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A class of bivariate exponential distributions

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ABSTRACT

We introduce a class of absolutely continuous bivariate exponential distributions, generated from quadratic forms of standard multivariate normal variates.

This class is quite flexible and tractable, since it is regulated by two parameters only, derived from the matrices of the quadratic forms: the correlation and the correlation of the squares of marginal components. A simple representation of the whole class is given in terms of 4-dimensional matrices. Integral forms allow evaluating the distribution function and the density function in most of the cases.

The class is introduced as a subclass of bivariate distributions with chi-square marginals; bounds for the dimension of the generating normal variable are underlined in the general case.

Finally, we sketch the extension to the multivariate case.

1. Introduction

The exponential distribution plays a fundamental role as a model in a variety of applications, typically connected with survival time, in some of its many forms of appearance. Its popularity is witnessed by the existence of a very broad literature in this area; the AMS MathSciNet lists over a thousand titles on the exponential distribution. Detailed reviews can be found in Johnson, Kotz, and Balakrishnan [1], in Balakrishnan and Basu [2] and, for the bivariate and the multivariate cases, in Barnett [3], Basu [4,5], and in Kotz, Balakrishnan and Johnson [6].

Unfortunately, unlike the normal distribution, the exponential distribution does not have a natural extension to the bivariate or the multivariate case. Therefore, a large number of classes of bivariate distributions with exponential marginals have been proposed since 1960 [7], and some examples can be found twenty years earlier as particular cases of bivariate gamma distributions [8]. None of the models proposed is completely satisfactory: some of them have a singular component (e.g. [9]); most proposals are indexed only by the correlation (e.g. [7,9–13]). Multiparameter bivariate exponential distributions have been suggested [14–17], but the parameters introduced in the construction of these models do not have a clear meaning. Moreover many bivariate exponential models are not easily extendible to the multivariate case.

In the present paper, we explore the probabilistic aspects of a class, $\mathcal{G}(2,2)$, of absolutely continuous bivariate exponential distributions: the full class of quadratic forms of a multivariate standard normal variable with given exponential marginal distributions. The expression of the moment generating function is derived from the structure of the underlying matrix. This construction leads to a flexible class indexed by two parameters: the correlation of the marginal components and the correlation of their squares. A representation theorem describes all the variates and exploits only 4-dimensional matrices, given in terms of the two parameters. Therefore, sampling from these distributions may be simulated easily.

This fact is especially relevant in practical work since the distribution function and the density function of the above class do not have a simple explicit expression, in general. For most of the elements in $\mathcal{G}(2,2)$, in Section 3.3, we obtain integral forms of the joint distribution function and the joint density function which allow a direct computation. Alternatively,

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one may follow the traditional way, which expresses the density function of quadratic forms of normal variates in terms of modified Bessel functions, power series or Laguerre polynomials; for work in this direction, see [8,18,19,16], and the reviews in [20,21]. All these authors only deal with particular cases of quadratic forms in normal variates, but possibly their techniques can be extended to the full class.

It may be noted that the variables studied in the present paper have non-negative correlation: this restriction is shared by most of the bivariate exponential models proposed in literature. In the full Frechet class with given exponential marginals, the correlation ranges in $[1-\frac{\pi^2}{6}, 1]$ (see [22]). As far as I know, none of the proposed bivariate exponential models is complete with respect to this range: very few have negative correlation only and just in two cases [7,17] there are models whose range is both positive and negative, but it is only a limited subset of the full admissible range.

The bivariate exponential class studied in this paper is presented in the framework of bivariate quadratic forms in standard normal variates whose marginal components have chi-square distributions, possibly with different degrees of freedom m and p; this class is denoted by $\mathcal{G}(m, p)$. Extending the result for the exponential case, we find the minimal dimension of the normal variable which is sufficient to describe the full class $\mathcal{G}(m, p)$.

Finally the natural multivariate extension is mentioned.

2. A general class of joint quadratic forms

Let $X = (X_1, \ldots, X_n)'$ be a random vector such that its components are mutually independent and each component X_i has standard normal distribution.

For the representation of quadratic forms, there is no loss of generality in considering only symmetric matrices. It is well-known that a symmetric matrix A is idempotent with rank m if and only if the random variable $X'AX$ has chi-square distribution with m degrees of freedom, denoted χ^2_m (e.g. Mathai and Provost [20, Chapter 5]).

The aim of the present section is to study the joint distribution of pairs of χ^2_m and χ^2_p, when they are both generated via quadratic forms of the same n-dimensional normal variate. Denote by $\mathcal{G}(m, p)_n$ this class of distributions, and by $\mathcal{G}(m, p)$ the union of these distributions as n varies.

The subsequent developments rest on a linear algebra result which is stated in the next lemma.

Lemma 1. *The following properties are equivalent for an $n \times n$ symmetric matrix A.*

(i) *Matrix A is idempotent with rank m.*
(ii) *The function $f(x) = Ax$ is an orthogonal projection on a subspace of dimension m.*
(iii) *There exist m vectors $c_{(i)}$, orthogonal with unit norm, such that $A = \sum_{s=1}^{m} c_{(s)}c'_{(s)}$, or equivalently an orthonormal matrix C of size $n \times m$, such that $A = CC'$.*
(iv) *There exist $n - m$ vectors $h_{(s)}$, orthogonal with unit norm, such that $A = I_n - \sum_s h_{(s)}h'_{(s)}$, or equivalently there exists an orthonormal matrix A_*, of size $n \times (n - m)$, such that $A = I_n - A_*(A_*)'$.*

A simple and direct proof of Lemma 1 is given in an Appendix. Alternatively, it could be deduced elaborating on and linking results which are in classical texts; see for instance Harville [23, mainly Section 12.3] or Ortega [24].

Consider a pair of matrices, $A = (a_{i,j})$ with rank m and $B = (b_{i,j})$ with rank p. Let $(U, V) \in \mathcal{G}(m, p)_n$, given by $U = X'AX$, $V = X'BX$. Recall that, for quadratic forms of standard normal variates, the covariance is given by (e.g. [20, p. 75])

$$\mathrm{Cov}(U, V) = 2\sum_{i,j} a_{ij}b_{ij} = 2\,\mathrm{tr}\,(AB).$$

Let us write $A = \sum_{s=1}^{m} c_{(s)}c'_{(s)}$ and $B = \sum_{s=1}^{p} d_{(s)}d'_{(s)}$, in the form of property (iii). Then we have

$$\mathrm{Cov}(U, V) = 2\sum_{s,t}(c'_{(s)}d_{(t)})^2$$

and the corresponding correlation coefficient is

$$\mathrm{Corr}(U, V) = \frac{\sum_{s,t}(c'_{(s)}d_{(t)})^2}{\sqrt{mp}} = \rho, \tag{1}$$

say. Further, write matrices A and B as indicated in property (iv), i.e. they are identified via A_* and B_*, respectively. On setting $A_*(A_*)' = \sum_{s=1}^{n-m} h_{(s)}h'_{(s)}$ and $B_*(B_*)' = \sum_{s=1}^{n-p} k_{(s)}k(s)'$, therefore the covariance is

$$\mathrm{Cov}(U, V) = 2\left[m + p - n + \sum_{s,t}(h'_{(s)}k_{(t)})^2\right]$$

leading to

$$\rho = \frac{m + p - n + \sum_{s,t}(h'_{(s)}k_{(t)})^2}{\sqrt{mp}}. \tag{2}$$

This correlation coefficient turns out to be always non-negative, and it varies in $[l, L]$, where

$$l = \frac{\max\{0, m+p-n\}}{\sqrt{mp}}, \qquad L = \frac{\min\{m,p\}}{\sqrt{mp}}.$$

In fact, looking for the minimal value, if $m + p \leq n$, two orthogonal matrices A and B can be selected. By Craig's theorem [20, p. 209], the corresponding quadratic forms are independent. Then the value 0 of the correlation is achieved. Conversely, $\rho = 0$ means $\operatorname{tr} AB = 0$, then A and B are orthogonal and the corresponding quadratic forms are independent. If $m+p > n$, there exist no orthogonal matrices A and B of rank m and p. On writing ρ as in (2), it is apparent that ρ is minimal when $\sum_{s,t}(h'_{(s)}k_{(t)})^2$ is minimal. Since $(n-m)+(n-p) < n$, all vectors $h_{(s)}$ can be chosen orthogonal to all $k_{(t)}$, and in that case $\rho = (m+p-n)/\sqrt{mp}$.

Looking for the maximal value, when $m \leq p$, the maximum of $\sum_{s,t}(c'_{(s)}d_{(t)})^2$ is m and it is achieved when all $c_{(s)}$'s lie on the subspace spanned by the $d_{(t)}$'s. In that case $\rho = m/\sqrt{mp}$, which corresponds to the fact that U is an additive component of V (see [20, p. 207]), or $U = V$ when $m = p$.

Therefore, $\mathcal{G}(m,p)_n$ contains independent pair if and only if $m + p \leq n$, in this case $\rho = 0$ is equivalent to the independence of the marginals. In $\mathcal{G}(m,p)_n$ there is the identical pair, (U, U), if and only if $m = p$.

3. A class of bivariate exponentials

When A and B have rank 2, the subclass $\mathcal{G}(2,2)_n$ is a set of bivariate distributions whose components are exponential with mean 2. The case with $n = 2$ is trivial since the identity is the only idempotent 2×2 matrix with rank 2; then $\mathcal{G}(2,2)_2 = \{U, U\}$, where $U = X^{\mathrm{T}}X$. Therefore in the following we study in detail $\mathcal{G}(2,2)_n$, assuming $n \geq 3$.

3.1. The moment generating function

The moment generating function of (U, V) is $M_{U,V}(u, v) = \{P(2u, 2v)\}^{-1/2}$ where

$$P(y,z) = \det(I - yA - zB) \tag{3}$$

is a polynomial which identifies $M_{U,V}$ (e.g. [20, p. 67]).

Consider first the subclass $\mathcal{G}(2,2)_3$, that is the set of bivariate exponential distributions generated by quadratic forms of a trivariate normal random variable. This class is especially simple to describe, but it is rather meagre too: the class is regulated by a single parameter, the correlation coefficient, ρ, and the marginal components are strongly correlated. In fact, from statement (iv) of Lemma 1, there exist two vectors h and k with unit norm such that $A = I_3 - hh'$ and $B = I_3 - kk'$. Then polynomial (3) takes the form

$$P(y,z) = 1 - 2(y+z) + (y+z)^2 + ((h'k)^2 - 1)(y^2z + yz^2 - yz).$$

Therefore the correlation coefficient identifies the joint distribution: in fact, $\rho = \{1 + (h'k)^2\}/2$. We have then proved the next statement.

Theorem 2. *In $\mathcal{G}(2,2)_3$, polynomial (3) is given by*

$$P(y,z) = 1 - 2(y+z) + (y+z)^2 + 2(\rho - 1)(y^2z + yz^2 - yz),$$

where $\rho = \operatorname{Corr}(U, V)$.

Finally, note that in this case, the marginal components are strongly correlated, since ρ ranges only within the interval $[2^{-1}, 1]$. The range of ρ becomes wider if the dimension of X increases: in fact, if $n > 3$, then $\mathcal{G}(2,2)_n$ will include pairs of independent exponential distributions, as specified at the end of previous section.

If $n > 3$, it is convenient to write the matrices in form of condition (iii) of Lemma 1. Notice that an orthogonal transformation Ψ of the space of $\mathbb{R}^n$ transforms a matrix A into $\Psi A \Psi'$. Hence, polynomial $P(y,z)$ and the moment generating function remain unchanged.

Therefore, the simplest way for generating all distributions in $\mathcal{G}(2,2)_n$ is to select an appropriate reference element. Without loss of generality, a pair of quadratic forms can be written via two matrices

$$A = cc' + dd', \qquad B = ee' + ff' \tag{4}$$

where vectors c, d are arbitrary orthonormal vectors, while $e = (1, 0, 0, \ldots 0)', f = (0, 1, 0, \ldots, 0)'$.

Lemma 3. *If A and B are as in (4), in $\mathcal{G}(2,2)_n$, polynomial (3) takes the form*

$$P(y,z) = 1 - 2(y+z) + (y+z)^2 + (a_{11} + a_{22} - 2)(y^2z + yz^2 - yz) + y^2z^2 \sum_{3 \leq i < j} (a_{ii}a_{jj} - a_{ij}^2).$$

and $\rho = \frac{1}{2}(a_{11} + a_{22})$.

The proof is given in an Appendix.

Let us consider now the general case, where both (c, d) and (e, f) are arbitrary pairs of orthonormal vectors in $\mathbb{R}^n$. Let l_1 and l_2 denote respectively the projections of c and d on the subspace $T = \mathcal{C}(B)$ spanned by the columns of B.

Theorem 4. *In $\mathcal{G}(2, 2)_n$, polynomial (3) is*

$$P(y, z) = 1 - 2(y + z) + (y + z)^2 + 2(\rho - 1)(y^2 z + yz^2 - yz) + \gamma y^2 z^2$$

where

$$\rho = \frac{1}{2}(\|l_1\|^2 + \|l_2\|^2), \qquad \gamma = 1 - \|l_1\|^2 - \|l_2\|^2 + \|l_1\|^2 \|l_2\|^2 \sin^2(l_1 l_2).$$

Proof. Assume first that hypotheses of Lemma 3 hold, and consider the corresponding polynomial. Since $2\rho = a_{11} + a_{22}$, clearly $\rho = (\|l_1\|^2 + \|l_2\|^2)/2$. Let $T^\perp$ be the subspace orthogonal complement of T; let g_1 and g_2 denote respectively the projections of c and d on $T^\perp$; namely $g_1 = (0, 0, c_3, \ldots, c_n)$ and $g_1 = (0, 0, d_3, \ldots, d_n)$. It is the easy to check that the coefficient of $y^2 z^2$ is

$$\gamma = \|g_1\|^2 \|g_2\|^2 \sin^2(g_1, g_2) = \|g_1\|^2 \|g_2\|^2 - (g_1' g_2)^2.$$

On recalling that c and d are orthonormal, and $c = l_1 + g_1$ and $d = l_2 + g_2$, where l_i is orthogonal to g_i $(i = 1, 2)$, we obtain that $\|g_i\|^2 = 1 - \|l_i\|^2$ and $(g_1' g_2) = -(l_1' l_2)$. Therefore

$$\gamma = 1 - \|l_1\|^2 - \|l_2\|^2 + \|l_1\|^2 \|l_2\|^2 - (l_1' l_2)^2 = 1 - 2\rho + \|l_1\|^2 \|l_2\|^2 \sin^2(l_1 l_2)$$

and the conclusion holds.

Consider next the general case and notice that ρ and γ depend on $\|l_1\|^2$, $\|l_2\|^2$ and $(l_1' l_2)^2$ only; because of the definition of l_1 and l_2, these terms are invariant under orthogonal transformations, then the conclusion holds in the general case. □

Notice that under the assumptions of Lemma 3, both ρ and γ depend only on the first two coordinates of c and d, being

$$2\rho = (c_1^2 + d_1^2 + c_2^2 + d_2^2), \qquad \gamma = 1 - (c_1^2 + d_1^2 + c_2^2 + d_2^2) + (c_1 d_2 - c_2 d_1)^2.$$

In $\mathcal{G}(2, 2)_n$, with $n > 3$, the correlation coefficient assumes all non-negative values. Of course the case of $\rho = 1$ correspond to the case of identity of the two marginals. The case $\rho = 0$ is equivalent to the independence (as for the bivariate normal).

In all other cases, one can build infinitely many different distributions with the same correlation coefficient. In fact

$$\gamma \in [\max\{1 - 2\rho, 0\}, (1 - \rho)^2],$$

namely γ is zero if the marginal components are equal, while it is 1 only in case of independence.

Notice that parameter γ appears in mixed moments of higher order, which can be obtained by differentiating $M_{U,V}(u, v)$ several times and then evaluating the result at $u = v = 0$. The lowest order term where γ occurs is $E(U^2 V^2)$, then γ characterizes the correlation of (U^2, V^2), such that

$$\mathrm{Corr}(U^2, V^2) = \frac{1 - \gamma + 6\rho + 3\rho^2}{10}.$$

3.2. Representation theorem

Increasing the dimension of X beyond 4 does not make the class any broader, as it is shown by the next result.

Theorem 5. *All bivariate exponential distributions generated by quadratic forms of an n-dimensional vector X can be obtained from a 4-dimensional vector, that is $\mathcal{G}(2, 2) = \mathcal{G}(2, 2)_4$.*

Proof. Since A and B have rank 2, they are identified by 2 pairs of vectors. Those 4 vectors generate a space which is at most 4-dimensional. Therefore, there exists an orthogonal transformation, Ψ, which maps $\mathcal{C}(A, B)$ into the subspace generated by the first 4 coordinate vectors. Hence the joint distribution of $(X'AX, X'BX)$ is the same of those of two quadratic forms involving only the first 4 normal variates of X. □

A simple representation for all distributions of $G(2, 2)$ is given in the following theorem which provides another parametrization, equivalent to the previous one, but more convenient.

Theorem 6. *Every bivariate exponential distribution in $G(2, 2)$ is the distribution of a pair (U, V), with $U = X'AX$ and $V = X'BX$, where X is a 4-variate standard normal,*

$$B = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

and, given $\rho \in [0, 1]$ and $\delta \in [0, \min(\rho, 1 - \rho)]$,

$$A(\delta)=\begin{pmatrix}\rho & -\delta & \alpha-\beta & \alpha+\beta\\ -\delta & \rho & \alpha+\beta & \alpha-\beta\\ \alpha-\beta & \alpha+\beta & 1-\rho & \delta\\ \alpha+\beta & \alpha-\beta & \delta & 1-\rho\end{pmatrix}$$

where

$$\alpha=\alpha(\rho,\delta)=\frac{\sqrt{(\rho-\delta)(1-\rho+\delta)}}{2},$$
$$\beta=\beta(\rho,\delta)=\frac{\sqrt{(\rho+\delta)(1-\rho-\delta)}}{2}.$$

Moreover parameter γ is given by

$$\gamma=\gamma(\delta)=(1-\rho)^2-\delta^2.$$

Proof. Due to Theorem 5 every joint distribution can be written by means of a 4-dimensional vector X; use the form (4), where the matrix A is defined as $A=cc'+dd'$, where
let

$$c=\left(\sqrt{(\rho-\delta)/2},\sqrt{(\rho-\delta)/2},\sqrt{\frac{1-\rho+\delta}{2}},\sqrt{\frac{1-\rho+\delta}{2}}\right),$$
$$d=\left(-\sqrt{(\rho+\delta)/2},\sqrt{(\rho+\delta)/2},\sqrt{\frac{1-\rho-\delta}{2}},-\sqrt{\frac{1-\rho-\delta}{2}}\right).$$

Since $\gamma=a_{33}a_{44}-a_{34}^2$, then $\gamma=\gamma(\delta)=(1-\rho)^2-\delta^2$. It assumes all possible values as δ varies. In particular we obtain:

$$\max\gamma=\gamma(0)=(1-\rho)^2,$$

and

$$\min\gamma=\begin{cases}\gamma(\rho)=1-2\rho & \rho\le 1/2\\ \gamma(1-\rho)=0 & \rho\ge 1/2.\end{cases}\quad\square$$

Under the above representation, polynomial (3) takes the form

$$P(y,z)=1-2(y+z)+(y+z)^2+2(\rho-1)(y^2z+yz^2-yz)+((1-\rho)^2-\delta^2)y^2z^2.$$

Notice that δ^2, as well as γ, is a function of $\mathrm{Corr}(U,V)$ and $\mathrm{Corr}(U^2,V^2)$.

3.3. *The distribution function*

It is very hard to give a general expression of a joint distribution in $\mathcal{G}(2,2)$, however, we may draw some conclusions, by distinguishing the case $\gamma>0$ and $\gamma=0$.

If $\gamma>0$, it is possible to find an explicit form both for the distribution and the density function, given by integral forms. Let $U=X'AX$, and $V=X'BX$, with X a 4-dimensional standard normal and A and B as in (4). Therefore $\gamma=a_{33}a_{44}-a_{34}^2$ and, given the structure of A, the assumption $\gamma>0$ implies $a_{33}>0$ and $a_{44}>0$. Given $s\in(0,1)$, let U_s be the subset of $\mathbb{R}^4$ where $U\le s$ occurs. Therefore U_s can be described as follows:

$$U_s=\{(x_1,x_2,x_3,x_4):z_1\le x_4\le z_2;\ w_1\le x_3\le w_2\},$$

where for $k=1,2$

$$z_k=z_k(s)=\frac{-\sum_{i=1}^{3}a_{i4}x_i\pm\sqrt{a_{44}s-\sum_{i,j=1}^{3}(a_{ij}a_{44}-a_{i4}a_{j4})x_ix_j}}{a_{44}}$$

and

$$w_k=w_k(s)=\frac{\sum_{i=1}^{2}(a_{i4}a_{34}-a_{i3}a_{44})x_i\pm\sqrt{\gamma a_{44}s}}{\gamma}.$$

Given $t\in(0,1)$, let V_t be the subset of $\mathbb{R}^4$ where $V\le t$ occurs. Being $V=X_1^2+X_2^2$, it results

$$V_t=\{(x_1,x_2,x_3,x_4):|x_1|\le\sqrt{t};\ y_1\le x_2\le y_2\},$$

where $y_1=y_1(t)=-\sqrt{t-x_1^2}$ and $y_2=y_2(t)=\sqrt{t-x_1^2}$.

Of course the joint distribution function of (U, V) is given by $F(s, t) = P((U, V) \in U_s \cap V_t)$. Therefore, if ϕ is the standard normal density, it can be expressed as

$$F(s,t) = \int_{-\sqrt{t}}^{\sqrt{t}} \mathrm{d}x_1 \int_{y_1(t)}^{y_2(t)} \mathrm{d}x_2 \int_{w_1(s)}^{w_2(s)} \mathrm{d}x_3 \int_{z_1(s)}^{z_2(s)} \prod_{i=1}^{4} \phi(x_i)\mathrm{d}x_4. \tag{5}$$

Since t and s are separated in $F(s, t)$, then the density is quite easily computable as the second mixed derivative of $F(s, t)$, as Example 1 shows.

Consider now the case $\gamma = 0$. In the expression of $V_t \cap U_s$ the two variables s, t cannot be separated; for some variable, let us say x_2, the limitation are

$$y_1(x_1, s, t) \le x_2 \le y_2(x_1, s, t)$$

where $y_1(x_1, s, t) = \max\{h(t), g_1(x_1, s)\}$ and $y_2(x_1, s, t) = \min\{k(t), g_2(x_1, s)\}$, for suitable h, k and g_i's. Therefore there is a set S where the derivative $f(s, t)$ does not exist. Namely, let $v = \sum a_{ij}x_ix_j$ and $u = \sum b_{ij}x_ix_j$, then it is

$$S = \{(x_1, x_2, x_3, x_4) : h(v) = g_1(x_1, u) \text{ or } k(v) = g_2(x_1, u)\}.$$

However set S has probability 0, as shown in the Example 2. In this well-known example it is easy find the density, while in the general case the expression of F is complex, its derivatives are intractable and it should be more convenient to resort to the techniques mentioned in the introduction.

Example 1. Let $\tilde{V} = X_1^2 + X_2^2$, and $\tilde{U} = \sum_{i,j=1}^{4} a_{ij}X_iX_j$, with $a_{2j} = a_{i2} = 0$. Due to Lemma 1, it is possible to find out h_1, h_3, h_4 such that $\sum h_i^2 = 1$ and $a_{ii} = 1 - h_i^2$, $a_{ij} = -h_ih_j$, for $i, j \neq 2$. From Lemma 3, we obtain $2\rho = a_{11} + a_{22}$ and $\gamma = a_{33}a_{44} - a_{34}^2$; therefore $\rho \le 1/2$ and $\gamma = 1 - 2\rho$. For every $\rho < 1/2$ (that is $\gamma > 0$), assume $h_1 = \sqrt{1-2\rho}$, $h_2 = h_3 = \sqrt{\rho}$. Then matrix A is

$$A = \begin{pmatrix} 2\rho & 0 & -\alpha & -\alpha \\ 0 & 0 & 0 & 0 \\ -\alpha & 0 & 1-\rho & -\rho \\ -\alpha & 0 & -\rho & 1-\rho \end{pmatrix}$$

where $\alpha = \sqrt{(1-2\rho)\rho}$.

Since the joint distribution of $(\tilde{U}, \tilde{V})$ is characterized only by ρ and γ, such form of A describes all the considered joint distributions. Now

$$V_t \cap U_s = \{|x_1| \le \sqrt{t};\ |x_2| \le \sqrt{t - x_1^2};\ w_1 \le x_3 \le w_2;\ z_1 \le x_4 \le z_2\}$$

where

$$w_k = \frac{1}{1-2\rho}\left(x_1\sqrt{(1-2\rho)\rho} \pm \sqrt{(1-2\rho)(1-\rho)s}\right)$$

and

$$z_k = \frac{1}{1-\rho}\left(x_1\sqrt{(1-2\rho)\rho} + x_3\rho \pm \sqrt{(1-\rho)s - \left(x_1\sqrt{\rho} - x_3\sqrt{1-2\rho}\right)^2}\right).$$

Express the joint distribution function of $(\tilde{V}, \tilde{U})$ as in (5), and compute its second derivative; then its density is given by

$$g(s,t) = \int_{-\sqrt{t}}^{\sqrt{t}} \mathrm{d}x_1 \int_{w_1}^{w_2} \phi(x_1)\phi(x_3) \frac{\psi(t - x_1^2)\left(\phi(z_2) + \phi(z_1)\right)}{2\sqrt{(1-\rho)s - \left(x_1\sqrt{\rho} - x_3\sqrt{1-2\rho}\right)^2}} \mathrm{d}x_3$$

where ψ is the density of a chi-squared variate with one degree of freedom. □

Example 2. Let $V = X_1^2 + X_2^2$, and $U = X_2^2 + X_3^2$. Here $a_{44} = 0$ and $\gamma = 0$. It is

$$V_t \cap U_s = \{|x_2| \le \min\{\sqrt{t}, \sqrt{s}\};\ |x_1| \le \sqrt{t - x_2^2};\ |x_3| \le \sqrt{s - x_2^2}\}.$$

The joint distribution function is calculated integrating the density of X on set $V_t \cap U_s$. Then when $s \neq t$, its second mixed derivative results

$$f(s,t) = \int_0^{\min\{s,t\}} \frac{\mathrm{e}^{-(s+t-y)/2}}{\sqrt{(2\pi)^3(s-y)(t-y)y}}\mathrm{d}y.$$

The set S where it does not exists has probability 0; in fact

$$P(U=V)=\int_S \Pi_i \phi(x_i)\mathrm{d}x_1\mathrm{d}x_2\mathrm{d}x_3\mathrm{d}x_4=0,$$

being $S=\{(x_1,x_2,x_3,x_4):\sum a_{ij}x_ix_j=\sum b_{ij}x_ix_j\}$.

4. Final remarks

We have introduced a class $\mathcal{G}(2,2)$ of absolutely continuous bivariate exponential distributions and explored its probabilistic properties. This class has a natural extension to the multivariate case, since it is a class of bivariate quadratic forms of standard normal vectors. A simple representation for all distributions of $\mathcal{G}(2,2)$ is given in terms of 4-dimensional matrices defined by two parameters only.

The aim of this paper was on the construction of the stochastic model and the study of its probabilistic properties, and we have not examined the statistical side. However, some remarks are in order. From the statistical viewpoint, the more critical aspect of this class is that the joint density does not have a simple expression. Excepted for the case $\gamma=0$, we have derived an integral form for computing the distribution function and the density function, but this does not lend itself to straightforward use of maximum likelihood estimation. On the other hand, the simple expressions connecting the parameters ρ and γ, or equivalently δ, with $\mathrm{Corr}(U,V)$ and $\mathrm{Corr}(U^2,V^2)$ allow immediate estimation of the parameters via the method of moments, once these correlations have been estimated by their sample counterparts. In addition, sampling from these distributions is straightforward, which makes them well suited for application of the MCMC methodology.

The class $\mathcal{G}(2,2)$ is studied as particular case of the bivariate class $\mathcal{G}(m,p)$, whose marginals are respectively χ^2_m and χ^2_p. For the simplest case, $\mathcal{G}(1,1)$, it is easy to check that the moment generating function of U,V is given by

$$M(u,v)=\frac{1}{(1-(2u+2v)+(1-\rho)4uv)^{1/2}}.$$

In the remaining cases, it is difficult to find a general formula for the moment generating function. Running the same argument presented in the proof of the Theorem 5, it can be stated that $\mathcal{G}(m,p)=\mathcal{G}(m,p)_{m+p}$, which means that an $(m+p)$-dimensional standard normal vector is sufficient to represent the whole set of bivariate chi-square distributions with m and p degrees of freedom respectively.

Higher-dimensional normal vectors are required to deal with the multivariate case, that is the k-dimensional variables whose marginal components are generated via quadratic forms of the same n-dimensional normal variate. The same argument of the proof of the Theorem 5 states that the class of distributions with marginals $\chi^2_{m_i}$, $i=1,2,\ldots,k$, is completely represented using m-dimensional normal vectors, where $m=\sum_i m_i$.

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Appendix

A.1. *Proof of* Lemma 1

(i) $\Longrightarrow$ (ii). If $A=(a_{i,j})$ has rank m, then the space $S=\{Ax:x\in\mathbb{R}^n\}$ is m-dimensional. Moreover, since A is symmetric idempotent, the projection on S turns out to be orthogonal.

(ii) $\Longrightarrow$ (iii). Consider an m-dimensional subspace S and $c_{(1)},\ldots,c_{(m)}$ a basis of S. If Ax is an orthogonal projection on S, then $Ax=\sum_{i=1}^m c_{(i)}c'_{(i)}x$. Therefore, we have $A=\sum_{i=1}^m c_{(i)}c'_{(i)}=CC'$, where C is the matrix whose ith column is $c_{(i)}$, and A satisfies the conditions required.

(iii) $\Longrightarrow$ (i). Clearly $A=CC'$ is idempotent since C is orthonormal, that is $C'C=I_m$. Moreover an algebraical routine gives that $\mathrm{rank}(A)=m$.

(ii) $\Longleftrightarrow$ (iv). Given the equivalence of the first three items, then the next statement is obvious: Let S be an m-dimensional subspace, $S^\perp$ its orthogonal complement. Given two matrices A and H, then Ax is the orthogonal projection on S and Hx is the orthogonal projection on $S^\perp$ if and only if matrices A and H are of type described in (iii) with $Ax+Hx=x$, that is $A+H=I_n$.

This concludes the proof. □

A.2. Proof of Lemma 3

Consider two $n \times n$ matrices A and B, given by $A = cc' + dd'$, $B = ee' + ff'$, where $e = (1, 0, 0, \dots 0)'$, $f = (0, 1, 0, \dots, 0)'$. Notice that $2\rho = a_{11} + a_{22} = c_1^2 + c_2^2 + d_1^2 + d_2^2$ and that $a_{ii}a_{jj} - a_{ij}^2 = (c_i d_j - c_j d_i)^2$. We are going to prove that the polynomial (3) is given by

$$P = 1 - 2(y+z) + y^2 + z^2 + yz(4 - 2\rho) + (y^2 z + yz^2)(2\rho - 2) + y^2 z^2 \sum_{3 \le i < j} (c_i d_j - c_j d_i)^2.$$

The proof works by direct computation, verifying that the above expression holds true. Recall that $P = \det(I_n - yA - zB)$ by definition. First of all, notice that all terms involving y^3 or z^3 have a null coefficient since the matrices are of rank 2. Moreover by direct computation one obtains that

$$\begin{aligned} P = {} & 1 - 2z + z^2 + y\left[-\sum (c_i^2 + d_i^2)\right] + y^2 \left[\sum_i \sum_{j \ne i} (c_j^2 d_i^2) - 2\sum_{i<j} (c_i c_j d_i d_j)\right] \\ & + yz\left[\sum (c_i^2 + d_i^2) + \sum_{i \ge 3} (c_i^2 + d_i^2)\right] - yz^2 \left[\sum_{i \ge 3} (c_i^2 + d_i^2)\right] \\ & + y^2 z\left[-\sum_{i\ge 3} (c_1^2 + d_1^2)(c_i^2 + d_i^2) - \sum_{i \ge 3} (c_2^2 + d_2^2)(c_i^2 + d_i^2) + \sum_{i \ge 3} (c_1 c_i + d_1 d_i)^2 \right. \\ & \left. + \sum_{i \ge 3} (c_2 c_i + d_2 d_i)^2 - 2\sum_{3 \le i < j} (c_i^2 + d_i^2)(c_j^2 + d_j^2) + 2\sum_{3 \le i<j} (c_i c_j + d_i d_j)^2\right] \\ & + y^2 z^2 \left[\sum_{3 \le i<j} (c_i^2 + d_i^2)(c_j^2 + d_j^2) - \sum_{3 \le i < j} (c_i c_j + d_i d_j)^2\right]. \end{aligned}$$

The coefficient of y^2 is

$$\begin{aligned} \sum_i d_i^2 \sum_{j \ne i} (c_j^2) - 2\sum_{i<j} (c_i c_j d_i d_j) &= \sum_i d_i^2 (1 - c_i^2) - 2\sum_{i<j} (c_i c_j d_i d_j) \\ &= \|d\|^2 - \left(\sum_i d_i c_i\right)^2 \end{aligned}$$

which equals 1 because d is of unit norm and orthogonal to c.

The coefficient of yz is

$$\begin{aligned} \|c\|^2 + \|d\|^2 + \sum_{i \ge 3} (c_i^2 + d_i^2) &= 2(\|c\|^2 + \|d\|^2) - (c_1^2 + d_1^2 + c_2^2 + d_2^2) \\ &= 4 - 2\rho. \end{aligned}$$

Of course the coefficient of yz^2 equals $2\rho - 2$.

With some algebra, the coefficient of $y^2 z$ turns into

$$\begin{aligned} & -\sum_{i \ge 3} \left[c_i^2 \left(\sum_{j \ne i} d_j^2\right) + d_i^2 \left(\sum_{j \ne i} c_j^2\right)\right] + 2[c_1 d_1 + c_2 d_2] \sum_{i \ge 3} c_i d_i + 4 \sum_{3 \le i<j} c_i c_j d_i d_j \\ & = -\sum_{i \ge 3} [c_i^2 (1 - d_i^2) + d_i^2 (1 - c_i^2)] + 2[c_1 d_1 + c_2 d_2] \sum_{i \ge 3} c_i d_i + 4 \sum_{3 \le i<j} c_i c_j d_i d_j \\ & = -\sum_{i \ge 3} [c_i^2 + d_i^2] + 2\sum_{i \ge 3} [c_i^2 d_i^2] + 4 \sum_{3 \le i<j} c_i c_j d_i d_j + 2[c_1 d_1 + c_2 d_2] \sum_{i \ge 3} c_i d_i \\ & = -2 + c_1 + c_2 + d_1 + d_2 + 2\left[\sum_{i \ge 3} c_i d_i\right]^2 + 2[c_1 d_1 + c_2 d_2] \sum_{i \ge 3} c_i d_i \\ & = 2\rho - 2 + 2\left[\sum_{i \ge 3} c_i d_i\right] \sum_{i=1}^n c_i d_i \\ & = 2\rho - 2, \end{aligned}$$

taking into account orthogonality of c and d.

Finally, the coefficient of $y^2 z^2$ is $\sum_{3 \le i<j} (c_i d_j - c_j d_i)^2$.

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