$$\text{essinf}_{\mathbb{P}\in\mathcal{P}(t,p)} Y_{t-}^{t,x,\mathbb{P}\circ[\tau_h]^{-1}} = \text{essinf}_{\mathbb{P}\in\mathcal{P}(t,p)} Y_{t-}^{t,x,\mathbb{P}} = W(t,x,p). \quad \square$$

In the following section we establish some regularity results and a dynamic programming principle. To this end we work with ϵ-optimal measures. Note that since we are taking the essential infimum over a family of random variables, existence of an ϵ-optimal $\mathbb{P}^\epsilon \in \mathcal{P}(t,p)$ is not standard. Therefore we provide a technical lemma, the proof of which is also strongly inspired by [3].

Lemma 4.3. *For any $(t,x,p) \in [0,T[\times\mathbb{R}^d \times \Delta(I)$ there is an ϵ-optimal $\mathbb{P}^\epsilon \in \mathcal{P}(t,p)$ in the sense that*

$$Y_{t-}^{t,x,\mathbb{P}^\epsilon} \leq W(t,x,p) + \epsilon \quad \mathbb{Q}\text{-a.s.}$$

Proof. Note that there exists a sequence $(\mathbb{P}^n)_{n\in\mathbb{N}}$, $\mathbb{P}^n \in \mathcal{P}(t,p)$, such that

$$W(t,x,p) = \text{essinf}_{\mathbb{P}\in\mathcal{P}(t,p)} Y_{t-}^{t,x,\mathbb{P}} = \inf_{n\in\mathbb{N}} Y_{t-}^{t,x,\mathbb{P}^n}.$$

For an $\epsilon > 0$ set $\Gamma_n := \{W(t,x,p) + \epsilon \geq Y_{t-}^{t,x,\mathbb{P}^n}\} \in \mathcal{F}_{t-}$ for any $n \in \mathbb{N}$. Then $\bar{\Gamma}_1 := \Gamma_1$, $\bar{\Gamma}_n := \Gamma_n \setminus (\cup_{m=1,\dots,n-1} \bar{\Gamma}_m)$ for $n \geq 2$ form a $\mathcal{F}_{t-}$ measurable partition of Ω.

We define $\mathbb{P}^\epsilon$, such that on $\Omega = \Omega_{0,t} \times \Omega_t$

$$\mathbb{P}^\epsilon = (\delta(p) \otimes \mathbb{P}^0) \otimes \left(\sum_{n\in\mathbb{N}} 1_{\bar{\Gamma}_n} \mathbb{P}^n|_{\Omega_t} \right),$$

where $\delta(p)$ denotes the measure under which $\mathbf{p}$ is constant and equal to p and $\mathbb{P}^0$ is a Wiener measure on $\Omega_{0,t}$.

So by definition $(B_s)_{s\in[t,T]}$ is a Brownian motion under $\mathbb{P}^\epsilon$ and $(\mathbf{p}_s)_{s\in[t,T]}$ is still a martingale under $\mathbb{P}^\epsilon$, since for all $t \leq s \leq r \leq T$

$$\mathbb{E}_{\mathbb{P}^\epsilon}[\mathbf{p}_r|\mathcal{F}_s] = \sum_{n\in\mathbb{N}} \mathbb{E}_{\mathbb{P}^n}[1_{\bar{\Gamma}_n}\mathbf{p}_r|\mathcal{F}_s] = \sum_{n\in\mathbb{N}} 1_{\bar{\Gamma}_n}\mathbb{E}_{\mathbb{P}^n}[\mathbf{p}_r|\mathcal{F}_s] = \sum_{n\in\mathbb{N}} 1_{\bar{\Gamma}_n}\mathbf{p}_s = \mathbf{p}_s.$$

Again the remaining conditions of Definition 3.1 are obviously met. Thus $\mathbb{P}^\epsilon \in \mathcal{P}(t,p)$ and it holds

$$W(t,x,p) + \epsilon \geq \sum_{n\in\mathbb{N}} 1_{\bar{\Gamma}_n} Y_{t-}^{t,x,\mathbb{P}^n} = Y_{t-}^{t,x,\mathbb{P}^\epsilon}. \quad \square$$

Furthermore for technical reasons we introduce the set $\mathcal{P}^f(t,p)$ as the set of all measures $\mathbb{P} \in \mathcal{P}(t,p)$, such that there exists a finite set $S \subset \Delta(I)$ with $\mathbf{p}_s \in S$ $\mathbb{P}$-a.s. for all $s \in [t,T]$.

Remark 4.4. Note that for any $(t,x,p) \in [0,T[\times\mathbb{R}^d \times \Delta(I)\epsilon > 0$ we can choose an ϵ-optimal $\mathbb{P}^\epsilon$ in the smaller class $\mathcal{P}^f(t,p)$. The idea of the proof is as follows: first choose $\frac{\epsilon}{2}$-optimal measure $\mathbb{P}^\epsilon \in \mathcal{P}(t,p)$. Since $\mathbf{p}$ progressively measurable we can approximate it by an elementary processes $\bar{\mathbf{p}}^\epsilon$, such that with BSDE estimates one has

$$|Y_{t-}^{t,x,\mathbb{P}^\epsilon} - Y_{t-}^{t,x,\bar{\mathbb{P}}^\epsilon}| \leq \frac{\epsilon}{2},$$

where $\bar{\mathbb{P}}^\epsilon$ distribution of $(B, \bar{\mathbf{p}}^\epsilon)$.

4.2. Some regularity results

For technical reasons we will consider the BSDE (10) with a slightly different notation. For any $t \in [0, T]$, $x \in \mathbb{R}^d$, $\mathbb{P} \in \mathcal{P}(t, p)$ let

$$Y_s^{t,x,\mathbb{P}} = \langle \mathbf{p}_T, g(X_T^{t,x}) \rangle + \int_s^T \tilde{H}(r, X_r^{t,x}, z_r^{t,x,\mathbb{P}}, \mathbf{p}_r) dr - \int_s^T z_r^{t,x,\mathbb{P}} dB_r - N_T^{\mathbb{P}} + N_s^{\mathbb{P}}, \quad (27)$$

where $\tilde{H}(t, x, p, \xi) = H(t, x, p, (\sigma^*(t, x))^{-1}\xi)$. Setting $Z_s^{t,x,\mathbb{P}} = (\sigma^*(s, X_s^{t,x}))^{-1} z_s^{t,x,\mathbb{P}}$ then gives the solution to (10).

In the following we will use the notation $Y_s^{t,x,\mathbb{P}} = Y_s^{t,x}$, $z^{t,x,\mathbb{P}} = z^{t,x}$, $N^{\mathbb{P}} = N$, whenever we work under a fixed $\mathbb{P} \in \mathcal{P}(t, p)$.

Remark 4.5. Observe that by (H) we have that $\tilde{H}$ is uniformly Lipschitz continuous in (ξ, p) uniformly in (t, x) and Lipschitz continuous in (t, x) with Lipschitz constant $c(1 + |\xi|)$, i.e. it holds for all $t, t' \in [0, T]$, $x, x' \in \mathbb{R}^d$, $\xi, \xi' \in \mathbb{R}^d$, $p, p' \in \Delta(I)$

$$|\tilde{H}(t, x, \xi, p)| \leq c(1 + |\xi|) \quad (28)$$

and

$$\begin{aligned}
|\tilde{H}(t, x, \xi, p) - \tilde{H}(t', x', \xi', p')| &\leq c(1 + |\xi|)(|x - x'| + |t - t'|) \\
&\quad + c|\xi - \xi'| + c|p - p'|.
\end{aligned} \quad (29)$$

Proposition 4.6. $W(t, x, p)$ *is uniformly Lipschitz continuous in x and uniformly Hölder continuous in t.*

Proof. The Lipschitz continuity can be shown by straightforward calculation using Proposition A.2. For the Hölder continuity in time, let $t, t' \in [0, T]$ such that $t' \leq t$ and assume $W(t', x, p) > W(t, x, p)$. Let $\mathbb{P}^\epsilon \in \mathcal{P}(t, p)$ be ϵ-optimal for $W(t, x, p)$ for a sufficiently small ϵ. Note that since $t' \leq t$ it holds $\mathbb{P}^\epsilon \in \mathcal{P}(t', p)$. Then we have with Hölder inequality and Proposition A.2

$$\begin{aligned}
0 &\leq W(t', x, p) - W(t, x, p) - \epsilon \\
&\leq \mathbb{E}_{\mathbb{P}^\epsilon} \left[\int_{t'}^t \tilde{H}(s, X_s^{t',x}, z_s^{t',x}, \mathbf{p}_s) ds \right] + \mathbb{E}_{\mathbb{P}^\epsilon} \left[\int_t^T \tilde{H}(s, X_s^{t',x}, z_s^{t',x}, \mathbf{p}_s) \right. \\
&\quad \left. - \tilde{H}(s, X_s^{t,x}, z_s^{t,x}, \mathbf{p}_s) ds + \langle \mathbf{p}_T, g(X_T^{t',x}) - g(X_T^{t,x}) \rangle \right] \\
&\leq c\,\mathbb{E}_{\mathbb{P}^\epsilon} \left[\int_{t'}^t (1 + |z_s^{t',x}|) ds \right] \\
&\quad + c\,\mathbb{E}_{\mathbb{P}^\epsilon} \left[\int_t^T \left((1 + |z_s^{t',x}|)|X_s^{t',x} - X_s^{t,x}| + |z_s^{t',x} - z_s^{t,x}| \right) ds + |X_T^{t',x} - X_T^{t,x}| \right] \\
&\leq c|t' - t|^{\frac{1}{2}} + c \left(\mathbb{E}_{\mathbb{P}^\epsilon} \left[\int_t^T |X_s^{t',x} - X_s^{t,x}|^2 ds + |X_T^{t',x} - X_T^{t,x}|^2 \right] \right)^{\frac{1}{2}} \\
&\leq c|t' - t|^{\frac{1}{2}}.
\end{aligned}$$

For the case $t' \leq t$, $W(t', x, p) < W(t, x, p)$ choose a $\mathbb{P}^\epsilon \in \mathcal{P}(t', p)$, which is ϵ-optimal for $W(t', x, p)$ for a sufficiently small ϵ. We define then the probability measure $\bar{\mathbb{P}}^\epsilon$, such that on $\Omega = \Omega_{0,t} \times \Omega_t$

$$\mathbb{P}^\epsilon = (\delta(p) \otimes \mathbb{P}^0) \otimes \mathbb{P}^\epsilon|_{\Omega_t},$$

where $\delta(p)$ denotes the measure under which $\mathbf{p}$ is constant and equal to p and $\mathbb{P}^0$ is a Wiener measure on $\Omega_{0,t}$. So by definition $(B_s)_{s \in [t,T]}$ is a Brownian motion under $\bar{\mathbb{P}}^\epsilon$. Furthermore the remaining conditions of Definition 3.1 are met, hence $\bar{\mathbb{P}}^\epsilon \in \mathcal{P}(t, p)$ and the same argument as above applies in that case. $\square$

Proposition 4.7. *$W(t, x, p)$ is convex and uniformly Lipschitz continuous with respect to p.*

Proof. 1. To show the convexity in p let $p_1, p_2 \in \Delta(I)$ and let $\mathbb{P}^1 \in \mathcal{P}(t, p_1)$, $\mathbb{P}^2 \in \mathcal{P}(t, p_2)$ be ϵ-optimal for $W(t, x, p_1)$, $W(t, x, p_2)$ respectively. For $\lambda \in [0, 1]$ define a martingale measure $\mathbb{P}^\lambda \in \mathcal{P}(t, p_\lambda)$, such that for all measurable $\phi : \mathcal{D}([0, T]; \Delta(I)) \times \mathcal{C}([0, T]; \mathbb{R}^d) \to \mathbb{R}_+$

$$\mathbb{E}_{\mathbb{P}^\lambda}[\phi(\mathbf{p}, B)] = \lambda \mathbb{E}_{\mathbb{P}^1}[\phi(\mathbf{p}, B)] + (1 - \lambda)\mathbb{E}_{\mathbb{P}^2}[\phi(\mathbf{p}, B)].$$

Observe that we just take two copies Ω_1, Ω_2 of the same space with weights λ, $(1 - \lambda)$. So for the respective solutions of the BSDE (27) it holds

$$Y^{t,x,\mathbb{P}^\lambda} = 1_{\{\Omega_1\}} Y^{t,x,\mathbb{P}^1} + 1_{\{\Omega_2\}} Y^{t,x,\mathbb{P}^2}.$$

Hence

$$W(t, x, p_\lambda) \leq Y^{t,x,\mathbb{P}^\lambda}_{t-} = 1_{\{\Omega_1\}} Y^{t,x,\mathbb{P}^1}_{t-} + 1_{\{\Omega_2\}} Y^{t,x,\mathbb{P}^2}_{t-}$$

$$\leq 1_{\{\Omega_1\}} W(t, x, p_1) + 1_{\{\Omega_2\}} W(t, x, p_2) + 2\epsilon$$

and the convexity follows by taking expectation, since ϵ can be chosen arbitrarily small.

2. Next we prove the uniform Lipschitz continuity in p. Since we have convexity in p, it suffices to show the Lipschitz continuity with respect to p on the extreme points e_i. Observe that $\mathcal{P}(t, e_i)$ consists in the single probability measure $\delta(e_i) \otimes \mathbb{P}^0$, where $\delta(e_i)$ is the measure under which $\mathbf{p}$ is constant and equal to e_i and $\mathbb{P}^0$ is a Wiener measure.
Assume $W(t, x, p) - W(t, x, e_i) > 0$ and let $\mathbb{P}^\epsilon \in \mathcal{P}^f(t, p)$ be ϵ-optimal for $W(t, x, p)$ for a sufficiently small ϵ. Then

$$0 \leq W(t, x, p) - W(t, x, e_i) - \epsilon$$

$$\leq \mathbb{E}_{\mathbb{P}^\epsilon}\left[\int_t^T \tilde{H}(s, X_s^{t,x}, z_s^{t,x}, \mathbf{p}_s) - \tilde{H}(s, X_s^{x,t}, z_s^{t,x,e_i}, e_i)ds + \langle \mathbf{p}_T - e_i, g(X_T^{x,t})\rangle \Big| \mathcal{F}_{t-}\right]$$

$$\leq Y_{t-}^{t,x} - Y_{t-}^{t,x,e_i}.$$

By the uniform Lipschitz continuity of $\tilde{H}$ in ξ and p it holds

$$Y_{t-}^{t,x} - Y_{t-}^{t,x,e_i} \leq \langle \mathbf{p}_T - e_i, g(X_T^{x,t})\rangle + c \int_t^T \left(|z_s^{t,x} - z_s^{t,x,e_i}| + |\mathbf{p}_s - e_i|\right) ds$$

$$- \int_t^T (z_s^{t,x} - z_s^{t,x,e_i})dB_s - (N - N^{e_i})_T + (N - N^{e_i})_{t-}$$

$$\leq c \left(\int_t^T (1 - (\mathbf{p}_s)_i) ds + 1 - (\mathbf{p}_T)_i \right) + c \int_t^T |z_s^{t,x} - z_s^{t,x,e_i}| ds$$

$$- \int_t^T (z_s^{t,x} - z_s^{t,x,e_i}) dB_s - (N - N^{e_i})_T + (N - N^{e_i})_{t-},$$

where we used the estimate

$$\int_t^T |\mathbf{p}_s - e_i| ds + |\mathbf{p}_T - e_i| \leq c \left(\int_t^T (1 - (\mathbf{p}_s)_i) ds + 1 - (\mathbf{p}_T)_i \right).$$

We define $\hat{Y}$ as the unique solution to the BSDE

$$\hat{Y}_s = c \left(\int_s^T (1 - (\mathbf{p}_r)_i) dr + 1 - (\mathbf{p}_T)_i \right) + c \int_s^T |\hat{z}_r| dr - \int_s^T \hat{z}_r dB_r - (\hat{N}_T - \hat{N}_s).$$

Then by comparison (Theorem A.3) we have

$$Y_{t-}^{t,x} - Y_{t-}^{t,x,e_i} \leq \hat{Y}_{t-}.$$

We claim that $\hat{Y}_s = (1 - (\mathbf{p}_s)_i) \tilde{Y}_s$, where $\tilde{Y}_s$ is on $s \in [t, T]$ the solution to

$$\tilde{Y}_s = c + c (T - s) + \int_s^T |\tilde{z}_r| dr - \int_s^T \tilde{z}_r dB_r. \tag{30}$$

This follows directly by applying the Itô formula

$$(1 - (\mathbf{p}_s)_i) \tilde{Y}_s = c (1 - (\mathbf{p}_T)_i) + c \int_s^T (1 - (\mathbf{p}_r)_i) dr + \int_s^T |(1 - (\mathbf{p}_r)_i) \tilde{z}_r| ds$$

$$- \int_s^T (1 - (\mathbf{p}_r)_i) \tilde{z}_r dB_r + \int_s^T \tilde{Y}_r d(\mathbf{p}_r)_i$$

and identifying $\hat{z}_s = (1 - (\mathbf{p}_s)_i) \tilde{z}_s$ and $\tilde{N}_s = \int_0^s \tilde{Y}_r d(\mathbf{p}_r)_i$ which is by the definition of $\mathcal{P}^f(t, p)$ purely discontinuous, hence strongly orthogonal to B. Furthermore

$$1 - (\mathbf{p}_{t-})_i = 1 - p_i \leq c \sum_j |(p)_j - \delta_{ij}| \leq c\sqrt{I}|p - e_i|,$$

hence

$$Y_{t-}^{t,x} - Y_{t-}^{t,x,e_i} \leq \hat{Y}_{t-} = (1 - (\mathbf{p}_{t-})_i)\tilde{Y}_{t-} \leq c\sqrt{I}|p - e_i|\tilde{Y}_{t-}.$$

It is well known (see e.g. [10]) that, the solution $\tilde{Y}$ to (30) is continuous, bounded in $\mathcal{L}^1$ and $\tilde{Y}_t$ is deterministic. So $\tilde{Y}_{t-} = \tilde{Y}_t \leq c$ and we have

$$Y_{t-}^{t,x} - Y_{t-}^{t,x,e_i} \leq c\sqrt{I}|p - e_i|. \quad \square$$

4.3. Dynamic programming principle

Next we show that a dynamic programming principle holds.

Theorem 4.8. *For all $(t, x, p) \in [0, T] \times \mathbb{R}^d \times \Delta(I), t' \in [t, T]$*

$$W(t, x, p) = \text{essinf}_{\mathbb{P} \in \mathcal{P}(t,p)} \mathbb{E}_{\mathbb{P}} \left[\int_t^{t'} \tilde{H}(s, X_s^{t,x}, z_s^{t,x,\mathbb{P}}, \mathbf{p}_s) ds + W(t', X_{t'}^{t,x}, \mathbf{p}_{t'-}) \Big| \mathcal{F}_{t-} \right].$$

Note that by Remark 4.4 it suffices to show

$$W(t, x, p) = \operatorname{essinf}_{\mathbb{P} \in \mathcal{P}^f(t,p)}$$

$$\times \mathbb{E}_{\mathbb{P}} \left[\int_t^{t'} \tilde{H}(s, X_s^{t,x}, z_s^{t,x,\mathbb{P}}, \mathbf{p}_s) ds + W(t', X_{t'}^{t,x}, \mathbf{p}_{t'-}) \bigg| \mathcal{F}_{t-} \right] \tag{31}$$

for all $(t, x, p) \in [0, T] \times \mathbb{R}^d \times \Delta(I)$, $t' \in [t, T]$.

For the proof of Theorem 4.8 we first show two lemmas.

Lemma 4.9. *Under any* $\mathbb{P} \in \mathcal{P}^f(t, p)$

$$Y_{t'-}^{t,x,\mathbb{P}} \geq W(t', X_{t'}^{t,x}, \mathbf{p}_{t'-}). \tag{32}$$

Proof. Fix $\mathbb{P} \in \mathcal{P}^f(t, p)$ and $t' \in [t, T]$. Let $(A_l)_{l \in \mathbb{N}}$ be a partition of $\mathbb{R}^d$ in Borel sets, such that $\operatorname{diam}(A_l) \leq \epsilon$ and choose for any $l \in \mathbb{N}$ some $y^l \in A_l$. Let z^{t',y^l} denote the z term of the solution of BSDE (27) with forward dynamics X^{t',y^l} instead of $X^{t,x}$. First observe that

$$Y_{t'-}^{t,x} = \mathbb{E}_{\mathbb{P}} \left[\int_{t'}^T \tilde{H}(s, X_s^{t,x}, z_s^{t,x}, \mathbf{p}_s) ds + \langle \mathbf{p}_T, g(X_T^{t,x}) \rangle \bigg| \mathcal{F}_{t'-} \right]$$

$$\geq \sum_{l=1}^{\infty} \mathbb{E}_{\mathbb{P}} \left[\int_{t'}^T \tilde{H}(s, X_s^{t',y^l}, z_s^{t',y^l}, \mathbf{p}_s) ds + \langle \mathbf{p}_T, g(X_T^{t',y^l}) \rangle \bigg| \mathcal{F}_{t'-} \right] 1_{\{X_{t'}^{t,x} \in A^l\}}$$

$$- c \sum_{l=1}^{\infty} \mathbb{E}_{\mathbb{P}} \left[\int_{t'}^T \left(|z_s^{t',y^l} - z_s^{x,t}| + (1 + |z_s^{x,t}|)|X_s^{t',y^l} - X_s^{t,x}| \right) ds \right.$$

$$\left. + |X_T^{t',y^l} - X_T^{t,x}| \bigg| \mathcal{F}_{t'-} \right] 1_{\{X_{t'}^{t,x} \in A^l\}}$$

where by Hölder inequality, Proposition A.2 and Gronwall inequality

$$\sum_{l=1}^{\infty} \mathbb{E}_{\mathbb{P}} \left[\int_{t'}^T \left(|z_s^{t',y^l} - z_s^{x,t}| + (1 + |z_s^{x,t}|)|X_s^{t',y^l} - X_s^{t,x}| \right) ds \right.$$

$$\left. + |X_T^{t',y^l} - X_T^{t,x}| \bigg| \mathcal{F}_{t'-} \right] 1_{\{X_{t'}^{t,x} \in A^l\}}$$

$$\leq c \sum_{l=1}^{\infty} \left(\mathbb{E}_{\mathbb{P}} \left[\int_{t'}^T |X_s^{t',y^l} - X_s^{t,x}|^2 ds + |X_T^{t',y^l} - X_T^{t,x}|^2 \bigg| \mathcal{F}_{t'-} \right] \right)^{\frac{1}{2}} 1_{\{X_{t'}^{t,x} \in A^l\}}$$

$$\leq c \sum_{l=1}^{\infty} |X_{t'}^{t,x} - y^l| 1_{\{X_{t'}^{t,x} \in A^l\}} \leq c\epsilon.$$

Hence

$$\sum_{l=1}^{\infty} Y_{t'-}^{t',y^l} 1_{\{X_{t'}^{t,x} \in A^l\}} - c\epsilon \leq Y_{t'-}^{t,x} \leq \sum_{l=1}^{\infty} Y_{t'-}^{t',y^l} 1_{\{X_{t'}^{t,x} \in A^l\}} + c\epsilon, \tag{33}$$

where the upper bound is given by a similar argumentation. Furthermore by assumption there exist $S = \{p^1, \ldots, p^k\}$, such that $\mathbb{P}[\mathbf{p}_{t'-} \in S] = 1$. We define for $m = 1, \ldots, k$ the probability

measures $\mathbb{P}^m$, such that on $\Omega = \Omega_{0,t} \times \Omega_t$

$$\mathbb{P}^m = (\delta(p^m) \otimes \mathbb{P}^0) \otimes \left(1_{\{\mathbf{p}_{t'-}=p^m\}} \mathbb{P}|_{\Omega_t} \right),$$

where $\delta(p^m)$ denotes the measure under which $\mathbf{p}$ is constant and equal to p^m and $\mathbb{P}^0$ is a Wiener measure on $\Omega_{0,t}$. So by definition $(B_s)_{s\in[t,T]}$ is a Brownian motion under $\mathbb{P}^m$ and $(\mathbf{p}_s)_{s\in[t,T]}$ is a martingale. We see this, since for $t' \leq s \leq T$

$$\mathbb{E}_{\mathbb{P}^m}[\mathbf{p}_s|\mathcal{F}_{t'-}] = \mathbb{E}_{\mathbb{P}}[1_{\{\mathbf{p}_{t'-}=p^m\}}\mathbf{p}_s|\mathcal{F}_{t'-}] = 1_{\{\mathbf{p}_{t'-}=p^m\}}\mathbf{p}_{t'-} = p^m.$$

Furthermore the remaining conditions of Definition 3.1 are met, hence $\mathbb{P}^m \in \mathcal{P}^f(t,p)$ for $m = 1, \ldots, k$ and

$$Y_{t'-}^{t',y^l,\mathbb{P}^m} 1_{\{\mathbf{p}_{t'-}=p^m\}} \geq W(t', y^l, p^m) 1_{\mathbf{p}_{t'-}=p^m}.$$

So it holds

$$Y_{t'-}^{t',y^l,\mathbb{P}} = \sum_{m=1}^k Y_{t'-}^{t',y^l,\mathbb{P}^m} 1_{\{\mathbf{p}_{t'-}=p^m\}} \geq \sum_{m=1}^k W(t', y^l, p^m) 1_{\{\mathbf{p}_{t'-}=p^m\}} = W(t', y^l, \mathbf{p}_{t'-}).$$

Since W is uniformly Lipschitz continuous in x, we have with (33)

$$Y_{t'-}^{t,x,\mathbb{P}} \geq W(t', X_{t'}^{x,t}, \mathbf{p}_{t'-}) - c\epsilon$$

for an arbitrarily small $\epsilon > 0$. $\square$

Lemma 4.10. *For any $\epsilon > 0$, $t' \in [t,T]$ and $\mathbb{P} \in \mathcal{P}^f(t,p)$ one can choose a $\mathbb{P}^\epsilon \in \mathcal{P}^f(t,p)$, such that*

(i) $\mathbb{P}^\epsilon = \mathbb{P}$ *on $\mathcal{F}_{t'-}$*
(ii) *and it holds*

$$Y_{t'-}^{t,x,\mathbb{P}^\epsilon} \leq W(t', X_{t'}^{t,x}, \mathbf{p}_{t'-}) + \epsilon. \tag{34}$$

Remark 4.11. Observe that by (i) it holds

$$\mathbb{E}_{\mathbb{P}}\left[\int_t^{t'} \tilde{H}(s, X_s^{t,x}, z_s^{t,x,\mathbb{P}}, \mathbf{p}_s)ds \,\middle|\, \mathcal{F}_{t-} \right] = \mathbb{E}_{\mathbb{P}^\epsilon}\left[\int_t^{t'} \tilde{H}(s, X_s^{t,x}, z_s^{t,x,\mathbb{P}}, \mathbf{p}_s)ds \,\middle|\, \mathcal{F}_{t-} \right],$$

while by (ii) and Lemma 4.9.

$$Y_{t'-}^{t,x,\mathbb{P}^\epsilon} \leq W(t', X_{t'}^{t,x}, \mathbf{p}_{t'-}) + \epsilon \leq Y_{t'-}^{t,x,\mathbb{P}} + \epsilon,$$

hence by comparison (Theorem A.3)

$$\mathbb{E}_{\mathbb{P}^\epsilon}\left[\int_t^{t'} \tilde{H}(s, X_s^{t,x}, z_s^{t,x,\mathbb{P}^\epsilon}, \mathbf{p}_s) - \tilde{H}(s, X_s^{t,x}, z_s^{t,x,\mathbb{P}}, \mathbf{p}_s)ds \,\middle|\, \mathcal{F}_{t-} \right] \leq \epsilon. \tag{35}$$

Proof of Lemma 4.10. Fix a $\mathbb{P} \in \mathcal{P}^f(t,p)$. Let $t' \in [t,T]$. By assumption there exist $S = \{p^1, \ldots, p^k\}$, such that $\mathbb{P}[\mathbf{p}_{t'-} \in S] = 1$. Furthermore let $(A_l)_{l\in\mathbb{N}}$ be a partition of $\mathbb{R}^d$ by Borel sets, such that $\text{diam}(A_l) \leq \bar{\epsilon}$ and choose for any $l \in \mathbb{N}$ some $y^l \in A_l$.

Define for any l, m measures $\mathbb{P}^{l,m} \in \mathcal{P}^f(t', p^m)$, such that

$$\mathbb{E}_{\mathbb{P}^{l,m}}\left[\int_{t'}^{T} \tilde{H}(s, X_s^{t',y^l}, z_s^{t',y^l,\mathbb{P}^{l,m}}, \mathbf{p}_s)ds + \langle \mathbf{p}_T, g(X_T^{t',y^l})\rangle \Big| \mathcal{F}_{t'-}\right]$$

$$\leq \inf_{\mathbb{P} \in \mathcal{P}^f(t',p^m)} \mathbb{E}_{\mathbb{P}}\left[\int_{t'}^{T} \tilde{H}(s, X_s^{t',y^l}, z_s^{t',y^l,\mathbb{P}}, \mathbf{p}_s)ds + \langle \mathbf{p}_T, g(X_T^{t',y^l})\rangle \Big| \mathcal{F}_{t'-}\right] + \epsilon$$

$$= W(t', y^l, p^m) + \epsilon.$$

We define the probability measures $\mathbb{P}^\epsilon$, such that on $\Omega = \Omega_{0,t'} \times \Omega_{t'}$

$$\mathbb{P}^\epsilon = (\mathbb{P}|_{\Omega_{0,t'}}) \otimes \left(\sum_{m=1}^{k}\sum_{l=1}^{\infty} 1_{\{X_{t'}^{t,x} \in A^l, \mathbf{p}_{t'-}=p^m\}}\mathbb{P}^{l,m}|_{\Omega_{t'}}\right).$$

So by definition $(B_s)_{s\in[t,T]}$ is a Brownian motion under $\mathbb{P}^\epsilon$. Also $(\mathbf{p}_s)_{s\in[t,T]}$ is a martingale, since for $t' \leq r \leq s \leq T$

$$\mathbb{E}_{\mathbb{P}^\epsilon}[\mathbf{p}_s|\mathcal{F}_r] = \sum_{m=1}^{k}\sum_{l=1}^{\infty} 1_{\{X_{t'}^{t,x} \in A^l, \mathbf{p}_{t'-}=p^m\}}\mathbb{E}_{\mathbb{P}^{l,m}}[\mathbf{p}_s|\mathcal{F}_r]$$

$$= \sum_{m=1}^{k}\sum_{l=1}^{\infty} 1_{\{X_{t'}^{t,x} \in A^l, \mathbf{p}_{t'-}=p^m\}}\mathbf{p}_r = \mathbf{p}_r.$$

Furthermore the remaining conditions of Definition 3.1 are obviously met, hence $\mathbb{P}^\epsilon \in \mathcal{P}^f(t, p)$. Note that by the uniform Lipschitz continuity of $\tilde{H}$ and Proposition A.2 we have as in (33)

$$Y_{t'-}^{t,x,\mathbb{P}^\epsilon} = \sum_{m=1}^{k}\sum_{l=1}^{\infty} 1_{\{X_{t'}^{t,x} \in A^l, \mathbf{p}_{t'-}=p^m\}}Y_{t'-}^{t,x,\mathbb{P}^\epsilon}$$

$$\leq \sum_{m=1}^{k}\sum_{l=1}^{\infty} 1_{\{X_{t'}^{t,x} \in A^l, \mathbf{p}_{t'-}=p^m\}}Y_{t'-}^{t',y^l,\mathbb{P}^\epsilon}$$

$$+ c\sum_{m=1}^{k}\sum_{l=1}^{\infty}\mathbb{E}_{\mathbb{P}}\left[1_{\{X_{t'}^{t,x} \in A^l, \mathbf{p}_{t'-}=p^m\}}|X_{t'}^{t,x} - y^l|\Big|\mathcal{F}_{t'-}\right]$$

$$\leq \sum_{m=1}^{k}\sum_{l=1}^{\infty} 1_{\{X_{t'}^{t,x} \in A^l, \mathbf{p}_{t'-}=p^m\}}Y_{t'-}^{t',y^l,\mathbb{P}^\epsilon} + c\bar{\epsilon}.$$

So it holds by the definition of $\mathbb{P}^\epsilon$

$$Y_{t'-}^{t,x,\mathbb{P}^\epsilon} \leq \sum_{m=1}^{k}\sum_{l=1}^{\infty} 1_{\{X_{t'}^{t,x} \in A^l, \mathbf{p}_{t'-}=p^m\}}Y_{t'-}^{t',y^l,\mathbb{P}^{l,m}} + c\bar{\epsilon}$$

$$\leq \sum_{m=1}^{k}\sum_{l=1}^{\infty} 1_{\{X_{t'}^{t,x} \in A^l, \mathbf{p}_{t'-}=p^m\}}W(t', y^l, p^m) + \epsilon + c\bar{\epsilon}$$

$$\leq W(t', X_{t'}^{t,x}, \mathbf{p}_{t'-}) + \epsilon + c\bar{\epsilon}$$

and the result follows, since $\bar{\epsilon}$ can be chosen arbitrarily small. $\quad\square$

We are now ready to prove Theorem 4.8.

Proof of Theorem 4.8. Let $\mathbb{P}^\epsilon \in \mathcal{P}^f(t, p)$ be ϵ-optimal for $W(t, x, p)$. Then by Lemma 4.9.

$$
\begin{aligned}
W(t, x, p) + \epsilon &\geq \mathbb{E}_{\mathbb{P}^\epsilon}\left[\int_t^T \tilde{H}(s, X_s^{t,x}, z_s^{t,x}, \mathbf{p}_s)ds + \langle \mathbf{p}_T, g(X_T^{t,x})\rangle \Big| \mathcal{F}_{t-}\right] \\
&= \mathbb{E}_{\mathbb{P}^\epsilon}\left[\int_t^{t'} \tilde{H}(s, X_s^{t,x}, z_s^{t,x}, \mathbf{p}_s)ds + Y_{t'-}^{t,x} \Big| \mathcal{F}_{t-}\right] \\
&\geq \mathbb{E}_{\mathbb{P}^\epsilon}\left[\int_t^{t'} \tilde{H}(s, X_s^{t,x}, z_s^{t,x}, \mathbf{p}_s)ds + W(t', X_{t'}^{t,x}, \mathbf{p}_{t'-}) \Big| \mathcal{F}_{t-}\right].
\end{aligned}
$$

To prove the reverse inequality choose $\mathbb{P}^{\epsilon_1} \in \mathcal{P}^f(t, p)$ to be ϵ_1 optimal for the RHS of (31), i.e.

$$
\begin{aligned}
\operatorname{essinf}_{\mathbb{P}\in\mathcal{P}^f(t,p)} &\mathbb{E}_{\mathbb{P}}\left[\int_t^{t'} \tilde{H}(s, X_s^{t,x}, z_s^{t,x,\mathbb{P}}, \mathbf{p}_s)ds + W(t', X_{t'}^{t,x}, \mathbf{p}_{t'-}) \Big| \mathcal{F}_{t-}\right] + \epsilon_1 \\
&\geq \mathbb{E}_{\mathbb{P}^{\epsilon_1}}\left[\int_t^{t'} \tilde{H}(s, X_s^{t,x}, z_s^{t,x,\mathbb{P}^{\epsilon_1}}, \mathbf{p}_s)ds + W(t', X_{t'}^{t,x}, \mathbf{p}_{t'-}) \Big| \mathcal{F}_{t-}\right]. \quad (36)
\end{aligned}
$$

Furthermore choose as in Lemma 4.10 for $\mathbb{P}^{\epsilon_1}$ a $\mathbb{P}^{\epsilon_{1,2}} \in \mathcal{P}^f(t, p)$ to be ϵ_2 optimal. Then by (34), (35)

$$
\begin{aligned}
\mathbb{E}_{\mathbb{P}^{\epsilon_1}}&\left[\int_t^{t'} \tilde{H}(s, X_s^{t,x}, z_s^{t,x,\mathbb{P}^{\epsilon_1}}, \mathbf{p}_s)ds + W(t', X_{t'}^{t,x}, \mathbf{p}_{t'-}) \Big| \mathcal{F}_{t-}\right] + 2\epsilon_2 \\
&\geq \mathbb{E}_{\mathbb{P}^{\epsilon_{1,2}}}\left[\int_t^{t'} \tilde{H}(s, X_s^{t,x}, z_s^{t,x,\mathbb{P}^{\epsilon_{1,2}}}, \mathbf{p}_s)ds + Y_{t'-}^{t,x,\mathbb{P}^{\epsilon_{1,2}}} \Big| \mathcal{F}_{t-}\right] = Y_{t-}^{t,x,\mathbb{P}^{\epsilon_{1,2}}}. \quad (37)
\end{aligned}
$$

Finally combining (36), (37) we have

$$
\begin{aligned}
\operatorname{essinf}_{\mathbb{P}\in\mathcal{P}^f(t,p)} &\mathbb{E}_{\mathbb{P}}\left[\int_t^{t'} \tilde{H}(s, X_s^{t,x}, z_s^{t,x,\mathbb{P}}, \mathbf{p}_s)ds + W(t', X_{t'}^{t,x}, \mathbf{p}_{t'}) \Big| \mathcal{F}_{t-}\right] + \epsilon_1 + 2\epsilon_2 \\
&\geq Y_{t-}^{t,x,\mathbb{P}^{\epsilon_{1,2}}} \geq W(t, x, p). \quad \square
\end{aligned}
$$

4.4. Viscosity solution property

To proof that W is a viscosity solution to (8) we first show the subsolution property which is an easy consequence of the Dynamic Programming Theorem 4.8.

Proposition 4.12. *W is a viscosity subsolution to (8) on $[0, T] \times \mathbb{R}^d \times \operatorname{Int}(\Delta(I))$.*

Proof. Let $\phi : [0, T] \times \mathbb{R}^d \times \Delta(I) \to \mathbb{R}$ be a test function such that $W - \phi$ has a strict global maximum at $(\bar{t}, \bar{x}, \bar{p})$ with $W(\bar{t}, \bar{x}, \bar{p}) - \phi(\bar{t}, \bar{x}, \bar{p}) = 0$ and $\bar{p} \in \operatorname{Int}(\Delta(I))$. We have to show, that

$$
\min\left\{\frac{\partial \phi}{\partial t} + \frac{1}{2}\operatorname{tr}(\sigma\sigma^*(t, x)D_x^2\phi) + H(t, x, D_x\phi, p), \lambda_{\min}\left(\frac{\partial^2 \phi}{\partial p^2}\right)\right\} \geq 0 \quad (38)
$$

holds at $(\bar{t}, \bar{x}, \bar{p})$.

By Proposition 4.7 W is convex in p. So since $\bar{p} \in \text{Int}(\Delta(I))$, it holds $\lambda_{\min}\left(\frac{\partial^2 \phi}{\partial p^2}(\bar{t}, \bar{x}, \bar{p})\right) \geq 0$. Furthermore

$$\phi(\bar{t}, \bar{x}, \bar{p}) = W(\bar{t}, \bar{x}, \bar{p})$$

$$= \text{essinf}_{\mathbb{P} \in \mathcal{P}^r(\bar{t}, \bar{p})} \mathbb{E}\left[\int_{\bar{t}}^t \tilde{H}(s, X_s^{\bar{t}, \bar{x}}, z_s^{\bar{t}, \bar{x}}, \mathbf{p}_s)ds + W(t, X_t^{\bar{t}, \bar{x}}, \mathbf{p}_{t-})\bigg|\mathcal{F}_{\bar{t}-}\right]$$

$$\leq \mathbb{E}\left[\int_{\bar{t}}^t \tilde{H}(s, X_s^{\bar{t}, \bar{x}}, z_s^{\bar{t}, \bar{x}}, \bar{p})ds + W(t, X_t^{\bar{t}, \bar{x}}, \bar{p})\bigg|\mathcal{F}_{\bar{t}-}\right].$$

Since by standard Markov arguments $\mathbb{E}\left[\int_{\bar{t}}^t \tilde{H}(s, X_s^{\bar{t}, \bar{x}}, z_s^{\bar{t}, \bar{x}}, \bar{p})ds + W(t, X_t^{\bar{t}, \bar{x}}, \bar{p})\bigg|\mathcal{F}_{\bar{t}-}\right]$ is deterministic and $W \leq \phi$ by construction, this yields

$$\phi(\bar{t}, \bar{x}, \bar{p}) \leq \mathbb{E}\left[\int_{\bar{t}}^t \tilde{H}(s, X_s^{\bar{t}, \bar{x}}, z_s^{\bar{t}, \bar{x}}, \bar{p})ds + \phi(t, X_t^{\bar{t}, \bar{x}}, \bar{p})\right],$$

which implies (38) as $t \downarrow \bar{t}$ by standard results (see e.g. [10]). $\square$

Proposition 4.13. *W is a viscosity supersolution to (8) on $[0, T] \times \mathbb{R}^d \times \Delta(I)$.*

Proof. Let $\phi : [0, T] \times \mathbb{R}^d \times \Delta(I) \to \mathbb{R}$ be a smooth test function, such that $W - \phi$ has a strict global minimum at $(\bar{t}, \bar{x}, \bar{p})$ with $W(\bar{t}, \bar{x}, \bar{p}) - \phi(\bar{t}, \bar{x}, \bar{p}) = 0$ and such that its derivatives are uniformly Lipschitz in p.
We have to show, that

$$\min\left\{\frac{\partial \phi}{\partial t} + \frac{1}{2}\text{tr}(\sigma \sigma^*(t, x)D_x^2\phi) + H(t, x, D\phi, p), \lambda_{\min}\left(\frac{\partial^2 \phi}{\partial p^2}\right)\right\} \leq 0 \tag{39}$$

holds at $(\bar{t}, \bar{x}, \bar{p})$. Observe that, if $\lambda_{\min}\left(\frac{\partial^2 \phi}{\partial p^2}\right) \leq 0$ at $(\bar{t}, \bar{x}, \bar{p})$, then (39) follows immediately.
We assume in the subsequent steps strict convexity of ϕ in p at $(\bar{t}, \bar{x}, \bar{p})$, i.e. there exist $\delta, \eta > 0$ such that for all $z \in T_{\Delta(I)(\bar{p})}$

$$\left\langle\frac{\partial^2 \phi}{\partial p^2}(t, x, p)z, z\right\rangle > 4\delta|z|^2 \quad \forall (t, x, p) \in B_\eta(\bar{t}, \bar{x}, \bar{p}). \tag{40}$$

Since ϕ is a test function for a purely local viscosity notion, one can modify it outside a neighborhood of $(\bar{t}, \bar{x}, \bar{p})$ such that for all $(s, x) \in [\bar{t}, T] \times \mathbb{R}^d$ the function $\phi(s, x, \cdot)$ is convex on the whole convex domain $\Delta(I)$. Thus for any $p \in \Delta(I)$ it holds

$$W(t, x, p) \geq \phi(t, x, p) \geq \phi(t, x, \bar{p}) + \left\langle\frac{\partial \phi}{\partial p}(t, x, p), p - \bar{p}\right\rangle. \tag{41}$$

We divide the proof in several steps. First we use an estimate which is stronger than (41) basing on assumption (40). In the second step we use the dynamic programming to establish estimates for $\mathbf{p}$. The subsequent steps are rather close to the standard case. We reduce the problem by considering a BSDE on a smaller time interval. Then we establish estimates for the auxiliary BSDE, which we use in the last step to show the viscosity supersolution property.

Step 1: As in [6] (36) one can show that there exist $\eta, \delta > 0$, such that for all $(t, x) \in B_\eta(\bar{t}, \bar{x}), p \in \Delta(I)$

$$W(t, x, p) \geq \phi(t, x, \bar{p}) + \left\langle\frac{\partial \phi}{\partial p}(t, x, p), p - \bar{p}\right\rangle + 2\delta|p - \bar{p}|^2. \tag{42}$$

Then by (42) we have for any $t > \bar{t}$ such that $(t - \bar{t})$ is sufficiently small and an $\eta' < \eta$

$$
\begin{aligned}
&W(t, X_t^{\bar{t}, \bar{x}}, \mathbf{p}_{t-}) \\
&\geq 1_{\{|X_t^{\bar{t},\bar{x}} - \bar{x}| < \eta'\}} \left(\phi(t, X_t^{\bar{t},\bar{x}}, \bar{p}) + \left\langle \frac{\partial \phi}{\partial p}(t, X_t^{\bar{t},\bar{x}}, \bar{p}), \mathbf{p}_{t-} - \bar{p} \right\rangle + \delta |\mathbf{p}_{t-} - \bar{p}|^2 \right) \\
&\quad + 1_{\{|X_t^{\bar{t},\bar{x}} - \bar{x}| \geq \eta'\}} \phi(t, X_t^{\bar{t},\bar{x}}, \mathbf{p}_{t-}) \\
&\geq \phi(t, X_t^{\bar{t},\bar{x}}, \bar{p}) + \left\langle \frac{\partial \phi}{\partial p}(t, X_t^{\bar{t},\bar{x}}, \bar{p}), \mathbf{p}_{t-} - \bar{p} \right\rangle + 1_{\{|X_t^{\bar{t},\bar{x}} - \bar{x}| < \eta'\}} \delta |\mathbf{p}_{t-} - \bar{p}|^2 \\
&\quad + 1_{\{|X_t^{\bar{t},\bar{x}} - \bar{x}| \geq \eta'\}} \left(\phi(t, X_t^{\bar{t},\bar{x}}, \mathbf{p}_{t-}) - \phi(t, X_t^{\bar{t},\bar{x}}, \bar{p}) - \left\langle \frac{\partial \phi}{\partial p}(t, X_t^{\bar{t},\bar{x}}, \bar{p}), \mathbf{p}_{t-} - \bar{p} \right\rangle \right).
\end{aligned}
$$

Recalling that ϕ is convex with respect to p, we get

$$
\begin{aligned}
W(t, X_t^{\bar{t}, \bar{x}}, \mathbf{p}_{t-}) &\geq \phi(t, X_t^{\bar{t},\bar{x}}, \bar{p}) + \left\langle \frac{\partial \phi}{\partial p}(t, X_t^{\bar{t},\bar{x}}, \bar{p}), \mathbf{p}_{t-} - \bar{p} \right\rangle \\
&\quad + \delta 1_{\{|X_t^{\bar{t},\bar{x}} - \bar{x}| < \eta'\}} |\mathbf{p}_{t-} - \bar{p}|^2.
\end{aligned} \tag{43}
$$

Step 2: Next we establish with the help of (43) an estimate for $\mathbf{p}$. By Theorem 4.8 we can choose for any $\epsilon > 0$, $t > \bar{t}$ a $\mathbb{P}^\epsilon \in \mathcal{P}^f(\bar{t}, \bar{p})$ such that we have

$$
\epsilon(t - \bar{t}) \geq \mathbb{E}_{\mathbb{P}^\epsilon} \left[\int_{\bar{t}}^t \tilde{H}(s, X_s^{\bar{t},\bar{x}}, z_s^{\bar{t},\bar{x}}, \mathbf{p}_s) ds + W(t, X_t^{\bar{t},\bar{x}}, \mathbf{p}_{t-}) - W(\bar{t}, \bar{x}, \bar{p}) \Big| \mathcal{F}_{\bar{t}-} \right]. \tag{44}
$$

Hence with (43) it holds for all $t > \bar{t}$, such that $(t - \bar{t})$ is sufficiently small,

$$
\begin{aligned}
\epsilon(t - \bar{t}) &\geq \mathbb{E}_{\mathbb{P}^\epsilon} \left[\int_{\bar{t}}^t \tilde{H}(s, X_s^{\bar{t},\bar{x}}, z_s^{\bar{t},\bar{x}}, \mathbf{p}_s) ds + \phi(t, X_t^{\bar{t},\bar{x}}, \bar{p}) - \phi(\bar{t}, \bar{x}, \bar{p}) \right. \\
&\quad \left. + \left\langle \frac{\partial \phi}{\partial p}(t, X_t^{\bar{t},\bar{x}}, \bar{p}), \mathbf{p}_{t-} - \bar{p} \right\rangle + \delta 1_{\{|X_t^{\bar{t},\bar{x}} - \bar{x}| < \eta'\}} |\mathbf{p}_{t-} - \bar{p}|^2 \Big| \mathcal{F}_{\bar{t}-} \right].
\end{aligned} \tag{45}
$$

With (28) we have for a generic constant c

$$
\left| \mathbb{E}_{\mathbb{P}^\epsilon} \left[\int_{\bar{t}}^t \tilde{H}(s, X_s^{\bar{t},\bar{x}}, z_s^{\bar{t},\bar{x}}, \mathbf{p}_s) ds \Big| \mathcal{F}_{\bar{t}-} \right] \right| \leq c \mathbb{E}_{\mathbb{P}^\epsilon} \left[\int_{\bar{t}}^t (1 + |z_s^{\bar{t},\bar{x}}|) ds \Big| \mathcal{F}_{\bar{t}-} \right]
$$

$$
\leq c(t - \bar{t})^{\frac{1}{2}}, \tag{46}
$$

since by Proposition A.2

$$
\mathbb{E}_{\mathbb{P}^\epsilon} \left[\int_{\bar{t}}^t |z_s^{\bar{t},\bar{x}}|^2 ds \Big| \mathcal{F}_{\bar{t}-} \right] \leq c \mathbb{E}_{\mathbb{P}^\epsilon} \left[\int_{\bar{t}}^t |X_s^{\bar{t},\bar{x}}|^2 ds \Big| \mathcal{F}_{\bar{t}-} \right] \leq c.
$$

Furthermore by Itô's formula and the assumptions on ϕ it holds

$$
\mathbb{E}_{\mathbb{P}^\epsilon} \left[|\phi(t, X_t^{\bar{t},\bar{x}}, \bar{p}) - \phi(\bar{t}, \bar{x}, \bar{p})| \Big| \mathcal{F}_{\bar{t}-} \right] \leq c(t - \bar{t})^{\frac{1}{2}}. \tag{47}
$$

Next, let $f : [\bar{t}, t] \times \mathbb{R}^n \to \mathbb{R}^n$ be a smooth bounded function, with bounded derivatives. Recall that under any $\mathbb{P} \in \mathcal{P}^f(\bar{t}, \bar{p})$ the process $\mathbf{p}$ is strongly orthogonal to B. So since under $\mathbb{P}^\epsilon$ the process $\mathbf{p}$ is a martingale with $\mathbb{E}_{\mathbb{P}^\epsilon}[\mathbf{p}_{t-} | \mathcal{F}_{\bar{t}-}] = \bar{p}$, it holds by Itô's formula

$$\mathbb{E}_{\mathbb{P}^\epsilon}\left[f_i(t, X_t^{\bar{t},\bar{x}})(\mathbf{p}_{t-} - \bar{p})_i\Big|\mathcal{F}_{\bar{t}-}\right]$$

$$= \mathbb{E}_{\mathbb{P}^\epsilon}\left[\int_{\bar{t}}^t f_i(s, X_s^{\bar{t},\bar{x}})d(\mathbf{p}_s)_i + \int_{\bar{t}}^t (\mathbf{p}_s - \bar{p})_i df_i(s, X_s^{\bar{t},\bar{x}})\right.$$

$$\left. + \left[f_i(\cdot, X_\cdot^{\bar{t},\bar{x}}), (\mathbf{p}_\cdot - \bar{p})_i\right]_{t-}\Big|\mathcal{F}_{\bar{t}-}\right]$$

$$= \mathbb{E}_{\mathbb{P}^\epsilon}\left[\int_{\bar{t}}^t \left(\frac{\partial}{\partial t}f_i(s, X_s^{\bar{t},\bar{x}}) + \langle D_x f_i(s, X_s^{\bar{t},\bar{x}}), b(X_s^{\bar{t},\bar{x}})\rangle\right.\right.$$

$$\left.\left. + \frac{1}{2}\mathrm{tr}(\sigma\sigma^*(s, X_s^{\bar{t},\bar{x}})D_x^2 f_i(s, X_s^{\bar{t},\bar{x}}))\right)(\mathbf{p}_s - \bar{p})_i ds\Big|\mathcal{F}_{\bar{t}-}\right].$$

Hence by (H)

$$\left|\mathbb{E}_{\mathbb{P}^\epsilon}\left[\left\langle\frac{\partial\phi}{\partial p}(t, X_t^{\bar{t},\bar{x}}, \bar{p}), \mathbf{p}_{t-} - \bar{p}\right\rangle\Big|\mathcal{F}_{\bar{t}-}\right]\right| \leq c\mathbb{E}_{\mathbb{P}^\epsilon}\left[\int_{\bar{t}}^t |\mathbf{p}_s - \bar{p}|ds\Big|\mathcal{F}_{\bar{t}-}\right] \leq c(t - \bar{t}). \quad (48)$$

Furthermore observe that, since $|\mathbf{p}_{t-} - \bar{p}| \leq 1$, it holds for $\epsilon' > 0$ by Young and Hölder inequality

$$\mathbb{E}_{\mathbb{P}^\epsilon}\left[1_{\{|X_t^{\bar{t},\bar{x}} - \bar{x}| < \eta'\}}|\mathbf{p}_{t-} - \bar{p}|^2\Big|\mathcal{F}_{\bar{t}-}\right]$$

$$\geq \mathbb{E}_{\mathbb{P}^\epsilon}\left[|\mathbf{p}_{t-} - \bar{p}|^2\Big|\mathcal{F}_{\bar{t}-}\right] - \frac{1}{\eta'}\mathbb{E}_{\mathbb{P}^\epsilon}\left[|X_t^{\bar{t},\bar{x}} - \bar{x}||\mathbf{p}_{t-} - \bar{p}|^2\Big|\mathcal{F}_{\bar{t}-}\right]$$

$$\geq \left(1 - \frac{\epsilon'}{\eta'}\right)\mathbb{E}_{\mathbb{P}^\epsilon}\left[|\mathbf{p}_{t-} - \bar{p}|^2\Big|\mathcal{F}_{\bar{t}-}\right] - \frac{1}{4\eta'\epsilon'}\mathbb{E}_{\mathbb{P}^\epsilon}\left[|X_t^{\bar{t},\bar{x}} - \bar{x}|^2\Big|\mathcal{F}_{\bar{t}-}\right],$$

hence

$$\mathbb{E}_{\mathbb{P}^\epsilon}\left[1_{\{|X_t^{\bar{t},\bar{x}} - \bar{x}| < \eta'\}}|\mathbf{p}_{t-} - \bar{p}|^2\Big|\mathcal{F}_{\bar{t}-}\right] \geq \left(1 - \frac{\epsilon'}{\eta'}\right)\mathbb{E}_{\mathbb{P}^\epsilon}\left[|\mathbf{p}_{t-} - \bar{p}|^2\Big|\mathcal{F}_{\bar{t}-}\right] - \frac{1}{4\eta'\epsilon'}(t - \bar{t}).$$
$$(49)$$

Choosing $0 < \epsilon' < \eta'$ and combining (45) with the estimates (46)–(49) there exists a constant c, such that

$$\mathbb{E}_{\mathbb{P}^\epsilon}\left[|\mathbf{p}_{t-} - \bar{p}|^2\Big|\mathcal{F}_{\bar{t}-}\right] \leq c(t - \bar{t})^{\frac{1}{2}}. \quad (50)$$

Since $\mathbf{p}$ is a martingale, it holds for all $s \in [\bar{t}, t[$

$$\mathbb{E}_{\mathbb{P}^\epsilon}\left[|\mathbf{p}_s - \bar{p}|^2\Big|\mathcal{F}_{\bar{t}-}\right] \leq c(t - \bar{t})^{\frac{1}{2}},$$

hence

$$\mathbb{E}_{\mathbb{P}^\epsilon}\left[\int_{\bar{t}}^t |\mathbf{p}_s - \bar{p}|ds\Big|\mathcal{F}_{\bar{t}-}\right] \leq (t - \bar{t})^{\frac{1}{2}}\left(\mathbb{E}_{\mathbb{P}^\epsilon}\left[\int_{\bar{t}}^t |\mathbf{p}_s - \bar{p}|^2 ds\Big|\mathcal{F}_{\bar{t}-}\right]\right)^{\frac{1}{2}} \leq c(t - \bar{t})^{\frac{5}{4}}. \quad (51)$$

Step 3: Note that under $\mathbb{P}^\epsilon \in \mathcal{P}^f(\bar{t}, \bar{p})$ the triplet $(Y_s^{\bar{t},\bar{x}}, z_s^{\bar{t},\bar{x}}, N_s)_{s\in[\bar{t},T]}$ is given by the unique solution to the BSDE

$$Y_s^{\bar{t},\bar{x}} = \langle \mathbf{p}_T, g(X_T^{\bar{t},\bar{x}})\rangle + \int_s^T \tilde{H}(r, X_r^{\bar{t},\bar{x}}, z_r^{\bar{t},\bar{x}}, \mathbf{p}_r)dr - \int_s^T z_r^{\bar{t},\bar{x}}dB_r - N_T + N_s.$$

To consider an auxiliary BSDE with terminal time t we define as in the standard case (see e.g. [10])

$$
\begin{aligned}
G(s, x, p) &= \frac{\partial \phi}{\partial t}(s, x, p) + \frac{1}{2}\mathrm{tr}(\sigma\sigma^*(s, x)D^2\phi(s, x, p)) \\
&\quad + \tilde{H}(t, x, \sigma^*(s, x)D\phi(s, x, p), p) \\
&= \frac{\partial \phi}{\partial t}(s, x, p) + \frac{1}{2}\mathrm{tr}(\sigma\sigma^*(s, x)D^2\phi(s, x, p)) + H(t, x, D\phi(s, x, p), p)
\end{aligned}
$$

and set

$$
\begin{aligned}
\tilde{Y}_s^{\bar{t},\bar{x}} &= Y_s^{\bar{t},\bar{x}} - \phi(s, X_s^{\bar{t},\bar{x}}, \mathbf{p}_s) - \int_s^t G(r, \bar{x}, \bar{p})dr \\
&\quad + \sum_{\bar{t} \le r \le s} \left(\phi(r, X_r^{\bar{t},\bar{x}}, \mathbf{p}_r) - \phi(r, X_r^{\bar{t},\bar{x}}, \mathbf{p}_{r-}) - \left\langle \frac{\partial}{\partial p}\phi(r, X_r^{\bar{t},\bar{x}}, \mathbf{p}_{r-}), \mathbf{p}_r - \mathbf{p}_{r-} \right\rangle \right) \\
\tilde{z}_s^{\bar{t},\bar{x}} &= z_s^{\bar{t},\bar{x}} - \sigma^*(s, X_s^{\bar{t},\bar{x}})D_x\phi(s, X_s^{\bar{t},\bar{x}}, \mathbf{p}_s).
\end{aligned}
$$

Then by Itô's formula $(\tilde{Y}^{\bar{t},\bar{x}}, \tilde{z}^{\bar{t},\bar{x}}, N)$ is on $[\bar{t}, t[$ the solution to the BSDE

$$
\begin{aligned}
\tilde{Y}_s^{\bar{t},\bar{x}} &= \xi + \int_s^t \left(\tilde{H}(r, X_r^{\bar{t},\bar{x}}, \tilde{z}_r^{\bar{t},\bar{x}} + \sigma^*(r, X_r^{\bar{t},\bar{x}})D_x\phi(r, X_r^{\bar{t},\bar{x}}, \mathbf{p}_r), \mathbf{p}_r) + \frac{\partial \phi}{\partial t}(r, X_r^{\bar{t},\bar{x}}, \mathbf{p}_r) \right. \\
&\quad \left. + \frac{1}{2}\mathrm{tr}(\sigma\sigma^*(r, X_r^{\bar{t},\bar{x}})D^2\phi(r, X_r^{\bar{t},\bar{x}}, \mathbf{p}_r)) - G(r, \bar{x}, \bar{p}) \right)dr \\
&\quad - \int_s^t \tilde{z}_r^{\bar{t},\bar{x}}dB_r - N_{t-} + N_s
\end{aligned}
$$

with the terminal value

$$
\begin{aligned}
\xi &= \bar{Y}_{t-}^{\bar{t},\bar{x}} - \phi(t, X_t^{\bar{t},\bar{x}}, \mathbf{p}_{t-}) + \sum_{\bar{t} \le r < t} \left(\phi(r, X_r^{\bar{t},\bar{x}}, \mathbf{p}_r) - \phi(r, X_r^{\bar{t},\bar{x}}, \mathbf{p}_{r-}) \right. \\
&\quad \left. - \left\langle \frac{\partial}{\partial p}\phi(r, X_r^{\bar{t},\bar{x}}, \mathbf{p}_{r-}), \mathbf{p}_r - \mathbf{p}_{r-} \right\rangle \right).
\end{aligned}
$$

Note that by the strict convexity assumption on ϕ it holds $\mathbb{P}^\epsilon$-a.s.

$$
\sum_{\bar{t} \le r < t} \left(\phi(r, X_r^{\bar{t},\bar{x}}, \mathbf{p}_r) - \phi(r, X_r^{\bar{t},\bar{x}}, \mathbf{p}_{r-}) - \left\langle \frac{\partial}{\partial p}\phi(r, X_r^{\bar{t},\bar{x}}, \mathbf{p}_{r-}), \mathbf{p}_r - \mathbf{p}_{r-} \right\rangle \right) \ge 0. \quad (52)
$$

Furthermore by Lemma 4.9 and the choice of ϕ we have $Y_{t-}^{\bar{t},\bar{x}} \ge W(t, X_t^{\bar{t},\bar{x}}, \mathbf{p}_{t-}) \ge \phi(t, X_t^{\bar{t},\bar{x}}, \mathbf{p}_{t-})$, hence $\xi \ge 0$.

Consider now the solution to the BSDE with the same driver but with the smaller target 0, i.e.

$$
\begin{aligned}
\bar{Y}_s^{\bar{t},\bar{x}} &= \int_s^t \left(\tilde{H}(r, X_r^{\bar{t},\bar{x}}, \tilde{z}_r^{\bar{t},\bar{x}} + \sigma^*(r, X_r^{\bar{t},\bar{x}})D_x\phi(r, X_r^{\bar{t},\bar{x}}, \mathbf{p}_r), \mathbf{p}_r) + \frac{\partial \phi}{\partial t}(r, X_r^{\bar{t},\bar{x}}, \mathbf{p}_r) \right. \\
&\quad \left. + \frac{1}{2}\mathrm{tr}(\sigma\sigma^*(r, X_r^{\bar{t},\bar{x}})D^2\phi(r, X_r^{\bar{t},\bar{x}}, \mathbf{p}_r)) - G(r, \bar{x}, \bar{p}) \right)dr \\
&\quad - \int_s^t \bar{z}_r^{\bar{t},\bar{x}}dB_r - \bar{N}_{t-} + \bar{N}_s. \quad (53)
\end{aligned}
$$

By Proposition A.2 it holds

$$\mathbb{E}_{\mathbb{P}^\epsilon}\left[\int_{\bar{t}}^t |\bar{z}_s^{\bar{t},\bar{x}}|^2 ds \,\Big|\, \mathcal{F}_{\bar{t}-}\right] \leq c\mathbb{E}_{\mathbb{P}^\epsilon}\left[\int_{\bar{t}}^t |\bar{f}_s|^2 ds \,\Big|\, \mathcal{F}_{\bar{t}-}\right] \tag{54}$$

with

$$\bar{f}_s := \tilde{H}(s, X_s^{\bar{t},\bar{x}}, \sigma^*(s, X_s^{\bar{t},\bar{x}})D_x\phi(s, X_s^{\bar{t},\bar{x}}, \mathbf{p}_s), \mathbf{p}_s) + \frac{\partial\phi}{\partial t}(s, X_s^{\bar{t},\bar{x}}, \mathbf{p}_s)$$

$$+ \frac{1}{2}\mathrm{tr}(\sigma\sigma^*(s, X_s^{\bar{t},\bar{x}})D^2\phi(s, X_s^{\bar{t},\bar{x}}, \mathbf{p}_s)) - G(s, \bar{x}, \bar{p}).$$

Because $\tilde{H}$ is uniformly Lipschitz continuous in p and the derivatives of ϕ with respect to p are uniformly bounded, we have

$$|\bar{f}_s| \leq \left| \tilde{H}(s, X_s^{\bar{t},\bar{x}}, \sigma^*(s, X_s^{\bar{t},\bar{x}})D_x\phi(s, X_s^{\bar{t},\bar{x}}, \bar{p}), \bar{p}) + \frac{\partial\phi}{\partial t}(s, X_s^{\bar{t},\bar{x}}, \bar{p}) \right.$$

$$\left. + \frac{1}{2}\mathrm{tr}(\sigma\sigma^*(s, X_s^{\bar{t},\bar{x}})D^2\phi(s, X_s^{\bar{t},\bar{x}}, \bar{p})) - G(s, \bar{x}, \bar{p}) \right| + c\,|\mathbf{p}_s - \bar{p}|$$

and it holds as in [10] by the estimate (51) for all $\epsilon' > 0$

$$\mathbb{E}_{\mathbb{P}^\epsilon}\left[\int_{\bar{t}}^t |\bar{f}_s|^2 ds \,\Big|\, \mathcal{F}_{\bar{t}-}\right] \leq \frac{1}{4\epsilon'}(t-\bar{t})O(t-\bar{t}) + \epsilon'c\,\mathbb{E}_{\mathbb{P}^\epsilon}\left[\int_{\bar{t}}^t |\mathbf{p}_s - \bar{p}|^2\,ds \,\Big|\, \mathcal{F}_{\bar{t}-}\right]$$

$$\leq \frac{1}{4\epsilon'}(t-\bar{t})O(t-\bar{t}) + \epsilon'c(t-\bar{t})^{\frac{3}{2}},$$

where $O(t-\bar{t}) \to 0$ as $t \to \bar{t}$. Hence we have by (54) and Cauchy inequality

$$\mathbb{E}_{\mathbb{P}^\epsilon}\left[\int_{\bar{t}}^t |\bar{z}_s^{\bar{t},\bar{x}}| ds \,\Big|\, \mathcal{F}_{\bar{t}-}\right] \leq c\left((t-\bar{t})O(t-\bar{t}) + (t-\bar{t})^{\frac{5}{4}}\right) \tag{55}$$

and

$$\bar{Y}_{\bar{t}-}^{\bar{t},\bar{x}} \geq -c\mathbb{E}_{\mathbb{P}^\epsilon}\left[\int_{\bar{t}}^t |\bar{f}_s| ds | \mathcal{F}_{\bar{t}-}\right] - c\mathbb{E}_{\mathbb{P}^\epsilon}\left[\int_{\bar{t}}^t |\bar{z}_s^{\bar{t},\bar{x}}| ds | \mathcal{F}_{\bar{t}-}\right]$$

$$\geq -c\left((t-\bar{t})O(t-\bar{t}) + (t-\bar{t})^{\frac{5}{4}}\right).$$

Since by the comparison theorem (Theorem A.3) it holds $\tilde{Y}_{\bar{t}-}^{\bar{t},\bar{x}} \geq \bar{Y}_{\bar{t}-}^{\bar{t},\bar{x}}$, this yields the following estimate for the auxiliary BSDE

$$\tilde{Y}_{\bar{t}-}^{\bar{t},\bar{x}} \geq -c\left((t-\bar{t})O(t-\bar{t}) + (t-\bar{t})^{\frac{5}{4}}\right). \tag{56}$$

Step 4: The theorem is proved, if we show $G(\bar{t}, \bar{x}, \bar{p}) \leq 0$. Note that by definition of $\tilde{Y}^{\bar{t},\bar{x}}$

$$\tilde{Y}_{\bar{t}-}^{\bar{t},\bar{x}} = Y_{\bar{t}-}^{\bar{t},\bar{x}} - \phi(\bar{t}, \bar{x}, \bar{p}) - \int_{\bar{t}}^t G(r, \bar{x}, \bar{p})dr. \tag{57}$$

Since $\phi(\bar{t}, \bar{x}, \bar{p}) = W(\bar{t}, \bar{x}, \bar{p})$, we have by the choice of $\mathbb{P}^\epsilon$ and the dynamic programming (Theorem 4.8)

$$Y_{\bar{t}-}^{\bar{t},\bar{x}} - \phi(\bar{t}, \bar{x}, \bar{p}) = Y_{\bar{t}-}^{\bar{t},\bar{x}} - W(\bar{t}, \bar{x}, \bar{p})$$

$$\leq Y_{\bar{t}-}^{\bar{t},\bar{x}} - \mathbb{E}_{\mathbb{P}^\epsilon}\left[\int_{\bar{t}}^{t} \tilde{H}(s, X_s^{\bar{t},\bar{x}}, z_s^{\bar{t},\bar{x}}, \mathbf{p}_s)ds + W(t, X_t^{\bar{t},\bar{x}}, \mathbf{p}_{t-})\Big|\mathcal{F}_{\bar{t}-}\right]$$

$$+ \epsilon(t - \bar{t})$$

$$= \mathbb{E}_{\mathbb{P}^\epsilon}\left[Y_{t-}^{\bar{t},\bar{x}} - W(t, X_t^{\bar{t},\bar{x}}, \mathbf{p}_{t-})\Big|\mathcal{F}_{\bar{t}-}\right] + \epsilon(t - \bar{t}).$$

Recall that by the choice of $\mathbb{P}^\epsilon$ according to Lemma 4.10 it holds

$$Y_{t-}^{\bar{t},\bar{x}} - W(t, X_t^{\bar{t},\bar{x}}, \mathbf{p}_{t-}) \leq c\epsilon(t - \bar{t}).$$

Hence

$$Y_{\bar{t}-}^{\bar{t},\bar{x}} - \phi(\bar{t}, \bar{x}, \bar{p}) \leq c\epsilon(t - \bar{t}). \tag{58}$$

Thus from (57) with (58) we have

$$\tilde{Y}_{\bar{t}-}^{\bar{t},\bar{x}} + \int_{\bar{t}}^{t} G(r, \bar{x}, \bar{p})dr \leq c\epsilon(t - \bar{t})$$

and finally by the estimate (56)

$$-c\left((t - \bar{t})O(t - \bar{t}) + (t - \bar{t})^{\frac{5}{4}}\right) + \int_{\bar{t}}^{t} G(r, \bar{x}, \bar{p})dr \leq c\epsilon(t - \bar{t}),$$

hence

$$\frac{1}{(t - \bar{t})}\int_{\bar{t}}^{t} G(s, \bar{x}, \bar{p})ds \leq c\left(O(t - \bar{t}) + (t - \bar{t})^{\frac{1}{4}}\right) + c\epsilon$$

which implies (39) as $t \downarrow \bar{t}$ since $\epsilon > 0$ can be chosen arbitrary small. $\quad\square$

Thus by Propositions 4.12 and 4.13 and comparison for (8) (see [6,5]) we now have the following result.

Theorem 4.14. *W is the unique viscosity solution to* (8).

Theorem 3.4 follows directly from Theorem 4.14 and the characterization of the value function in Theorem 2.8.

5. Concluding remarks

In this paper we have shown an alternative representation of the value function in terms of a minimization of solutions of certain BSDEs over some specific martingale measures. These BSDEs correspond to the dynamics of a stochastic differential game with the beliefs of the uninformed player (modulo a Girsanov transformation) as an additional forward dynamic. We used this to show how to explicitly determine the optimal reaction of the informed player under some rather restrictive assumptions. To have a representation like

$$V(t, x, p) = Y_{t-}^{t,x,\bar{\mathbb{P}}}$$

in a more general case a careful analysis of the optimal measure is necessary. In the simpler framework of [6] the existence of a weak limit $\mathbb{P}^*$ for a minimizing sequence is straightforward using [16]. In our case any limiting procedure needs to take into account the BSDE structure. The

question of existence of an optimal measure under which there is a representation by a solution to a BSDE poses therefore a rather delicate problem, which shall be addressed in a subsequent work.

Acknowledgments

Appendix. Results for BSDE on $\mathcal{D}([0, T]; \Delta(I)) \times \mathcal{C}([0, T]; \mathbb{R}^d)$

Here we restate existence and uniqueness results of for BSDEs adapted to our setting. More general results can be found in [9].

Let $\Omega := \mathcal{D}([0, T]; \Delta(I)) \times \mathcal{C}([0, T]; \mathbb{R}^d)$ and $(\Omega, \mathcal{F}, (\mathcal{F}_s)_{s \in [t, T]})$ be defined as in Section 3.1. We fix a $\mathbb{P} \in \mathcal{P}(t, p)$ and denote $\mathbb{E}_{\mathbb{P}}[\cdot] = \mathbb{E}[\cdot]$. Let $\xi \in \mathcal{L}^2_T(\mathbb{P})$, i.e. ξ is a square integrable $\mathcal{F}_T$-measurable random variable. Let $f : \Omega \times [0, T] \times \mathbb{R}^d \to \mathbb{R}$ be $\mathcal{P} \otimes \mathcal{B}(\mathbb{R}^d)$ measurable, such that $f(\cdot, 0) \in \mathcal{H}^2(\mathbb{P})$ and such that, there exists a constant c, such that $\mathbb{P} \otimes dt$ a.e.

$$|f(\omega, s, z^1) - f(\omega, s, z^2)| \leq c|z^1 - z^2| \quad \forall z^1, z^2 \in \mathbb{R}^d. \tag{A.1}$$

We consider on $\mathcal{D}([0, T]; \Delta(I)) \times \mathcal{C}([0, T]; \mathbb{R}^d)$ the BSDE

$$Y_s = \xi + \int_s^T f(r, z_r) ds + \int_s^T z_r dB_r - (N_T - N_s). \tag{A.2}$$

The existence and uniqueness can be shown by a combination of the proof for the solvability of BSDE via a fixed point argument as in [10] and the Galtchouk–Kunita–Watanabe decomposition (see e.g. [1]).

Theorem A.1. *For any fixed* $\mathbb{P} \in \mathcal{P}(t, x)$ *there exists a solution* $(Y, z, N) \in \mathcal{H}^2 (\mathbb{P}) \times \mathcal{H}^2(\mathbb{P}) \times \mathcal{M}^2_0(\mathbb{P})$ *to* (A.2)*, such that N is strongly orthogonal to the Brownian motion B. Furthermore* (Y, z) *are unique in* $\mathcal{H}^2(\mathbb{P}) \times \mathcal{H}^2(\mathbb{P})$ *and* $N \in \mathcal{M}^2_0(\mathbb{P})$ *is unique up to indistinguishability.*

Furthermore we note that we have the following dependence on the data.

Proposition A.2. *For* $i = 1, 2$, *let* $\xi^i \in \mathcal{L}^2_T(\mathbb{P})$. *Let* $f^i : \Omega \times [0, T] \times \mathbb{R}^d \to \mathbb{R}$ *be two generators for the BSDE* (A.2)*, i.e.* $\mathcal{P} \otimes \mathcal{B}(\mathbb{R}^d)$ *measurable,* $f^i(\cdot, 0) \in \mathcal{H}^2(\mathbb{P})$ *and* f^i *are uniformly Lipschitz continuous in z.*

Let $(Y^i, z^i, N^i) \in \mathcal{H}^2(\mathbb{P}) \times \mathcal{H}^2(\mathbb{P}) \times \mathcal{M}^2_0(\mathbb{P})$ *be the respective solutions. Set* $\delta z = z^1 - z^2$ *and* $\delta \xi = \xi^1 - \xi^2, \delta f = f^1(\cdot, z^2) - f^2(\cdot, z^2)$. *Then it holds for any* $s \in [0, T]$

$$\mathbb{E}\left[\sup_{r\in[s,T]}|\delta Y_r|^2\Big|\mathcal{F}_s\right] + \mathbb{E}\left[\int_s^T |\delta z_r|^2 dr|\mathcal{F}_s\right]$$

$$\leq c\left(\mathbb{E}\left[|\delta\xi|^2|\mathcal{F}_s\right] + \mathbb{E}\left[\int_s^T |\delta f_r|^2 dr\Big|\mathcal{F}_s\right]\right). \tag{A.3}$$

Also we have a comparison principle which can be established as in [10].

Theorem A.3. *For $i = 1, 2$, let $\xi^i \in \mathcal{L}_T^2(\mathbb{P})$. Let $f^i : \Omega \times [0, T] \times \mathbb{R}^d \to \mathbb{R}$ be two generators for the BSDE* (A.2), *i.e. f^i is $\mathcal{P} \otimes \mathcal{B}(\mathbb{R}^d)$ measurable, uniformly Lipschitz continuous in z and $f^i(\cdot, 0) \in \mathcal{H}^2(\mathbb{P})$.*

Let $(Y^i, z^i, N^i) \in \mathcal{H}^2(\mathbb{P}) \times \mathcal{H}^2(\mathbb{P}) \times \mathcal{M}_0^2(\mathbb{P})$ be the respective solutions. Assume

(i) $\delta\xi = \xi^1 - \xi^2 \geq 0$ *holds $\mathbb{P}$-a.s.*
(ii) $\delta f = f^1(\cdot, z_\cdot^2) - f^2(\cdot, z_\cdot^2) \geq 0$ *holds $\mathbb{P} \otimes dt$-a.s.*

Then for any time $s \in [0, T]$ it holds $Y_s^1 - Y_s^2 \geq 0$ $\mathbb{P}$-a.s.

References

[1] J.P. Ansel, C. Stricker, Décomposition de Kunita–Watanabe, in: Séminaire de Probabilités, XXVII, in: Lecture Notes in Math., vol. 1557, Springer, Berlin, 1993, pp. 30–32.
[2] R.J. Aumann, M.B. Maschler, Repeated Games with Incomplete Information, with the Collaboration of Richard E. Stearns, MIT Press, Cambridge, MA, 1995.
[3] R. Buckdahn, J. Li, Stochastic differential games and viscosity solutions of Hamilton–Jacobi–Bellman–Isaacs equations, SIAM J. Control Optim. 47 (2008) 444–475.
[4] I. Capuzzo Dolcetta, P.L. Lions, Hamilton–Jacobi equations with state constraints, Trans. Amer. Math. Soc. (ISSN: 0002-9947) 318 (1990).
[5] P. Cardaliaguet, A double obstacle problem arising in differential game theory, J. Math. Anal. Appl. 360 (1) (2009) 95–107.
[6] P. Cardaliaguet, C. Rainer, On a continuous time game with incomplete information, Math. Oper. Res. 34 (4) (2009).
[7] P. Cardaliaguet, C. Rainer, Stochastic differential games with asymmetric information, Appl. Math. Optim. 59 (1) (2009) 1–36.
[8] M.G. Crandall, H. Ishii, P.L. Lions, User's guide to viscosity solutions of second order partial differential equations, Bull. Amer. Math. Soc. 27 (1992) 1–67.
[9] N. El Karoui, S.-J. Huang, A general result of existence and uniqueness of backward stochastic differential equations, in: Backward Stochastic Differential Equations, Paris, 1995–1996, in: Pitman Res. Notes Math. Ser., vol. 364, Longman, Harlow, 1997, pp. 27–36.
[10] N. El Karoui, S. Peng, M.C. Quenez, Backward stochastic differential equations in finance, Math. Finance 1 (1997) 1–71.
[11] L.C. Evans, P.E. Souganidis, Differential games and representation formulas for solutions of Hamilton–Jacobi–Isaacs equations, Indiana Univ. Math. J. 33 (1984) 773–797.
[12] W.H. Fleming, P.E. Souganidis, On the existence of value functions of two-players, zero-sum stochastic differential games, Indiana Univ. Math. J. 38 (1989) 293–314.
[13] S. Hamadène, J.-P. Lepeltier, Zero-sum stochastic differential games and backward equations, Systems Control Lett. 24 (1995) 259–263.
[14] S. Hamadène, J.-P. Lepeltier, S. Peng, BSDE with continuous coefficients and applications to Markovian non zero sum stochastic differential games, in: Pitman Res. Notes in Math. Series, vol. 364, Longman, 1997, pp. 161–175.
[15] J. Jacod, A.N. Shiryaev, Limit Theorems for Stochastic Processes, in: Grundlagen der Mathematischen Wissenschaften, vol. 288, 1980.
[16] P.A. Meyer, W.A. Zeng, Tightness criteria for laws of semimartingales, Ann. Inst. H. Poincaré Probab. Statist. 20 (4) (1984) 353–372.
[17] S. Peng, A generalized dynamic programming principle and Hamilton–Jacobi–Bellman equations, Stochastics 38 (1992) 119–134.