Diagonalization of Matrices over Graded Principal Ideal Domains

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ABSTRACT

The main result of this paper is the analogue of the classical diagonal reduction of matrices over PIDs, for graded principal ideal domains. A method for diagonalizing graded matrices over a graded principal ideal domain is obtained. In Section 2 we emphasis on some applications. A procedure is given to decide whether or not a matrix defined over an ordinary Dedekind domain (i.e. nongraded), with cyclic class group, is diagonalizable. In case the answer is positive the diagonal form can be calculated. This can be done by taking a suitable graded PID which has the Dedekind domain as its part of degree zero. It turns out that, even in the case where diagonalization of a matrix over the part of degree zero is not possible, the diagonal representation over the graded ring contains useful information. The main reason for this is that the graded ring hasn't essentially more units than its part of degree zero. We illustrate this by considering the problem of von Neumann regularity of a matrix over a Gr-PID and to matrices over Dedekind domains with cyclic class group. These problems were the original motivation for studying diagonalization over graded rings.

0. INTRODUCTION

For details concerning the theory of graded rings we refer to [5]. We recall some definitions, facts, and notation for further use in the text.

A ring R is said to be a graded ring if there is a family of additive subgroups $\{R_n,\ n \in \mathbb{Z}\}$ of R such that $R = \oplus_{n \in \mathbb{Z}} R_n$ and $R_i R_j \subset R_{i+j}$ for

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$i, j \in \mathbb{Z}$. It follows from this that $1 \in R_0$ and that R_0 is a subring of R. The elements of the R_i are called homogeneous elements of R. A left R-module M over the graded ring R is said to be a graded left R-module if there is a family $\{M_n,\ n \in \mathbb{Z}\}$ of additive subgroups of M such that $M = \oplus_{n \in \mathbb{Z}} M_n$ and $R_i M_j \subset M_{i+j}$ for all $i, j \in \mathbb{Z}$. The ideals of a graded ring R which are graded R-modules are called homogeneous ideals, i.e., a homogeneous ideal is generated by its homogeneous elements. A graded ring in which all homogeneous ideals are generated by one homogeneous element is called a graded principal ideal domain (Gr-PID).

Let M, N be graded modules over a graded ring R. A morphism $f\colon M \to N$ is said to be homogeneous of degree p if $f(M_i) \subset N_{i+p}$ for any $i \in \mathbb{Z}$. Morphisms of degree p form an additive subgroup of $\operatorname{Hom}_R(M, N)$. A graded module M is called graded-free if it has a basis consisting of homogeneous elements.

Two matrices A, B over a ring R are called R-equivalent if there exist invertible matrices P, Q over R such that $PAQ = B$, denoted by $A \underset{R}{\sim} B$. The following elementary matrices are used:

$$D_i(u) = \begin{pmatrix} 1 & & & & & & \\ & \ddots & & & & & \\ & & 1 & & & & \\ & & & u & & & \\ & & & & 1 & & \\ & & & & & \ddots & \\ & & & & & & 1 \end{pmatrix},$$

$$P_{ij} = \begin{pmatrix} 1 & & & & & & & & & \\ & \ddots & & & & & & & & \\ & & 1 & & & & & & & \\ & & & 0 & 0 & \cdots & 0 & 1 & & \\ & & & 0 & 1 & & & 0 & & \\ & & & \vdots & & \ddots & & \vdots & & \\ & & & 0 & & & 1 & 0 & & \\ & & & 1 & 0 & \cdots & 0 & 0 & & \\ & & & & & & & & 1 & \\ & & & & & & & & & \ddots \\ & & & & & & & & & & 1 \end{pmatrix}$$

1. DIAGONAL REDUCTION FOR MATRICES OVER GRADED PRINCIPAL IDEAL DOMAINS

Throughout this section R is a graded principal (left and right) ideal domain.

Let M, N be graded-free (left) R-modules, $(e_j)_{j=1,\dots,n}$ a homogeneous basis for M, and $(f_i)_{i=1,\dots,m}$ a homogeneous basis for N. Consider a homogeneous morphism α in $\mathrm{Hom}_R(M, N)$, say of degree d. Then

$$\begin{aligned} \alpha(e_1) &= a_{11}f_1 + a_{21}f_2 + \cdots + a_{m1}f_m, \\ &\vdots \\ \alpha(e_n) &= a_{1n}f_1 + a_{2n}f_2 + \cdots + a_{mn}f_m \end{aligned} \tag{1}$$

with all a_{ij} homogeneous elements of R, since the f_i's are homogeneous and the $\alpha(e_j)$'s are homogeneous. So, for all i, j, we have

$$\deg \alpha(e_j) = \deg e_j + d = \deg(a_{ij}f_i) = \deg a_{ij} + \deg(f_i). \tag{2}$$

If $A = (a_{ij})$ is the matrix of α with respect to the basis (e_j) and (f_i), then for all (i, j), (k, l) the following equalities hold:

$$\deg e_j - \deg e_l = \deg a_{ij} - \deg a_{il},$$

$$\deg f_i - \deg f_k = \deg a_{kj} - \deg a_{ij}.$$

Together they yield

$$\deg a_{ij} + \deg a_{kl} = \deg a_{il} + \deg a_{kj}. \tag{4}$$

Conversely, let $A = (a_{ij})$ be an $n \times m$ matrix over R with homogeneous entries and such that for all (i, j), (k, l) the relation (4) holds. Grade the free R-modules R^n, R^m by choosing bases (e_j) in R^n and (f_i) in R^m such that

$$\deg e_j - \deg e_l = \deg a_{ij} - \deg a_{il},$$

$$\deg f_i - \deg f_k = \deg a_{kj} - \deg a_{ij}$$

holds for all (i, j), (k, l), which is possible in view of (4) implies that A can be considered as the matrix associated to a homogeneous morphism α in $\mathrm{Hom}(R^n, R^m)$.

It follows from the above that an $m \times n$ matrix satisfies the relation (4) iff it is homogeneous in a graded matrix ring $M_n^{\mathrm{gr}}(R)$, i.e. homogeneous for some suitable gradation on $M_n(R)$. We therefore make the following

DEFINITION 1.1. An $m \times n$ matrix A over R is *graded* iff all entries are homogeneous elements of R and for all (i, j), (k, l) the relations

$$\deg a_{ij} + \deg a_{kl} = \deg a_{il} + \deg a_{kj} \tag{4}$$

hold.

If R is commutative, then A is graded iff all entries and all minors of A are homogeneous in R.

In order to reduce a graded $m \times n$ matrix A over R to a diagonal form, the following operations are used [2]:

I. *Interchanging rows and columns.* This is done by multiplying on the left or right by the elementary matrix P_{ij}.

II. *Multiplying a column (row) on the right (left) with a unit element of R.* This is done by multiplying on the right (left) by the elementary matrix $D_i(u)$, u a unit of R.

Noting that a unit of R is homogeneous [9], it is clear that the matrices P_{ij} and $D_i(u)$ are graded and that the result obtained by operation I and II is again a graded matrix.

III. *Replacing the first element in each of two given columns (rows) by their highest common left (right) factor (HCLF) and* 0 *respectively.* In this case it is not obvious that this can be done by a graded invertible matrix, so

LEMMA 1.2. *If A is a graded $m \times n$ matrix over R, then operation* III *can be performed by multiplying A by invertible graded matrices. Moreover the result obtained by operation* III *is again a graded matrix.*

Proof. Suppose we want to replace the entries a_{1j}, a_{1l} of A by their HCLF and 0 respectively. Consider the 2×2 matrix

$$\begin{pmatrix} a_{1j} & a_{1l} \\ a_{2j} & a_{2l} \end{pmatrix}.$$

Since R is a Gr-PID and the a_{ij}'s are homogeneous, we have

$$a_{1j}R + a_{1l}R = gR,$$

with g a homogeneous HCLF of a_{1j} and a_{1l}. So $a_{1j} = ga$ and $a_{1l} = gb$ with $aR + bR = R$ and $aR \cap bR$ a principal right ideal. Therefore, there exists an invertible matrix C with homogeneous (a, b) as its first row, say

$$C = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Let

$$C^{-1} = \begin{pmatrix} d' & -b' \\ -c' & a' \end{pmatrix}.$$

Operation III can be performed by multiplying A on the right by

$$G = \begin{pmatrix} 1 & & & & & & & & & \\ & \ddots & & & & & & & & \\ & & 1 & & & & & & & \\ & & & d' & 0 & \cdots & 0 & -b' & & \\ & & & 0 & 1 & & & 0 & & \\ & & & \vdots & & \ddots & & \vdots & & \\ & & & 0 & & & 1 & 0 & & \\ & & & -c' & 0 & \cdots & 0 & a & & \\ & & & & & & & & 1 & & \\ & & & & & & & & & \ddots & \\ & & & & & & & & & & 1 \end{pmatrix}.$$

It remains to prove that the matrix G and $A' = AG$ are graded matrices.

To show that G is a graded matrix it suffices to prove that C^{-1} is a graded matrix. Since $CC^{-1} = I$, the following relations hold:

$$\begin{aligned} ad' - bc' &= 1, \\ ba' &= ab', \\ cd' &= dc', \\ da' - cb' &= 1. \end{aligned} \tag{5}$$

Since there is a homogeneous solution of the equation $ax + by = 1$, by considering the expansion in homogeneous elements of any solution and taking the terms of degree zero on both sides of the equation, we see that the entries c', d' are homogeneous. Therefore the other elements of C^{-1} can also be taken homogeneous. We then have, in view of (5),

$$\begin{aligned} \deg a &= -\deg d', \qquad \deg d = -\deg a', \\ \deg b &= -\deg c', \qquad \deg c = -\deg b', \end{aligned} \tag{6}$$

and

$$\begin{aligned} \deg a - \deg a' &= \deg b - \deg b' = \deg c - \deg c' \\ &= \deg d - \deg d'. \end{aligned}$$

Replacing $\deg a, \deg b$ in the first equation of (6′) by $-\mathit{deg}\, d'$, $-\mathit{deg}\, c'$, and using (6), we get

$$\deg a' + \deg d' = \deg b' + \deg c',$$

which shows that C^{-1} is a graded matrix.

For the elements in the jth column of A' we find

$$a'_{ij} = a_{ij} d' - a_{il} c';$$

for the elements in the lth column of A' we find

$$a'_{il} = -a_{ij} b' + a_{il} a'.$$

Now,

$$\deg a_{ij} + \deg d' = \deg a_{ij} - \deg a = \deg a_{ij} + \deg g,$$

$$\begin{aligned} \deg a_{1j} &= \deg a_{il} - \deg a_{1l} + \deg g = \deg a_{il} - \deg b \\ &= \deg a_{il} + \deg c', \end{aligned}$$

and

$$\begin{aligned} \deg a_{ij} - \deg a_{il} &= \deg a_{1j} - \deg a_{1l} \\ &= \deg a - \deg b = \deg a' - \deg b'. \end{aligned}$$

This yields that all a'_{ik}'s are homogeneous. Furthermore, $\deg a'_{ik} - \deg a'_{il} = \deg a_{1k} - \deg a_{1l}$, i.e., it is constant for all i, which shows that A' is graded.

Analogous calculation for operations on the rows can be made. ■

Recall that in an integral domain, an element a is said to be a total divisor of $b \neq 0$, denoted by $a \parallel b$, if there exists an invariant element c (i.e. $cR = Rc$), such that $a \mid c \mid b$.

THEOREM 1.3. *Let R be a* Gr-PID, *and let A be an $m \times n$ graded matrix over R. Then we can find graded invertible matrices $P \in \mathrm{GL}_m(R)$ and $Q \in \mathrm{Gl}_n(R)$ such that $PAQ = \mathrm{diag}(e_1, \ldots, e_r, 0, \ldots, 0)$ and*

$$e_i \parallel e_{i+1},\ e_r \neq 0. \tag{*}$$

Proof. Since all homogeneous elements have a finite number of factors (R being a Gr-PID), it follows, by induction on the number of factors and on $\max(m, n)$, from Lemma 1.2, that a graded matrix may be reduced to a diagonal form; say $\mathrm{diag}(a_1, \ldots, a_r, 0, \ldots, 0)$. (Compare with the classical procedure of reducing matrices over a PID.) This form can be reduced to a diagonal form satisfying (*), in the following way.

For all $d \in R_h$, with h an arbitrary element of $\mathbb{Z}$, we have

$$\begin{pmatrix} 1 & d \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix} = \begin{pmatrix} a_1 & da_2 \\ 0 & a_2 \end{pmatrix}.$$

Again applying Lemma 1.2 (i.e. operation III), we can diminish the length of a_2 (i.e. its number of factors) unless a_1 is a left factor of da_2 for all $d \in R_h$, i.e., unless $a_1 R \supset R_h a_2$. But then $a_1 R \supset R_h a_2 R$. Let c be the homogeneous invariant generator of Ra_2R, i.e. $Rc = cR = Ra_2R$; then for all $n \in \mathbb{Z}$,

$$R_h a_2 R_n = cR_{h + \deg a_2 + n - \deg c}$$

(because a_2, c are homogeneous elements).

Take $n = \deg c - h - \deg a_2$; then from $R_h a_2 R_n = cR_0$ it follows that $c \in R_h a_2 R \subset a_1 R$, and thus $a_1 \mid c$. By definition of c, $c \mid a_2$, yielding $a_1 \parallel a_2$. By repeating the argument, we finally construct P, Q invertible matrices such that $\mathrm{PAQ} = \mathrm{diag}(e_1, \ldots, e_r, 0, \ldots, 0)$ and $e_i \parallel e_{i+1}$, $e_r \neq 0$.

We still have to prove that P and Q are graded matrices. Let $A_0 = A$ and A_n be the reduction of A after n steps in the procedure, and P_n, Q_n the invertible matrices such that $P_n A Q_n = A_n$, i.e. $a_{ij}^{(n)} = \Sigma_{t,r} p_{it}^{(n)} a_{tr} q_{rj}^{(n)}$. We claim that P_n, Q_n have homogeneous elements and that

$$\deg a_{ij}^{(n)} = \deg p_{it}^{(n)} + \deg a_{tr} + \deg q_{rj}^{(n)} \tag{7}$$

for all t, r, from which it follows that P_n and Q_n are graded. If $n = 1$, P_1 (or Q_1) is either of the form P_{ij}, $D_i(u)$ (u a unit of R), or $G(a', b', c', d')$ with a', b', c', d' satisfying the relations (5). Straightforward calculations shows that all these matrices are graded and that (7) is fulfilled in this case.

Suppose the claim is true for $n = 1$. Since A_n is obtained by applying the first step ($n = 1$) to A_{n-1}, we obtain

$$a_{ij}^{(n)} = \sum p'_{ik} \sum p_{kt}^{(n-1)} a_{tr} q_{rb}^{(n-1)} \Big) q'_{lj}$$

with

$$\deg a_{ij}^{(n)} = \deg p_{ik} \Big(\sum p_{kt}^{(n-1)} a_{tr} q_{rl}^{(n-1)} \Big) q_{lj} = \deg p_{ik} a_{hl}^{(n-1)} q_{lj}$$

for all k, l, t, r. The induction hypothesis yields that

$$\deg a_{ij}^{(n)} = \deg p_{ik} + \deg p_{kt}^{(n-1)} + \deg a_{tr} + \deg q_{rl}^{(n-1)} + \deg q_{lj}$$

for all k, l, t, r. It follows that $\Sigma_k p_{ik} p_{kt}^{(n-1)}$ and $\Sigma_l q_{rl}^{(n-1)} q_{lj}$ have homogeneous terms of the same degree, i.e.,

$$p_{it}^{(n)} = \sum_k p_{ik} p_{kt}^{(n-1)} \quad \text{and} \quad q_{rj}^{(n)} = \sum_l q_{rl}^{(n-1)} q_{lj}$$

are homogeneous, and (7) follows from the above calculations of $\deg a_{ij}^{(n)}$.

Finally we have to verify products of the form

$$\left(\begin{array}{cc|c} 1 & d & \\ 0 & 1 & 0 \\ \hline \multicolumn{2}{c|}{0} & I \end{array}\right) \cdot P_n = P_{n+1}.$$

It is easy to see that in order to obtain a graded matrix P_{n+1} it is necessary and sufficient to take $d \in R_{\deg p_{11}^{(n)} - dp_{21}^{(n)}}$. This is possible because in the proof d was chosen in some arbitrarily chosen R_h.

2. APPLICATIONS AND EXAMPLES

I. The Case $R_0[X, X^{-1}, \varphi]$

Let R_0 be a PID. The skew polynomial ring $R = R_0[X, X^{-1}, \varphi]$, where X is a variable and φ is an automorphism of R_0, is a graded PID. This follows immediately from the observation that every homogeneous ideal of R is generated by its part of degree zero [8]. The diagonalization procedure in that case may be reduced to the classical procedure over a PID. This follows from the following result.

LEMMA 2.1. *Any graded matrix over* $R = R_0[X, X^{-1}, \varphi]$, *where* R_0 *is a* PID., *is equivalent to a matrix over* R_0.

Proof. Let $A = (a_{ij})$ be a graded matrix over R. Multiplying the ith column by u_i, with $\deg u_i = -\deg a_{1i}$ (e.g. $u_i = x^{-\deg a_{1i}}$), and multiplying the jth row, $j \neq 1$, by a unit v_j, where $\deg v_j = -\deg a_{jj} - \deg u_j$, gives a matrix

$$A' = \begin{pmatrix} a_{11}u_1 & \cdots & a_{1n}u_n \\ v_2 a_{21} u_1 & \cdots & v_2 a_{2n} u_n \\ \vdots & & \vdots \\ v_m a_{m1} u_1 & \cdots & v_m a_{mn} u_n \end{pmatrix} =: (a'_{ij}),$$

which is graded and equivalent to A. Since A' is graded, it follows, considering the 1st jth rows and the jth kth columns, that

$$\deg a_{1j} + \deg a_{jk} + \deg u_j + \deg u_k + \deg v_j$$
$$= \deg a_{jj} + \deg a_{1k} + \deg u_j + \deg u_k + \deg v_j.$$

By construction it follows that the left hand side is equal to $\deg a'_{jk}$ and the right hand side is equal to zero, so $\deg a'_{jk} = 0$ for all j, k. ■

For completeness sake we give a characterization of Gr-PIDs which are skew polynomial rings.

LEMMA 2.2. *A Gr-PID R is of the form $R_0[X, X^{-1}, \varphi]$, where X is an homogeneous element of lowest strict positive degree in R, if and only if there is a unit element in every $R_i \neq 0$ and R_0 is a* PID.

Proof. The "only if" part is trivial.

Let R be a graded PID for which every $R_i \neq 0$ contains an invertible element. We first show that R_0 is a PID. If I_0 is any right ideal in R, then $I_0 R$ is a homogeneous right ideal in R, so it is principal, say $I_0 R = aR$. Take a unit element u of degree equal to $-\deg a$. Then $uaR = aR = I_0 R$ and ua is an element of R_0; therefore $uaR_0 = I_0$. The same argument holds for left ideals. Now choose a unit element of smallest strict positive degree in R, say u with $\deg u = s$. Since $R(R_0 u) = R$, u being a unit, it follows that $R_{ms} = R_0 u^m$ for all $m \in \mathbb{Z}$. If $s \nmid n$, say $n = sq + r$, $0 \neq r < s$; then R must be zero, for otherwise it would contain a unit element v, and this implies that vu^{-q} is a unit of degree $r < s$, which contradicts the assumption on u. It follows that every element of R may be written as a polynomial in u and u^{-1}. In order to have $R \cong R_0[u, u^{-1}, \varphi]$, we must verify the multiplication rule. Since $R_0 u = uR_0 = Rs$, we have for every $a \in R_0$ that $au = ua^{\varphi}$ for some $a^{\varphi} \in R_0$. It is easy to verify that the map $\varphi: R_0 \to R_0: a \mapsto a^{\varphi}$ defines an automorphism of R_0. A straightforward calculation now shows that $R \cong R[X, X^{-1}, \varphi]$, where the isomorphism is defined by sending $u \mapsto X$.

II. Generalized Rees Rings

There is an important class of Gr-PIDs which are not of the form $R_0[X, X^{-1}, \varphi]$. Let R_0 be a (commutative) Dedekind domain with cyclic class group, i.e., every ideal J of R_0 is equal to $I^n\,(a)$ for some fixed ideal I of R_0, $a \in K$. If K is the quotient field of R_0 then consider the subring $R = \oplus_{n \in \mathbb{Z}} I^n X^n$ of $K[X, X^{-1}]$, X a variable.

Note that I^n is defined for negative n, since every ideal J of a Dedekind domain is invertible, its inverse being $J^{-1} = \{r \in K | rJ \subset R_0\}$. For more details about Dedekind domains and class groups we refer to [3]. The rings $R = \oplus_{n \in \mathbb{Z}} I^n X^n$ were introduced in [8] where they are called generalized Rees rings. There it is shown that generalized Rees rings are graded Dedekind domains (i.e., the homogeneous ideals form a multiplicative group), and that the homogeneous ideals are generated by their part of degree zero.

In the case where the class group of R is cyclic, as assumed, we claim that $R = \oplus_{n \in \mathbb{Z}} I^n X^n$ is a Gr-PID. For, since $I^{-1}X^{-1} \subset R$, we have $I^{-1}X^{-1}I \subset RI$, so $RX^{-1} \subset RI$. But $(RI)_{-1} = R_{-1}I = I^{-1}X^{-1}I = R_0 X^{-1}$; therefore

$RI = R(RI)_{-1} = RX^{-1}$, i.e., RI is a principal ideal in R. Now if J is any homogeneous ideal of R, then $J = RJ_0$; therefore $J = RIR_0a = RX^{-1}R_0a = RX^{-1}a$, i.e., J is also a principal ideal of R.

In view of Lemma 2.2 it is clear that R is not of the form $R_0[X, X^{-1}]$, unless R_0 is a PID, which is not necessarily the case.

III. Diagonalization of Matrices over Dedekind Domains with Cyclic Class Group

If R_0 is a Dedekind domain, then it is not possible to diagonalize every matrix over R. Indeed, let I be an ideal of R_0 which is not principal; then $I = R_0a + R_0b$ [31]. Consider the matrix $(a\ b)$. It is not possible to diagonalise this matrix, since otherwise I would be a principal ideal.

Even if a matrix is diagonalizable, there is no way to obtain a diagonalization, since pairs of elements do not allow greatest common divisors. We now show that, using graded techniques, it is possible to obtain a diagonal form for matrices over Dedekind domains with cyclic class group.

Let $R = \oplus_{n \in \mathbf{Z}} I^n X^n$, I a generator of the class group of R_0. We saw that R is a Gr-PID. Consider now any matrix $A = (a_{ij})$ over R_0. This is obviously a graded matrix over R, so it is possible to diagonalize it over R. There are two possibilities:

(1) The matrix A is R-equivalent to a diagonal matrix over R_0.
(2) The matrix A is not R-equivalent to a diagonal matrix over R_0.

We first show that in the first case the matrix is also R_0-equivalent to the diagonal matrix.

PROPOSITION 2.3. *Let R_0 and R be as above. A matrix A over R_0 is R_0-equivalent to a diagonal matrix over R iff it is R-equivalent to a diagonal matrix over R_0.*

Proof. We use the following known result for matrices over Dedekind domains:

(**) Given an $m \times n$ matrix A over a Dedekind domain S, let M_A denote the S-submodule of $S^{(n)}$ generated by the rows of A; and let $N_A = S^{(n)}/M_A$. Then for $m \times n$ matrices A and B over S, A is equivalent to B iff $N_A \cong N_B$ as S-modules [4], Theorem 1.1].

Since the "only if" part is trivial we only have to prove that if $A \underset{R}{\sim} \operatorname{diag}(a_1, \ldots, a_r, 0, \ldots, 0)$ with $a_i \in R_0$, then $A \underset{R_0}{\sim} \operatorname{diag}(a_1, \ldots, a_r, 0, \ldots, 0)$.

As in the nongraded case, it follows that $N_A = R^{(n)}/M_A$ is graded isomorphic to a direct sum of cyclic matrices, say $R/\operatorname{ann} z_1 \oplus \cdots \oplus R/\operatorname{ann} z_r$, where $\operatorname{ann} z_i = Ra_i$. If the entries of A as well as the a_i are of degree zero, it follows that $(R^{(n)}/M_A)_0 = R_0^{(n)}/(M_A)_0$ and $(R/\operatorname{ann} z_i)_0 = R/(\operatorname{ann} z_i)_0$; therefore $R_0^{(n)}/(M_A)_0 = R_0/\operatorname{ann} z_1 \oplus \cdots \oplus R_0/\operatorname{ann} z_r$. But since the latter is also isomorphic to $R_0^{(n)}/M_{\operatorname{diag}(a_1,\ldots,a_r,0,\ldots,0)}$ it follows from (**) that $A \underset{R_0}{\sim} \operatorname{diag}(a_1,\ldots,a_r,0,\ldots,0)$. ■

EXAMPLE. Consider $\mathbb{Z}[\sqrt{-5}]$. This is a Dedekind domain which is not a PID; its class group is $\mathbb{Z}/2\mathbb{Z}$. The ideal $I = (3, 1+2\sqrt{-5})$ is not a principal ideal of $\mathbb{Z}[\sqrt{-5}]$; therefore it generates the class group of $\mathbb{Z}[\sqrt{-5}]$, i.e. $\langle \bar{I} \rangle = \operatorname{Cl}(\mathbb{Z}[\sqrt{-5}])$. It is now possible to reduce the matrix

$$A = \begin{pmatrix} 3 & 1+2\sqrt{-5} \\ 1-2\sqrt{-5} & 3 \end{pmatrix}$$

to a diagonal matrix of $R = \oplus_{n \in \mathbb{Z}} I^n X^n$.

1st step. Since $3R + (1+2\sqrt{-5})R = X^{-1}R$, X^{-1} is the HCF of 3 and $1+2\sqrt{-5}$. We now look for a solution of $3\alpha + (1+2\sqrt{-5})\beta = 1$ with $\alpha, \beta \in I^{-1}$. Then $\alpha X^{-1}, \beta X^{-1}$ are elements of R and a solution of the equation $3Xz_1 + (1+2\sqrt{-5})Xz_2 = 1$, with $z_1, z_2 \in R$. Take $\alpha = 77/(1+2\sqrt{-5})$, $\beta = 10+2\sqrt{-5}$; then β is trivially in I^{-1}, since it is in $\mathbb{Z}[\sqrt{-5}]$. Since 21 has two decompositions in irreducible factors, namely $3\times 7 = (1+2\sqrt{-5})(1-2\sqrt{-5})$, it follows that $\alpha \times 3 = 7\times 21/(1+2\sqrt{-5}) = 7(1-2\sqrt{-5}) \in \mathbb{Z}[\sqrt{-5}]$ and $\alpha(1+2\sqrt{-5}) = 77 \in \mathbb{Z}[\sqrt{-5}]$, so $\alpha \in I^{-1}$. In view of the general method used in Lemma 1.2, we obtain

$$\begin{pmatrix} 3 & 1+2\sqrt{-5} \\ 1-2\sqrt{-5} & 3 \end{pmatrix} \begin{pmatrix} \dfrac{77}{1+2\sqrt{-5}} X^{-1} & 1+2\sqrt{-5} & X \\ (10+2\sqrt{-5})X^{-1} & -3 & X \end{pmatrix}$$

$$= \begin{pmatrix} X^{-1} & 0 \\ \dfrac{-119-26\sqrt{-5}}{3} & 12X \end{pmatrix}.$$

We call the last matrix A_1.

2nd step. We now simplify A_1:

$$\begin{pmatrix} X^{-1} & 0 \\ \dfrac{-119-26\sqrt{-5}}{3}X^{-1} & 12X \end{pmatrix}\begin{pmatrix} 1 & 0 \\ \frac{9}{3}X^{-2} & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ \dfrac{3\sqrt{-5}}{3}X^{-2} & 1 \end{pmatrix}$$

$$= \begin{pmatrix} X^{-1} & 0 \\ \dfrac{-11+10\sqrt{-5}}{3}X^{-1} & 12X \end{pmatrix}.$$

3rd step. We have to calculate the HCF of

$$RX^{-1} + R\frac{-11+10\sqrt{-5}}{3}X^{-1} = 1.$$

Since

$$RX^{-1} \subset RX^{-1} + R\frac{-11+10\sqrt{-5}}{3}X^{-1},$$

and since RX^{-1} is a graded maximal ideal, the HCF is either 1 or X^{-1}. A straightforward calculation shows that X^{-1} is not possible, so

$$RX^{-1} + R\,\frac{-11+10\sqrt{-5}}{3}X^{-1} = 1.$$

Again we have to solve the equation

$$\alpha X^{-1} + \beta\frac{-11+10\sqrt{-5}}{3}X^{-1} = 1$$

with $\alpha, \beta \in R$, i.e.,

$$x + y\frac{11+10\sqrt{-5}}{3} = 1 \qquad \text{with} \quad x, y \in I.$$

The solution $\alpha = [-15+5(1+2\sqrt{-5})]X$, $\beta = -3X$ will do. Then

$$\begin{pmatrix} -15+5(1+2\sqrt{-5}) & X & -3X \\ \dfrac{-11+10\sqrt{-5}}{3} & X^{-1} & -X^{-1} \end{pmatrix} \begin{pmatrix} X^{-1} & 0 \\ \dfrac{-11+10\sqrt{-5}}{3} & 12X \end{pmatrix}$$

$$= \begin{pmatrix} 1 & -36X^2 \\ 0 & -12 \end{pmatrix},$$

and this matrix is denoted A_2.

4th step. Since the 2nd entry of the first row of A_2 is a multiple of the 1*st* entry of this row, we finally obtain

$$\begin{pmatrix} 1 & -36X^2 \\ 0 & -12 \end{pmatrix} \begin{pmatrix} 1 & 36X^2 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -12 \end{pmatrix}$$

Conclusion: The result obtained is a diagonal matrix over $\mathbb{Z}[\sqrt{-5}]$, and by Proposition 2.3 it follows that

$$A \underset{\mathbb{Z}[\sqrt{-5}]}{\sim} \operatorname{diag}(1, -12).$$

Indeed, it was found with a computer by K. Coppieters that

$$\begin{pmatrix} -8+\sqrt{-5} & 2+\sqrt{-5} \\ -1+\sqrt{-5} & 1 \end{pmatrix} \begin{pmatrix} 3 & 1+2\sqrt{-5} \\ 1-2\sqrt{-5} & 3 \end{pmatrix}$$

$$\begin{pmatrix} -8-\sqrt{-5} & -1-\sqrt{-5} \\ 2-\sqrt{-5} & 1 \end{pmatrix} = \begin{pmatrix} 12 & 0 \\ 0 & -1 \end{pmatrix}.$$

REMARKS.

(1) Note that in this first case the method only allows you to find the diagonal form of the matrix over the ground ring (in this case $\mathbb{Z}[\sqrt{-5}]$) and not the matrices P and Q over the ground ring for which PAQ $= \operatorname{diag}(e_1, \ldots, e_r, 0, \ldots, 0)$.

(2) If the matrix A over R_0 is not R-equivalent to a diagonal matrix over R_0, we do find a reduction of A over $R = \oplus_{n \in \mathbb{Z}} I^n X^n$, which can be useful in some cases, illustrated by the following application.

IV. Von Neumann Regular Inverses of Graded Matrices over Gr-PIDs and of Matrices over Dedekind Domains with Cyclic Class Group

An $m \times n$ matrix A over a ring R is called von Neumann regular iff there exists an $n \times m$ matrix X over R such that $AXA = A$. In the general case it is well known that the $m \times n$ matrix A is von Neumann regular iff $\operatorname{Ker} A$ and $\operatorname{Im} A$ are direct summands of R^n, R^m respectively. In case A is R-equivalent with a diagonal matrix D, then it is easy to see that A is von Neumann regular iff D is von Neumann regular.

If now A is a graded $m \times n$ matrix over a Gr-PID R, then there exist invertible graded matrices P and Q such that $\mathrm{PAQ} = \operatorname{diag}(e_1, \ldots, e_r, 0, \ldots, 0)$. Since R has no zero divisors different from zero, A is von Neumann regular iff the e_i's are unit elements of R. Since unit elements are homogeneous, there exists a von Neumann regular inverse which is graded. We therefore have:

A graded $m \times n$ matrix over a Gr-PID *R has a graded von Neumann regular inverse iff A is R-equivalent to the matrix $I_r \oplus 0$, with I_r the $r \times r$ unit matrix. Moreover, all graded von Neumann regular inverses of A are given by*

$$Q\begin{pmatrix} I_r & U \\ V & W \end{pmatrix}P,$$

if P and Q are given by

$$PAQ = \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}$$

with U, V, W arbitrarily chosen such that $\begin{pmatrix} I_r & U \\ V & W \end{pmatrix}$ is graded.

As we have seen, matrices over Dedekind domains R_0 with cyclic class group can be considered as graded matrices over generalized Rees rings R. Although the ring R is very complicated, it is possible to determine all its units. It turns out that the new units can be easily calculated from the units in R_0. A unit in a graded domain must be a homogeneous element, so if z is a unit in R, then $z = aX^n$ for some n and $a \in R_0$. Then $z^{-1} = a'X^{-n}$ and $aa' = 1$, with $a \in I^n$ and $a' \in I^{-n}$ (or vice versa). Therefore $a'I^n \subset R$ and $I^n \subset Ra$, but also $Ra \subset I^n$, i.e., I^n is a principal ideal. So n must be a multiple of e, the order of the class group of R_0. It follows that $U(R) = \{ua^{ne}X^{ne} \mid n \in \mathbb{Z}, u \in U(R_0)$ and $aR = I^e\}$. So the degree of a unit in R is always a multiple of e. [If the class group is of infinite order, then $U(R) = U(R_0)$.]

EXAMPLES. The matrix

$$\begin{pmatrix} 3 & 0 & 1+2\sqrt{-5} \\ 1-2\sqrt{-5} & 0 & 3 \\ 0 & 0 & 0 \end{pmatrix}$$

is equivalent over R and R_0 to the matrix

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & -12 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

So, since -12 is not invertible in $\mathbb{Z}[\sqrt{-5}]$, the matrices are not von Neumann regular.

The matrix

$$\begin{pmatrix} 3 & 1+2\sqrt{-5} & 0 \\ 1+2\sqrt{-5} & 3 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

is not von Neumann regular, since the diagonal form

$$\begin{pmatrix} X^{-1} & 0 & 0 \\ 0 & (28-4\sqrt{-5})X & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

has diagonal entries of degree -1 and $+1$, which are not multiples of 2 (the order of the class group of $R_0 = \mathbb{Z}[\sqrt{-5}]$), so they can not be units in R.

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