Advances in Applied Mathematics 46 (2011) 312–362

Contents lists available at ScienceDirect

Advances in Applied Mathematics

www.elsevier.com/locate/yaama

Cellularity and the Jones basic construction

Frederick M. Goodman *, John Graber

Department of Mathematics, University of Iowa, Iowa City, IA, United States

ARTICLE INFO

Article history:
Available online 27 October 2010

Dedicated to Dennis Stanton on the occasion of his 60th birthday

MSC:
20C08
16G99
81R50

Keywords:
Cellular algebras
Jones basic construction
Brauer algebra
BMW algebra
Diagram algebras
Tangle algebras

ABSTRACT

We establish a framework for cellularity of algebras related to the Jones basic construction. Our framework allows a uniform proof of cellularity of Brauer algebras, ordinary and cyclotomic BMW algebras, walled Brauer algebras, partition algebras, and others. Our cellular bases are labeled by paths on certain branching diagrams rather than by tangles. Moreover, for the class of algebras that we study, we show that the cellular structures are compatible with restriction and induction of modules.

1. Introduction

Cellularity is a concept due to Graham and Lehrer [23] that is useful for studying non-semisimple specializations of certain algebras such as Hecke algebras, q-Schur algebras, etc. A number of important examples of cellular algebras, including the Hecke algebras of type A and the Birman–Wenzl–Murakami (BMW) algebras, actually occur in towers $A_0 \subseteq A_1 \subseteq A_2 \subseteq \cdots$ with coherent cellular structures. Coherence means that the cellular structures are well behaved with respect to induction and restriction.

* Corresponding author.
E-mail addresses: goodman@math.uiowa.edu (F.M. Goodman), jgraber@math.uiowa.edu (J. Graber).

0196-8858/$ – see front matter
doi:10.1016/j.aam.2010.10.003

This paper establishes a framework for proving cellularity of towers of algebras $(A_n)_{n\geqslant 0}$ that are obtained by repeated Jones basic constructions from a coherent tower of cellular algebras $(Q_n)_{n\geqslant 0}$.

Examples that fit in our framework include: Temperley–Lieb algebras, Brauer algebras, walled Brauer algebras, Birman–Wenzl–Murakami (BMW) algebras, cyclotomic BMW algebras, partition algebras, and contour algebras. We give a uniform proof of cellularity for all of these algebras.

We should alert the reader that we use a definition of cellular algebras that is slightly weaker than the original definition of Graham and Lehrer. The two definitions are equivalent in case 2 is invertible in the ground ring, and we know of no consequence of cellularity that would not also hold with the weaker definition; in particular, all results of Graham and Lehrer [23] go through with the modified definition. See Section 2.2 for details. Our contention is that the relaxed definition is in fact superior, as it allows one to deal more naturally with extensions of cellular algebras. For this reason, we have retained the terminology "cellularity" for our weaker definition, rather than inventing some new terminology such as "weak cellularity."

Once we have proved our abstract result (Theorem 3.2), it is generally very easy to check that each example fits our framework, and thus that the tower $(A_n)_{n\geqslant 0}$ in the example is a coherent tower of cellular algebras. What we need is, for the most part, already in the literature, or completely elementary. The application of our method to the cyclotomic BMW algebras depends on a very recent result of Mathas regarding induced modules of cyclotomic Hecke algebras [46].

For most of our examples, cellularity has been established previously (but coherence of the cellular structures is a new result). Many of the existing proofs of cellularity for these algebras follow the pattern made explicit by Xi in his paper on cellularity of the partition algebras [61]. The cellular bases obtained are pieced together from cellular bases of the (quotient) algebras Q_k and bases of certain R-modules V_k of tangles or diagrams, where R is the ground ring for A_n; a formal method for piecing the parts together is König and Xi's method of "inflation" [39]. It is not evident that the resulting "tangle bases" yield coherent cellular structures. By contrast, the cellular bases that we produce are indexed by paths on the branching diagram (Bratteli diagram) for the generic semisimple representation theory of the tower $(A_n)_{n\geqslant 0}$ over a field, and coherence is built into the construction.

For example, for the Brauer algebras, the BMW algebras, and the cyclotomic BMW algebras, our cellular basis of the n-th algebra is indexed by up-down tableaux of length n, and may be regarded as an analogue of Murphy's cellular basis [50] for the Hecke algebra, or the basis of Dipper, James and Mathas [12] for the cyclotomic Hecke algebras. A Murphy type basis for the BMW and Brauer algebras has been constructed by Enyang [15], and by Rui and Si [52] for cyclotomic BMW algebras.

Let us remark on the role played by the generic ground ring for our examples. For each of our examples $(A_n)_{n\geqslant 0}$, there is a generic ground ring R such that any specialization A_n^S to a ground ring S is obtained as $A_n^S = A_n^R \otimes_R S$. Moreover, R is an integral domain, and if F denotes the field of fractions of R, then the algebras $(A_n^F)_{n\geqslant 0}$ are split semisimple with a known representation theory and branching diagram. It suffices for us to prove that the sequence of algebras defined over the generic ground ring R is a coherent cellular tower, and we find that we can use the structure of the algebras defined over F as a tool to accomplish this.

Our approach is influenced by the work of König and Xi [39] as well as by the work of Cox et al. on "towers of recollement" [8]. In fact, the idea behind our approach is roughly the following: Each algebra A_n (over the generic ground ring R) contains an essential idempotent e_{n-1} with the properties that $e_{n-1}A_ne_{n-1} \cong A_{n-2}$ and $A_n/(A_ne_{n-1}A_n) \cong Q_n$, where Q_n is a cellular algebra. Assuming that A_{n-2} and A_{n-1} are cellular, we show that the (generally non-unital) ideal $I_n = A_ne_{n-1}A_n$ is a "cellular ideal" in A_n by relating ideals of A_{n-2} to ideals of A_n contained in I_n. This proof involves a new basis-free characterization of cellularity and also involves showing that $I_n \cong A_{n-1} \otimes_{A_{n-2}} A_{n-1}$ as A_{n-1} bimodules; thus I_n is a sort of Jones basic construction for the pair $A_{n-2} \subseteq A_{n-1}$. Since our version of cellularity behaves well under extensions, we can conclude that A_n is cellular. Our method is related to ideas introduced by König and Xi in their treatment of cellularity and Morita equivalence [39].

Following Cox et al. [8], our approach employs the interaction between induction and restriction functors relating A_{n-1}-mod and A_n-mod, on the one hand, and localization and globalization functions relating A_n-mod and A_{n-2}-mod, on the other hand. (Write $e = e_{n-1} \in A_n$. The localization functor $F : A_n\text{-mod} \to eA_ne\text{-mod} \cong A_{n-2}\text{-mod}$ is $F : M \mapsto eM$. The globalization function $G : A_{n-2}\text{-mod} \cong eA_ne\text{-mod} \to A_n\text{-mod}$ is $G : N \mapsto A_ne \otimes_{eA_ne} N$.)

Our framework and that of Cox et al. dovetail nicely; in fact, our main result (Theorem 3.2) says that if (A_n), (Q_n) are two sequences of algebras satisfying our framework axioms, then (A_n) satisfies a cellular version of the axioms for towers of recollement; see [9] for a discussion of cellularity and towers of recollement.

Although our techniques do not seem to be adaptable to proving "strict" cellularity in the sense of [23], by combining our results with previous proofs of "strict" cellularity for our examples, we can show the existence of "strictly" cellular Murphy type bases, i.e. bases indexed by paths on the generic branching diagram for the sequence of algebras $(A_n)_{n\geqslant 0}$. We will indicate how this can be done for the cyclotomic BMW algebras; other examples are similar.

Several other general frameworks have been proposed for cellularity which also successfully encompass many of our examples; see [39,24,57].

In a companion paper [19], we refine the framework of this paper to take into account the role played by Jucys–Murphy elements. At the same time, we modify Andrew Mathas's theory [45] of cellular algebras with Jucys–Murphy elements to take into account coherent sequences of such algebras.

2. Preliminaries

2.1. Algebras with involution

Let R be a commutative ring with identity. In the following, assume A is an R-algebra with an involution i (that is, an R-linear algebra anti-automorphism of A with $i^2 = \mathrm{id}$).

If M is a left A-module, we define a right A-module $i(M)$ as follows. As a set, $i(M)$ is a copy of M, with elements marked with the symbol i, $i(M) = \{i(m)\colon\ m \in M\}$. The R-module structure of $i(M)$ is given by $i(m_1)+i(m_2) = i(m_1+m_2)$, and $ri(m) = i(rm)$. Finally, the right A-module structure is defined by $i(m)a = i(i(a)m)$. If $\alpha : M \to N$ is a homomorphism of left A-modules, define $i(\alpha) : i(M) \to i(N)$ by $i(\alpha)(i(m)) = i(\alpha(m))$. Then $i : A\text{-mod} \to \text{mod-}A$ is a functor. For any fixed M, $i : M \to i(M)$ given by $m \mapsto i(m)$ is, by definition, an isomorphism of R-modules.

If Δ is a left ideal in A, we have two possible meanings for $i : \Delta \to i(\Delta)$, namely the restriction to Δ of the involution i, whose image is a right ideal in A, or the application of the functor i. However, there is no problem with this, as the right A-module obtained by applying the functor i can be identified with the right ideal $i(\Delta)$.

The same construction gives a map from right A-modules to left A-modules. Moreover, if A and B are R-algebras with involutions i_A and i_B, and M is an A–B-bimodule, then $i(M)$, defined as above as an R-module has the structure of a B–A-bimodule with $bi(m)a = i(i_A(a)mi_B(b))$. Note that $i \circ i(M)$ is naturally isomorphic to M, so i is an equivalence between the categories of A–B-bimodules and the category of B–A-bimodules.

Lemma 2.1. *Suppose A, B, and C are R-algebras with involutions i_A, i_B, and i_C. Let ${}_BP_A$ and ${}_AQ_C$ be bimodules. Then*

$$i(P \otimes_A Q) \cong i(Q) \otimes_A i(P),$$

as C–B-bimodules.

Proof. It is straightforward to check that there is a well-defined R-linear isomorphism $f_0 : P \otimes_A Q \to i(Q) \otimes_A i(P)$ such that $f_0(p \otimes q) = i(q) \otimes i(p)$. Then

$$f = f_0 \circ i^{-1} : i(P \otimes_A Q) \to i(Q) \otimes_A i(P)$$

is an R-linear isomorphism. Finally, one can check that f is a C–B-bimodule map. $\square$

Remark 2.2. Note that if we identify $i(P \otimes_A Q)$ with $i(Q) \otimes_A i(P)$ via f, then we have the formula $i(p \otimes q) = i(q) \otimes i(p)$. In particular, let M be a B–A-bimodule, and identify $i \circ i(M)$ with M, and $i(M \otimes_A i(M))$ with $i \circ i(M) \otimes_A i(M) = M \otimes_A i(M)$. Then we have the formula $i(x \otimes i(y)) = y \otimes i(x)$. We will use these identifications throughout the paper.

2.2. Cellularity

We recall the definition of *cellularity* from [23]; see also [44]. The version of the definition given here is slightly weaker than the original definition in [23]; we justify this below.

Definition 2.3. Let R be an integral domain and A a unital R-algebra. A *cell datum* for A consists of an algebra involution i of A; a partially ordered set $(\Lambda, \geqslant)$ and for each $\lambda \in \Lambda$ a set $\mathcal{T}(\lambda)$; and a subset $\mathcal{C} = \{c_{s,t}^{\lambda}\colon\ \lambda \in \Lambda \text{ and } s, t \in \mathcal{T}(\lambda)\} \subseteq A$; with the following properties:

(1) $\mathcal{C}$ is an R-basis of A.
(2) For each $\lambda \in \Lambda$, let $\breve{A}^{\lambda}$ be the span of the $c_{s,t}^{\mu}$ with $\mu > \lambda$. Given $\lambda \in \Lambda$, $s \in \mathcal{T}(\lambda)$, and $a \in A$, there exist coefficients $r_v^s(a) \in R$ such that for all $t \in \mathcal{T}(\lambda)$:

$$ac_{s,t}^{\lambda} \equiv \sum_v r_v^s(a) c_{v,t}^{\lambda} \bmod \breve{A}^{\lambda}.$$

(3) $i(c_{s,t}^{\lambda}) \equiv c_{t,s}^{\lambda} \bmod \breve{A}^{\lambda}$ for all $\lambda \in \Lambda$ and $s, t \in \mathcal{T}(\lambda)$.

A is said to be a *cellular algebra* if it has a cell datum.

For brevity, we will write that $(\mathcal{C}, \Lambda)$ is a cellular basis of A.

Remark 2.4.

(1) The original definition in [23] requires that $i(c_{s,t}^{\lambda}) = c_{t,s}^{\lambda}$ for all λ, s, t. However, one can check that the results of [23] remain valid with our weaker axiom. In fact, we are not aware of any consequence of cellularity that would not also hold with our weaker definition.
(2) In case $2 \in R$ is invertible, our definition is equivalent to the original. Here is the proof: Suppose that 2 is invertible in the ground ring and that $\{c_{s,t}^{\lambda}\}$ is a cellular basis in the sense of Definition 2.3. We want to produce a new cellular basis $\{a_{s,t}^{\lambda}\}$ satisfying the strict equality $i(a_{s,t}^{\lambda}) = a_{t,s}^{\lambda}$ for all λ, s, t. By hypothesis, for each λ, s, t there is a unique $f(\lambda, s, t) \in \breve{A}^{\lambda}$ such that $i(c_{s,t}^{\lambda}) = c_{t,s}^{\lambda} + f(\lambda, s, t)$. One easily checks that $i(f(\lambda, s, t)) = -f(\lambda, t, s)$. Declare $a_{s,t}^{\lambda} = c_{s,t}^{\lambda} + (1/2) f(\lambda, t, s)$ for all λ, s, t. Then $\{a_{s,t}^{\lambda}\}$ has the desired properties.

We recall some basic structures related to cellularity, see [23]. Given $\lambda \in \Lambda$. Let A^{λ} denote the span of the $c_{s,t}^{\mu}$ with $\mu \geqslant \lambda$. It follows that both A^{λ} and $\breve{A}^{\lambda}$ (defined above) are i-invariant two sided ideals of A. If $t \in \mathcal{T}(\lambda)$, define C_t^{λ} to be the R-submodule of $A^{\lambda}/\breve{A}^{\lambda}$ with basis $\{c_{s,t}^{\lambda} + \breve{A}^{\lambda} : s \in \mathcal{T}(\lambda)\}$. Then C_t^{λ} is a left A-module by Definition 2.3(2). Furthermore, the action of A on C_t^{λ} is independent of t, i.e. $C_u^{\lambda} \cong C_t^{\lambda}$ for any $u, t \in \mathcal{T}(\lambda)$. The *left cell module* Δ^{λ} is defined as follows: as an R-module, Δ^{λ} is free with basis $\{c_s^{\lambda}\colon\ s \in \mathcal{T}(\lambda)\}$; for each $a \in A$, the action of a on Δ^{λ} is defined by $ac_s^{\lambda} = \sum_v r_v^s(a) c_v^{\lambda}$ where $r_v^s(a)$ is as in Definition 2.3(2). Then $\Delta^{\lambda} \cong C_t^{\lambda}$, for any $t \in \mathcal{T}(\lambda)$. For all $s, t \in \mathcal{T}(\lambda)$, we have a canonical A–A-bimodule isomorphism $\alpha : A^{\lambda}/\breve{A}^{\lambda} \to \Delta^{\lambda} \otimes_R i(\Delta^{\lambda})$ defined by $\alpha(c_{s,t}^{\lambda} + \breve{A}^{\lambda}) = c_s^{\lambda} \otimes_R i(c_t^{\lambda})$. Moreover, we have $i \circ \alpha = \alpha \circ i$, using Remark 2.2 and point (3) of Definition 2.3.

Definition 2.5. Suppose A is a unital R-algebra with involution i, and J is an i-invariant ideal; then we have an induced algebra involution i on A/J. Let us say that J is a *cellular ideal* in A if it satisfies the axioms for a cellular algebra (except for being unital) with cellular basis

$$\left\{c_{s,t}^{\lambda}\colon \lambda \in \Lambda_J \text{ and } s, t \in \mathscr{T}(\lambda)\right\} \subseteq J$$

and we have, as in point (2) of the definition of cellularity,

$$ac_{s,t}^{\lambda} \equiv \sum_{v} r_v^s(a) c_{v,t}^{\lambda} \bmod \breve{J}^{\lambda}$$

not only for $a \in J$ but also for $a \in A$.

Remark 2.6. (On extensions of cellular algebras.) If J is a cellular ideal in A, and $H = A/J$ is cellular (with respect to the involution induced from the involution on A), then A is cellular. In fact, let $(\Lambda_J, \geqslant)$ be the partially ordered set in the cell datum for J and $\mathscr{C}_J$ the cellular basis. Let $(\Lambda_H, \geqslant)$ be the partially ordered set in the cell datum for H and $\{\bar{h}_{u,v}^{\mu}\}$ the cellular basis. Then A has a cell datum with partially ordered set $\Lambda = \Lambda_J \cup \Lambda_H$, with partial order agreeing with the original partial orders on Λ_J and on Λ_H and with $\lambda > \mu$ if $\lambda \in \Lambda_J$ and $\mu \in \Lambda_H$. A cellular basis of A is $\mathscr{C}_J \cup \{h_{s,t}^{\mu}\}$, where $h_{s,t}^{\mu}$ is any lift of $\bar{h}_{s,t}^{\mu}$.

With the original definition of [23], the assertions of this remark would be valid only if the ideal J has an i-invariant R-module complement in A. The ease of handling extensions is our motivation for using the weaker definition of cellularity.

2.3. Basis-free formulations of cellularity

König and Xi have given a basis-free definition of cellularity [39]. We describe a slight weakening of their definition, which corresponds exactly to our weaker form of Graham–Lehrer cellularity.

Definition 2.7 *(König and Xi).* Let R be an integral domain and A a unital R-algebra with involution i. An i-invariant two sided ideal J in A is called a *split ideal* if, and only if, there exists a left ideal Δ of A contained in J, with Δ finitely generated and free over R, and there is an isomorphism of A–A-bimodules $\alpha : J \to \Delta \otimes_R i(\Delta)$ making the following diagram commute:

$$\begin{tikzcd} J \arrow[r, "\alpha"] \arrow[d, "i"] & \Delta \otimes_R i(\Delta) \arrow[d, "i"] \\ J \arrow[r, "\alpha"] & \Delta \otimes_R i(\Delta) \end{tikzcd}$$

A finite chain of i-invariant two sided ideals

$$0 = J_0 \subset J_1 \subset J_2 \subset \cdots \subset J_n = A$$

is called a *cell chain* if for each j $(1 \leqslant j \leqslant n)$, the quotient J_j/J_{j-1} is a split ideal of A/J_{j-1} (with respect to the involution induced by i on A/J).

Remark 2.8.

(1) König and Xi call a split ideal a “cell ideal.” We changed the terminology to avoid confusion with other concepts.

(2) The definition of a cell chain differs from the one given by Konig and Xi in that we dropped the requirement that J_{j-1} have an i-invariant R-module complement in J_j.

Lemma 2.9. *Let R be an integral domain and let A be a unital R-algebra with involution i. An ideal J of A is split if, and only if, there exists a left A-module M that is finitely generated and free as an R-module, and there exists an isomorphism of A–A-bimodules $\gamma : J \to M \otimes_R i(M)$ making the following diagram commute*:

$$\begin{tikzcd} J \arrow[r, "\gamma"] \arrow[d, "i"] & M \otimes_R i(M) \arrow[d, "i"] \\ J \arrow[r, "\gamma"] & M \otimes_R i(M) \end{tikzcd}$$

Proof. If J is split, it clearly satisfies the condition of the lemma. Conversely, suppose the condition of the lemma is satisfied. Fix some element b_0 of the basis of M over R and define a left A-module map $\beta : M \to A$ by $\beta(m) = \gamma^{-1}(m \otimes b_0)$. Then β is an isomorphism of M onto a left ideal Δ of A contained in J.

Now we have $\beta \otimes i(\beta) : M \otimes_R i(M) \to \Delta \otimes_R i(\Delta)$ is an isomorphism satisfying $(\beta \otimes i(\beta)) \circ i = i \circ (\beta \otimes i(\beta))$. It follows that $\alpha = (\beta \otimes i(\beta)) \circ \gamma : J \to \Delta \otimes_R i(\Delta)$ is an isomorphism of A–A-bimodules satisfying the requirement for a split ideal, namely, $\alpha \circ i = i \circ \alpha$. $\square$

Lemma 2.10 *(König and Xi). Let A be an R-algebra with involution. A is cellular if, and only if, A has a finite cell chain.*

Proof. We sketch the proof from [38], p. 372.

Suppose A has a cell datum with partially ordered set $(\Lambda, \geqslant)$ and cell basis $\{c^\lambda_{s,t}\}$. Write Λ as a sequence $(\lambda_1, \lambda_2, \dots, \lambda_n)$, where λ_1 is maximal in Λ, and, for $1 \leqslant j < n$, λ_{j+1} is maximal in $\Lambda \setminus \{\lambda_1, \dots, \lambda_j\}$. Then for each $j \geqslant 1$, $\Gamma_j = \{\lambda_1, \dots, \lambda_j\}$ is an order ideal in Λ. Set $\Gamma_0 = \emptyset$. Define $A(\Gamma_j)$ to be the R-submodule of A spanned by the basis elements $c^\lambda_{u,v}$, with $\lambda \in \Gamma_j$. Then $A(\Gamma_j)$ is an i-invariant two sided ideal in A, and

$$0 = A(\Gamma_0) \subset A(\Gamma_1) \subset \cdots \subset A(\Gamma_n) = A.$$

Moreover (see [23], p. 6),

$$A(\Gamma_j)/A(\Gamma_{j-1}) \cong A^{\lambda_j}/\breve{A}^{\lambda_j} \cong \Delta^{\lambda_j} \otimes_R i\big(\Delta^{\lambda_j}\big),$$

and the isomorphism $\alpha : A(\Gamma_j)/A(\Gamma_{j-1}) \to \Delta^{\lambda_j} \otimes_R i(\Delta^{\lambda_j})$ satisfies $\alpha \circ i = i \circ \alpha$. Thus $(A(\Gamma_j))_{1 \leqslant j \leqslant n}$ is a cell chain.

Conversely, suppose $(J_j)_{0 \leqslant j \leqslant n}$ is a cell chain in A. Then for each $j \geqslant 1$, we have an A-module Δ_j that is finitely generated and free as an R-module, and an isomorphism of A–A-bimodules $\alpha_j : J_j/J_{j-1} \to \Delta_j \otimes_R i(\Delta_j)$ satisfying $i \circ \alpha_j = \alpha_j \circ i$. Let $\{b^j_s\colon\ s \in \mathscr{T}(j)\}$ be an R-basis of Δ_j and let $c^{\lambda_j}_{s,t}$ be any lift in J_j of $\alpha_j^{-1}(b^j_s \otimes i(b^j_t))$. Now take Λ' to be Λ with the order $\lambda_1 > \lambda_2 > \cdots > \lambda_n$. Let $\mathscr{C} = \{c^{\lambda_j}_{s,t}\colon\ 1 \leqslant j \leqslant n;\ s, t \in \mathscr{T}(j)\}$. Then $(\mathscr{C}, \Lambda')$ is a cellular basis of A. $\square$

Remark 2.11. In the lemma, A has a cellular basis $\{c^\lambda_{s,t}\}$ with $i(c^\lambda_{s,t}) = c^\lambda_{t,s}$ if and only if, A has a finite cell chain (J_j) such that for each $j \geqslant 1$, J_{j-1} has an i-invariant R-module complement in J_j.

Note that if we follow the procedure of the proof, starting with a cell datum on A with partially ordered set $(\Lambda, \geqslant)$, then the only information that we retain about Λ is that λ_{j+1} is maximal in $\Lambda \setminus \Gamma_j$; we cannot recover the partial order on Λ from this. Moreover, if we continue to produce a cellular basis $\{c^j_{s,t}\}$ from the cell chain $(A(\Gamma_j))_{0 \leqslant j \leqslant n}$, the result will not necessarily have the properties of a cellular basis with respect to the original partially ordered set $(\Lambda, \geqslant)$.

In order to prove our main results, we will need a different basis-free formulation of cellularity that allows us to pass back and forth between the formulation of Definition 2.3 and the basis-free formulation without losing information about the partially ordered set.

Definition 2.12. Let A be an R-algebra with involution i. Let $(\Lambda, \geqslant)$ be a finite partially ordered set. For $\lambda \in \Lambda$, let $\Gamma_{\geqslant\lambda}$ denote the order ideal $\{\mu\colon \mu \geqslant \lambda\}$ and $\Gamma_{>\lambda}$ the order ideal $\{\mu\colon \mu > \lambda\}$.

A Λ-*cell net* is a map from the set of order ideals of Λ to the set of i-invariant two sided ideals of A, $\Gamma \mapsto A_\Gamma$, with the following properties:

(1) $A_\emptyset = \{0\}$. If $\Gamma_1 \subseteq \Gamma_2$, then $A_{\Gamma_1} \subseteq A_{\Gamma_2}$.
(2) For $\lambda \in \Lambda$, write $A_{\geqslant\lambda} = A_{\Gamma_{\geqslant\lambda}}$ and $A_{>\lambda} = A_{\Gamma_{>\lambda}}$. Then

$$A = \mathrm{span}\{A_{\geqslant\mu}\colon \mu \in \Lambda\},$$

and for all $\lambda \in \Lambda$,

$$A_{>\lambda} = \mathrm{span}\{A_{\geqslant\mu}\colon \mu > \lambda\}.$$

(3) For each $\lambda \in \Lambda$, there is an A-module M^λ, finitely generated and free as an R-module, such that whenever $\Gamma \subseteq \Gamma'$ are order ideals of Λ, with $\Gamma' \setminus \Gamma = \{\lambda\}$, then there exists an isomorphism of A–A-bimodules

$$\alpha : A_{\Gamma'}/A_\Gamma \to M^\lambda \otimes_R i\big(M^\lambda\big),$$

satisfying $i \circ \alpha = \alpha \circ i$.

Proposition 2.13. *Let A be an R-algebra with involution, and let $(\Lambda, \geqslant)$ be a finite partially ordered set. Then A has a cell datum with partially ordered set Λ if, and only if, A has a Λ-cell net.*

Proof. Suppose that A has a cell datum with partially ordered set Λ and cell basis $\{c^\lambda_{s,t}\}$. For each order ideal Γ of Λ, let $A(\Gamma)$ denote the span of those $c^\lambda_{s,t}$ with $\lambda \in \Gamma$. Then $\Gamma \mapsto A(\Gamma)$ is a Λ-cell net.

Conversely, suppose that A has a Λ-cell net, $\Gamma \mapsto A_\Gamma$. For each $\lambda \in \Lambda$, we have an isomorphism of A–A-bimodules $\alpha_\lambda : A_{\geqslant\lambda}/A_{>\lambda} \to M^\lambda \otimes_R i(M^\lambda)$. Let $\{b^\lambda_s\colon s \in \mathscr{T}(\lambda)\}$ be an R-basis of M^λ and let $c^\lambda_{s,t}$ be any lift of $\alpha_\lambda^{-1}(b^\lambda_s \otimes i(b^\lambda_t))$ to $A_{\geqslant\lambda}$. We claim that

$$\mathscr{C} = \big\{c^\lambda_{s,t}\colon \lambda \in \Lambda;\ s, t \in \mathscr{T}(\lambda)\big\}$$

is an R-basis of A.

Let A^λ be the span of those $c^\mu_{s,t}$ with $\mu \geqslant \lambda$ and $\breve{A}^\lambda$ the span of those $c^\mu_{s,t}$ with $\mu > \lambda$. If $\mu \geqslant \lambda$, then for all $s, t \in \mathscr{T}(\mu)$, $c^\mu_{s,t} \in A_{\geqslant\mu} \subseteq A_{\geqslant\lambda}$, using point (1) of Definition 2.12. Hence $A^\lambda \subseteq A_{\geqslant\lambda}$. Similarly, $\breve{A}^\lambda \subseteq A_{>\lambda}$.

We claim that

$$\text{for all } \lambda \in \Lambda, \quad A_{\geqslant\lambda} = A^\lambda. \tag{2.1}$$

This is clear if λ is a maximal element of Λ. (Note that $A_{>\lambda} = A_\emptyset = \{0\}$.) Now suppose that λ is not maximal and that for all $\mu > \lambda$, $A_{\geqslant\mu} = A^\mu$. Then

$$A_{>\lambda} = \mathrm{span}\{A_{\geqslant\mu}\colon \mu > \lambda\} = \mathrm{span}\big\{A^\mu\colon \mu > \lambda\big\} = \breve{A}^\lambda,$$

where the first equality comes from (2) of Definition 2.12 and the second from the induction hypothesis. By definition of $\{c_{s,t}^{\lambda}\}$, we have

$$A_{\geqslant\lambda} = \mathrm{span}\{c_{s,t}^{\lambda}\} + A_{>\lambda} = \mathrm{span}\{c_{s,t}^{\lambda}\} + \breve{A}^{\lambda} = A^{\lambda}.$$

Assertion (2.1) now follows by induction. Point (2) of Definition 2.12 and (2.1) imply that $A_{>\lambda} = \breve{A}^{\lambda}$ for all $\lambda \in \Lambda$, and that $A = \mathrm{span}(\mathscr{C})$.

We now proceed to establish linear independence of $\mathscr{C}$. Write Λ as a sequence $(\lambda_1, \lambda_2, \ldots, \lambda_K)$ with λ_1 maximal and λ_{j+1} maximal in $\Lambda \setminus \{\lambda_1, \ldots, \lambda_j\}$ for $1 \leqslant j < K$. Put $\Gamma_j = \{\lambda_1, \ldots, \lambda_j\}$ for $j \geqslant 1$ and $\Gamma_0 = \emptyset$. Then $(\Gamma_j)_{0\leqslant j\leqslant K}$ is a maximal chain of order ideals. Since $\Gamma_j \setminus \Gamma_{j-1} = \{\lambda_j\}$, we have an isomorphism $\gamma_j : A_{\Gamma_j}/A_{\Gamma_{j-1}} \to M^{\lambda_j} \otimes_R i(M^{\lambda_j})$ with $i \circ \gamma_j = \gamma_j \circ i$. Thus $(A_{\Gamma_j})_{0\leqslant j\leqslant K}$ is a cell chain in A. So by the proof of Lemma 2.10, A has a cellular basis

$$\mathscr{B} = \{b_{s,t}^{\lambda}\colon \lambda \in \Lambda;\ s, t, \in \mathscr{T}(\lambda)\},$$

but with respect to the "wrong" partial order on Λ. Since $\mathscr{C}$ is a spanning set of the same cardinality as the basis $\mathscr{B}$, it follows that $\mathscr{C}$ is linearly independent over R, and thus an R-basis of A.

Because $A_{>\lambda} = \breve{A}^{\lambda}$ for all $\lambda \in \Lambda$, it is now easy to see that properties (2) and (3) of Definition 2.3 are satisfied by $\mathscr{C}$. □

Remark 2.14. In the proposition, the following are equivalent:

(1) A has a cellular basis $\{c_{s,t}^{\lambda}\}$ with $i(c_{s,t}^{\lambda}) = c_{t,s}^{\lambda}$.
(2) A has a Λ cell net $\Gamma \to A_\Gamma$ such that for each pair $\Gamma \subseteq \Gamma'$, A_Γ has an i-invariant R-module complement in $A_{\Gamma'}$.
(3) A has a Λ cell net $\Gamma \to A_\Gamma$ suchthat for each $\lambda \in \Lambda$, $A_{>\lambda}$ has an i-invariant R-module complement in $A_{\geqslant\lambda}$.

The implications (1) ⇒ (2) ⇒ (3) are evident. For (3) ⇒ (1), let B_λ denote the i-invariant R-module complement of $A_{>\lambda}$ in $A_{\geqslant\lambda}$, and, in the second paragraph of the proof of the proposition, let $c_{s,t}^{\lambda}$ be the unique lift of $\alpha_\lambda^{-1}(b_s^{\lambda} \otimes i(b_t^{\lambda}))$ in B_λ.

2.4. Coherent towers of cellular algebras

Definition 2.15. Let $H_0 \subseteq H_1 \subseteq H_2 \subseteq \cdots$ be an increasing sequence of cellular algebras, with a common multiplicative identity element, over an integral domain R. Let Λ_n denote the partially ordered set in the cell datum for H_n. We say that $(H_n)_{n\geqslant 0}$ is a *coherent tower of cellular algebras* if the following conditions are satisfied:

(1) The involutions are consistent; that is, the involution on H_{n+1}, restricted to H_n, agrees with the involution on H_n.
(2) For each $n \geqslant 0$ and for each $\lambda \in \Lambda_n$, the induced module $\mathrm{Ind}_{H_n}^{H_{n+1}}(\Delta^{\lambda})$ has a filtration by cell modules of H_{n+1}. That is, there is a filtration

$$\mathrm{Ind}_{H_n}^{H_{n+1}}(\Delta^{\lambda}) = M_t \supseteq M_{t-1} \supseteq \cdots \supseteq M_0 = (0)$$

such that for each $j \geqslant 1$, there is a $\mu_j \in \Lambda_{n+1}$ with $M_j/M_{j-1} \cong \Delta^{\mu_j}$.

(3) For each $n \geqslant 0$ and for each $\mu \in \Lambda_{n+1}$, the restriction $\mathrm{Res}^{H_{n+1}}_{H_n}(\Delta^{\mu})$ has a filtration by cell modules of H_n. That is, there is a filtration

$$\mathrm{Res}^{H_{n+1}}_{H_n}\big(\Delta^{\mu}\big) = N_s \supseteq N_{s-1} \supseteq \cdots \supseteq N_0 = (0)$$

such that for each $i \geqslant 1$, there is a $\lambda_i \in \Lambda_n$ with $N_j/N_{j-1} \cong \Delta^{\lambda_i}$.

The modification of the definition for a *finite* tower of cellular algebras is obvious.

We call a filtration as in (2) and (3) a *cell filtration*. In the examples that we study, we will also have *uniqueness of the multiplicities* of the cell modules appearing as subquotients of the cell filtrations, and *Frobenius reciprocity* connecting the multiplicities in the two types of filtrations. We did not include uniqueness of multiplicities and Frobenius reciprocity as requirements in the definition, as they will follow from additional assumptions that we will impose later; see Lemma 2.22.[1]

Example 2.16. *The tower of Hecke algebras of type A is a coherent tower of cellular algebras.* Let R be an integral domain and q an invertible element of R. Let $H_n(R,q)$ denote the Hecke algebra of type A generated by elements $T_1, \dots, T_{n-1}$ satisfying the braid relations and the quadratic relations $(T_j - q)(T_j + 1) = 0$ for $1 \leqslant j \leqslant n-1$. When $q = 1$, $H_n(R,q)$ is the group algebra $R\mathfrak{S}_n$ of the symmetric group $\mathfrak{S}_n$. As is well known, $H_n(R,q)$ has a basis T_w ($w \in \mathfrak{S}_n$) given by $T_w = T_{j_1} \dots T_{j_\ell}$ for any reduced expression $w = s_{j_1} \dots s_{j_\ell}$. The map defined by $i(T_w) = T_{w^{-1}}$ is an algebra involution. The map defined by $(T_w)^{\#} = (-q)^{\ell(w)}(T_{w^{-1}})^{-1}$ is an algebra automorphism. The assignment $T_w \mapsto T_w$ is an embedding of $H_n(R,q)$ into $H_{n+1}(R,q)$. The algebra involutions are consistent on $(H_n)_{n \geqslant 0}$.

Dipper and James [10,11] studied the representation theory of the Hecke algebras, defining Specht modules S^{λ} which generalize Specht modules for symmetric groups. They showed that induced modules of Specht modules have a filtration by Specht modules [10]. Jost [35] showed that restrictions of Specht modules have Specht filtrations.

Murphy [50] showed that the Hecke algebras are cellular (before the formalization of the notion of cellularity in [23]). Murphy shows that his cell modules Δ^{λ} satisfy $\Delta^{\lambda} \cong (S^{\lambda'})^{\#}$, where λ' is the transpose of λ and the superscript # means that the module is twisted by the automorphism #. Thus it follows from the results of Dipper, James, and Jost cited above that restricted modules and induced modules of Murphy's cell modules have cell filtrations.

2.5. Inclusions of split semisimple algebras and branching diagrams

A general source for the material in this section is [18].

A finite dimensional split semisimple algebra over a field F is one which is isomorphic to a finite direct sum of full matrix algebras over F.

Suppose $A \subseteq B$ are finite dimensional split semisimple algebras over F (with the same identity element). Let $A(i)$, $i \in I$, be the minimal ideals of A and $B(j)$, $j \in J$, the minimal ideals of B. We associate a $J \times I$ *inclusion matrix* Ω to the inclusion $A \subseteq B$, as follows. Let W_j be a simple $B(j)$-module. Then W_j becomes an A-module via the inclusion, and $\Omega(j,i)$ is the multiplicity of a simple A_i-module in the decomposition of W_j as an A-module. An equivalent characterization of the inclusion matrix is the following. Let q_i be a minimal idempotent in $A(i)$ and let z_j be the identity of $B(j)$ (a minimal central idempotent in B). Then $q_i z_j$ is the sum of $\Omega(j,i)$ minimal idempotents in $B(j)$.

It is convenient to encode an inclusion matrix by a bipartite graph, called the *branching diagram*; the branching diagram has vertices labeled by I arranged on one horizontal line, vertices labeled by J arranged along a second (higher) horizontal line, and $\Omega(j,i)$ edges connecting $j \in J$ to $i \in I$.

[1] Hemmer and Nakano [29] have obtained remarkable general results about uniqueness of multiplicities in Specht filtrations of modules over Hecke algebras of type A. Hartmann and Paget [28] obtained analogous results for modules over Brauer algebras. The assertions that we require here are much more special, applying only to induced modules of cell modules and restrictions of cell modules.

If $A_1 \subseteq A_2 \subseteq A_3 \cdots$ is a (finite or infinite) sequence of inclusions of finite dimensional split semisimple algebras over F, then the branching diagram for the sequence is obtained by stacking the branching diagrams for each inclusion, with the upper vertices of the diagram for $A_i \subseteq A_{i+1}$ being identified with the lower vertices of the diagram for $A_{i+1} \subseteq A_{i+2}$.

For our purposes, it suffices to restrict our attention to the case that $A_0 \cong F$. In most of our examples, the entries in each inclusion matrix are all 0 or 1; thus in the branching diagram there are no multiple edges between vertices.

Definition 2.17. An (infinite) abstract branching diagram $\mathfrak{B}$ is an infinite graph with vertex set $V = \coprod_{i \geqslant 0} V_i$, with the following properties

(1) V_0 is a singleton and V_i is finite for all i.
(2) Two vertices $v \in V_i$ and $w \in V_j$ are adjacent only if $|i - j| = 1$. Multiple edges are allowed between adjacent vertices.
(3) If $i \geqslant 1$ and $v \in V_i$, then v is adjacent to at least one vertex in V_{i-1} and to at least one vertex in V_{i+1}.

The definition can be modified in the obvious way for a *finite* abstract branching diagram. When we treat the walled Brauer algebra in Section 5.6, we will loosen the definition by dropping the requirement that V_0 is a singleton.

The branching diagram for a sequence of finite dimensional split semisimple algebras (with the restrictions mentioned above) is an abstract branching diagram, and conversely, given an abstract branching diagram $\mathfrak{B}$, one can construct a sequence of finite dimensional split semisimple algebras (over any given field) whose branching diagram is (isomorphic to) $\mathfrak{B}$.

Let $\mathfrak{B}$ be an abstract branching diagram with vertex set $V = \coprod_{i \geqslant 0} V_i$. We usually denote the unique element of V_0 by $\emptyset$. We picture $\mathfrak{B}$ with the elements of V_i arranged on the horizontal line $y = i$ in the plane, and we call V_i the i-th *row* of vertices in $\mathfrak{B}$. If $v \in V_i$ and $w \in V_{i+1}$ are adjacent, we write $v \nearrow w$. The subgraph of $\mathfrak{B}$ consisting of V_i and V_{i+1} and the edges connecting them is called the i-th *level* of $\mathfrak{B}$.

Now suppose we are given an abstract branching diagram $\mathfrak{B}_0$ with vertex set $V^{(0)} = \coprod_{i \geqslant 0} V_i^{(0)}$. We construct a new abstract branching diagram $\mathfrak{B}$ as follows: The vertex set of $\mathfrak{B}$ is $V = \coprod_{k \geqslant 0} V_k$, where

$$V_k = \coprod_{\substack{i \leqslant k \\ k-i \text{ even}}} V_i^{(0)} \times \{k\}.$$

Thus the k-th row of vertices of $\mathfrak{B}$ consists of copies of rows $k, k-2, k-4, \ldots$ of vertices of $\mathfrak{B}_0$. Now if $(\lambda, k) \in V_k$ and $(\mu, k+1) \in V_{k+1}$, there exist $i \leqslant k$ with $k - i$ even such that $\lambda \in V_i^{(0)}$, and $j \leqslant k+1$ with $k+1-j$ even such that $\mu \in V_j^{(0)}$. We declare $(\lambda, k) \nearrow (\mu, k+1)$ if, and only if, $|i - j| = 1$ and λ and μ are adjacent in $\mathfrak{B}_0$. The number of edges connecting (λ, k) and $(\mu, k+1)$ is the same as the number of edges connecting λ and μ in $\mathfrak{B}_0$.

The first few levels of $\mathfrak{B}$ is picture schematically in Fig. 2.1, where each diagonal line represents all the edges connecting vertices in $V_i^{(0)}$ with vertices in $V_{i\pm1}^{(0)}$. Note that the k-th level of $\mathfrak{B}$ is a folded copy of the first k levels of $\mathfrak{B}_0$. We call $\mathfrak{B}$ the *branching diagram obtained by reflections from* $\mathfrak{B}_0$.

Example 2.18. Take $\mathfrak{B}_0$ to be Young's lattice. Thus $V_k^{(0)}$ consists of Young diagrams of size k, and $\lambda \nearrow \mu$ in $\mathfrak{B}_0$ if μ is obtained from λ by adding one box. Then the k-th row of vertices in the abstract branching diagram $\mathfrak{B}$ obtained from $\mathfrak{B}_0$ by reflections consists of all pairs (λ, k), where λ is a Young diagram of size $i \leqslant k$, with $k - i$ even. Moreover, $(\lambda, k) \nearrow (\mu, k+1)$ in $\mathfrak{B}$ if, and only if, μ is obtained from λ either by adding one box or by removing one box.

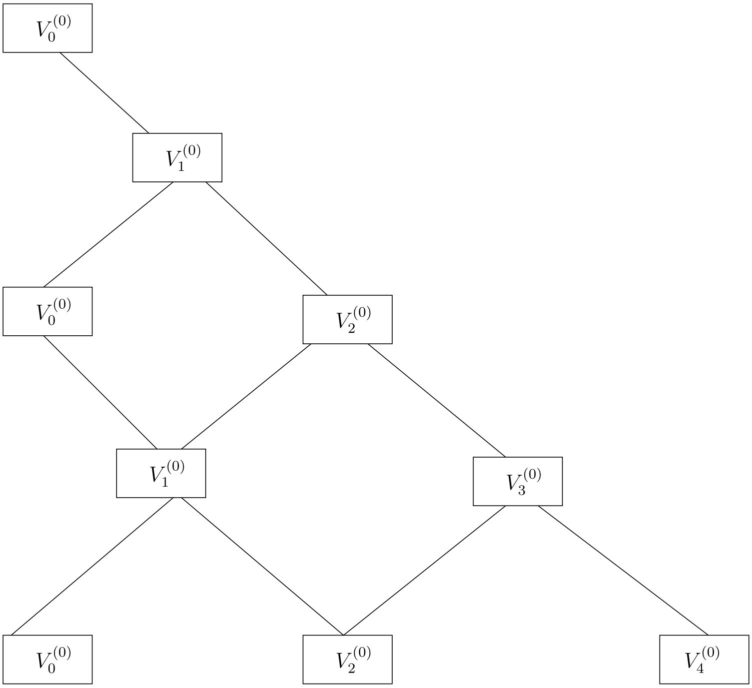


Fig. 2.1. Branching diagram obtained by reflections.

2.6. The Jones basic construction

This paper could be written without ever mentioning the Jones basic construction. Nevertheless, in our view, the basic construction plays an essential role behind the scenes.

The Jones basic construction was introduced [32] in the theory of von Neumann algebras and is crucial in the analysis of von Neumann subfactors. Translated to the context of finite dimensional split semisimple algebras over a field, the basic construction was a fundamental ingredient in Wenzl's analysis of the generic structure of the Brauer algebras and the BMW algebras [55,6,56].

The basic construction for finite dimensional split semisimple algebras can be described as follows (see [18]): let $A \subseteq B$ be finite dimensional split semisimple algebras over field F, with the same multiplicative identity element. The basic construction for the pair $A \subseteq B$ is the algebra $\mathrm{End}(B_A)$. This algebra is also split semisimple and the inclusion matrix for the pair $B \subseteq \mathrm{End}(B_A)$ is a transpose of that for the pair $A \subseteq B$. Suppose now that B has a faithful F-valued trace ε with faithful restriction to A. Here faithful means that the bilinear form $(x, y) \mapsto \varepsilon(xy)$ is non-degenerate. In this case there is a unique trace preserving conditional expectation $\varepsilon_A : B \to A$, i.e. a unital A–A-bimodule map satisfying $\varepsilon \circ \varepsilon_A = \varepsilon$. Identify B with its image in $\mathrm{End}_F(B)$ under the left regular representation. The basic construction $\mathrm{End}(B_A)$ is equal to $B\varepsilon_A B = \{\sum_{i=1}^n b_i' \varepsilon_A b_i'':\ n \geqslant 1,\ b_i', b_i'' \in B\}$. Moreover, $B\varepsilon_A B \cong B \otimes_A B$, where the latter is given the algebra structure determined by $(b_1 \otimes b_2)(b_3 \otimes b_4) = b_1 \otimes \varepsilon_A(b_2 b_3) b_4$. Note that we have three realizations for the basic construction,

$$\mathrm{End}(B_A) \cong B\varepsilon_A B \cong B \otimes_A B,$$

any of which could serve as a potential definition of the basic construction in a more general setting.

Suppose in addition that we are given an algebra C with $B \subseteq C$ and that C contains an idempotent e such that $exe = \varepsilon_A(x)e$ for $x \in B$, and $x \mapsto xe$ is injective from B to $Be \subseteq C$. Note that BeB is a possibly non-unital subalgebra of C. By [55], Theorem 1.3, $BeB \cong B\varepsilon_A B \cong \mathrm{End}(B_A)$, and, in particular, BeB is unital and semisimple.

Let's now describe how Wenzl used these ideas to show the generic semisimplicity of the Brauer algebras. We refer the reader to Section 5.2.1 for the definition of the Brauer algebras. Consider the Brauer algebras $B_n = B_n(F, \delta)$ over $F = \mathbb{C}$ or $F = \mathbb{Q}(\delta)$, in the first case with parameter δ a non-integer complex number, and in the second case with parameter δ an indeterminant over $\mathbb{Q}$. The Brauer algebras have a canonical F-valued trace ε and conditional expectations $\varepsilon_n : B_n \to B_{n-1}$ preserving the trace. Each Brauer algebra B_n contains an essential idempotent e_{n-1} with $e_{n-1}^2 = \delta e_{n-1}$ and $e_{n-1} x e_{n-1} = \delta \varepsilon_{n-1}(x) e_{n-1}$ for $x \in B_{n-1}$. Moreover, $x \mapsto x e_{n-1}$ is injective from B_{n-1} to B_n and one has $B_n / B_n e_{n-1} B_n \cong F\mathfrak{S}_n$, which is semisimple, since F has characteristic 0. Let $f_{n-1} = \delta^{-1} e_{n-1}$; then f_{n-1} is an idempotent with $f_{n-1} x f_{n-1} = \varepsilon_{n-1}(x) f_{n-1}$ for $x \in B_{n-1}$. We have $B_0 \cong B_1 \cong F$.

Suppose it is known for some n that B_k is split semisimple and that the trace ε is faithful on B_k for $k \leqslant n$. By Wenzl's observation applied to $B_{n-1} \subseteq B_n \subseteq B_{n+1}$ and the idempotent $f_n \in B_{n+1}$, we have $B_n e_n B_n = B_n f_n B_n \cong B_n \varepsilon_n B_n \cong \mathrm{End}((B_n)_{B_{n-1}})$. But it is elementary to check that $B_n e_n B_n = B_{n+1} e_n B_{n+1}$. Thus we have that the ideal $B_{n+1} e_n B_{n+1} \subseteq B_{n+1}$ is split semisimple, and the quotient of B_{n+1} by this ideal ($\cong F\mathfrak{S}_{n+1}$) is also split semisimple, so B_{n+1} is split semisimple. To continue the inductive argument, it is necessary to verify that the trace ε is faithful on B_{n+1}. Wenzl uses a Lie theory argument for this.

In this paper, we develop a cellular analog of this argument. Let's continue to use the example of the Brauer algebras to illustrate this. Cellularity is a property that is preserved under specializations, so it suffices to consider the Brauer algebras over the generic ring $R = \mathbb{Z}[\delta]$. Let F denote the field of fractions of R, $F = \mathbb{Q}(\delta)$. Write B_n for $B_n(R, \delta)$ and B_n^F for $B_n(F, \delta)$. By Wenzl's theorem, B_n^F is split semisimple. We have $B_0 \cong B_1 \cong R$.

Suppose it is known for some n that B_k is cellular for $k \leqslant n$. We want to show that $B_{n+1} e_n B_{n+1} = B_n e_n B_n$ is a cellular ideal in B_{n+1}. It will then follow that B_{n+1} is cellular, because the quotient $B_{n+1} / B_{n+1} e_n B_{n+1} \cong R\mathfrak{S}_{n+1}$ is cellular. Let Λ_{n-1} denote the partially ordered set in the cell datum for B_{n-1}. For each order ideal Γ of Λ_{n-1}, write $J(\Gamma)$ for the span in B_{n-1} of all $c_{s,t}^{\lambda}$ with $\lambda \in \Gamma$. The crucial point is to show that $\Gamma \mapsto B_n e_n J(\Gamma) B_n = B_{n+1} e_n J(\Gamma) B_{n+1}$ is a Λ_{n-1}-cell net in $B_{n+1} e_n B_{n+1}$. Along the way to doing this, we show that

$$J'(\Gamma) := B_n \otimes_{B_{n-1}} J(\Gamma) \otimes_{B_{n-1}} B_n \cong B_n e_n J(\Gamma) B_n \tag{2.2}$$

via $b' \otimes x \otimes b'' \mapsto b' e_n x b''$; consequently, if $\Gamma_1 \subseteq \Gamma_2$, then $J'(\Gamma_1)$ imbeds in $J'(\Gamma_2)$. In particular,

$$B_n \otimes_{B_{n-1}} B_n \cong B_n e_n B_n = B_{n+1} e_n B_{n+1}, \tag{2.3}$$

and $J'(\Gamma)$ imbeds as an ideal in the (non-unital) algebra $B_n \otimes_{B_{n-1}} B_n$. Essentially, what we show is that $B_{n+1} e_n B_{n+1} = B_n e_n B_n$ is isomorphic to the basic construction $B_n \otimes_{B_{n-1}} B_n$, and that $\Gamma \mapsto J'(\Gamma)$ is a Λ_{n-1}-cell net in $B_n \otimes_{B_{n-1}} B_n$.

We note that B_n is *not* a projective B_{n-1}-module, but the isomorphisms (2.2) and the embeddings $J'(\Gamma_1) \hookrightarrow J'(\Gamma_2)$ reflect the projectivity of B_n^F over B_{n-1}^F.

2.7. Coherent cellular towers and extension of the ground ring

Let R be an integral domain and let F denote the field of fractions of R. We will be interested in coherent towers $(H_n)_{n \geqslant 0}$ of cellular algebras over R such that for all n, the F-algebra $H_n^F := H_n \otimes_R F$ is (split) semisimple. We will see that in this situation we have uniqueness of multiplicities in the filtrations of induced and restricted modules by cell modules, and Frobenius reciprocity connecting these multiplicities.

For any algebra A over R, write A^F for the F-algebra $A \otimes_R F$. Moreover, for a left (or right) A-module M, write M^F for the left (or right) A^F module $M \otimes_R F$.

Lemma 2.19. *Let R be an integral domain and F its field of fractions. Let A and B be R-algebras. For modules M_A and ${}_A N$, we have*

$$M \otimes_A N \otimes_R F \cong M^F \otimes_{A^F} N^F \tag{2.4}$$

as F-vector spaces. The isomorphism

$$M \otimes_A N \otimes_R F \to M^F \otimes_{A^F} N^F$$

is determined by $(x \otimes_A y \otimes_R f) \mapsto (x \otimes_R \mathbf{1}_F) \otimes_{A^F} (y \otimes_R f)$. *If ${}_A N_B$ is a bimodule, then the isomorphism in* (2.4) *is an isomorphism of right B^F-modules, and similarly, if ${}_B M_A$ is a bimodule, then the isomorphism is an isomorphism of left B^F-modules.*

Proof. Note that

$$\begin{aligned} M \otimes_A (N \otimes_R F) &\cong M \otimes_A A^F \otimes_{A^F} (N \otimes_R F) \\ &= (M \otimes_A A \otimes_R F) \otimes_{A^F} (N \otimes_R F) \\ &\cong (M \otimes_R F) \otimes_{A^F} (N \otimes_R F) \\ &= M^F \otimes_{A^F} N^F. \end{aligned}$$

If we track a simple tensor through these equalities and isomorphisms, we see that

$$\begin{aligned} x \otimes_A y \otimes_R f &\mapsto x \otimes_A \mathbf{1}_{A^F} \otimes_{A^F} (y \otimes_R f) \\ &= x \otimes_A \mathbf{1}_A \otimes_R \mathbf{1}_F \otimes_{A^F} (y \otimes_R f) \mapsto (x \otimes_R \mathbf{1}_F) \otimes_{A^F} (y \otimes_R f). \end{aligned}$$

The final statement follows from this. □

Lemma 2.20. *Let R be an integral domain and F its field of fractions. If M is a free R-module, then the map $M \to M \otimes_R F$ determined by $x \mapsto x \otimes 1_F$ is injective.*

Proof. It follows from [30], Propositions 3.2 and 3.3 that the map $x \mapsto x \otimes 1$ takes an R-basis of M to an F-basis of $M \otimes_R F$. In particular, the map is injective. □

Lemma 2.21. *Let R be an integral domain and F its field of fractions. Let $N_1 \subseteq N_2$ be R-modules with N_2 free. Let $\iota : N_1 \to N_2$ denote the injection. Then $\iota \otimes \mathrm{id}_F : N_1 \otimes_R F \to N_2 \otimes_R F$ is injective.*

Proof. Any element of $N_1 \otimes_R F$ can be written as $y = (1/q)(x \otimes 1_F)$, with $q \in R^\times$ and $x \in N_1$. Then $\iota \otimes \mathrm{id}_F(y) = (1/q)(\iota(x) \otimes 1_F) = (1/q)\gamma \circ \iota(x)$, where $\gamma : N_2 \to N_2 \otimes_R F$ is determined by $z \mapsto z \otimes 1_F$. Because N_2 is a free R-module, γ is injective, by Lemma 2.20, and it follows that $\iota \otimes \mathrm{id}_F$ is injective. □

Lemma 2.22. *Let R be an integral domain with field of fractions F. Suppose that $(H_n)_{n \geqslant 0}$ is a coherent tower of cellular algebras over R and that H_n^F is split semisimple for all n. Let Λ_n denote the partially ordered set in the cell datum for H_n. Then*

(1) *$\{(\Delta^\lambda)^F\colon \lambda \in \Lambda_n\}$ is a complete family of simple H_n^F-modules.*
(2) *Let $[\omega(\mu, \lambda)]_{\mu \in \Lambda_{n+1}, \lambda \in \Lambda_n}$ denote the inclusion matrix for $H_n^F \subseteq H_{n+1}^F$. Then for any $\lambda \in \Lambda_n$ and $\mu \in \Lambda_{n+1}$, and any cell filtration of $\mathrm{Res}_{H_n}^{H_{n+1}}(\Delta^\mu)$, the number of subquotients of the filtration isomorphic to Δ^λ is $\omega(\mu, \lambda)$.*
(3) *Likewise, for any $\lambda \in \Lambda_n$ and $\mu \in \Lambda_{n+1}$, and any cell filtration of $\mathrm{Ind}_{H_n}^{H_{n+1}}(\Delta^\lambda)$, the number of subquotients of the filtration isomorphic to Δ^μ is $\omega(\mu, \lambda)$.*

Proof. For point (1), $(\Delta^\lambda)^F$ is a cell module for H_n^F, and, for a semisimple cellular algebra, the cell modules are precisely the simple modules.

We have

$$\left(\mathrm{Res}_{H_n}^{H_{n+1}}\left(\Delta^\mu\right)\right)^F = \mathrm{Res}_{H_n^F}^{H_{n+1}^F}\left(\left(\Delta^\mu\right)^F\right) \cong \bigoplus_{\lambda\in\Lambda_n} \omega(\mu,\lambda)\left(\Delta^\lambda\right)^F, \tag{2.5}$$

by definition of the inclusion matrix. On the other hand, if

$$\mathrm{Res}_{H_n}^{H_{n+1}}\left(\Delta^\mu\right) = N_s \supseteq N_{s-1} \supseteq \cdots \supseteq N_0 = (0)$$

is a cell filtration, with $N_j/N_{j-1} \cong \Delta^{\lambda_j}$, then

$$\left(\mathrm{Res}_{H_n}^{H_{n+1}}\left(\Delta^\mu\right)\right)^F = N_s^F \supseteq N_{s-1}^F \supseteq \cdots \supseteq N_0^F = (0),$$

by Lemma 2.21, because all the modules N_j are free as R-modules. Moreover, $N_j^F/N_{j-1}^F \cong (N_j/N_{j-1})^F \cong (\Delta^{\lambda_j})^F$ by right exactness of tensor products. Since H_n^F modules are semisimple,

$$\left(\mathrm{Res}_{H_n}^{H_{n+1}}\left(\Delta^\mu\right)\right)^F \cong \bigoplus_{j=1}^{s}\left(\Delta^{\lambda_j}\right)^F. \tag{2.6}$$

Comparing (2.5) and (2.6) and taking into account that $\Delta^\lambda \mapsto (\Delta^\lambda)^F$ is injective, we obtain conclusion (2).

Likewise,

$$\left(\mathrm{Ind}_{H_n}^{H_{n+1}}\left(\Delta^\lambda\right)\right)^F = H_{n+1} \otimes_{H_n} \Delta^\lambda \otimes_R F \cong H_{n+1}^F \otimes_{H_n^F} \left(\Delta^\lambda\right)^F,$$

by Lemma 2.19. But

$$H_{n+1}^F \otimes_{H_n^F} \left(\Delta^\lambda\right)^F = \mathrm{Ind}_{H_n^F}^{H_{n+1}^F}\left(\left(\Delta^\lambda\right)^F\right) \cong \bigoplus_{\mu\in\Lambda_{n+1}} \omega(\mu,\lambda)\left(\Delta^\mu\right)^F,$$

using (2.5) and Frobenius reciprocity. The rest of the argument for point (3) is similar to that for point (2). □

Lemma 2.23. *Adopt the assumptions and notation of Lemma* 2.22. *Assume in addition that the branching diagram* $\mathfrak{B}$ *for* $(H_n^F)_{n\geqslant 0}$ *has no multiple edges and that* $H_0^F = F$. *It follows that each* H_n *has a cell datum* (*perhaps different from the one initially given*) *with the same partially ordered set* Λ_n *but with* $\mathscr{T}(\lambda)$ *equal to the set of paths on* $\mathfrak{B}$ *from* $\emptyset$ *to* λ.

Proof. Referring to the proof of Proposition 2.13, it suffices to show that, for each n and for each $\lambda \in \Lambda_n$, the cell module Δ^λ has an R-basis indexed by the set $\mathscr{P}(\lambda)$ of paths in $\mathfrak{B}$ from $\emptyset$ to λ. But this says only that the rank of Δ^λ over R is $|\mathscr{P}(\lambda)|$, and this is true because $\mathrm{rank}_R(\Delta^\lambda) = \dim_F(\Delta^\lambda \otimes_R F) = |\mathscr{P}(\lambda)|$. See also the following remark. □

Remark 2.24. In principle, in the situation of Lemma 2.23, we can recursively build bases of cell modules, using the cell filtrations of restrictions. Suppose we have bases of Δ^λ for all $\lambda \in \Lambda_n$ for some n. Let $\mu \in \Lambda_{n+1}$. Then Δ^μ, regarded as an H_n-module, has a filtration by cell modules of H_n,

$$\Delta^\mu = N_s \supseteq N_{s-1} \supseteq \cdots \supseteq N_0 = (0),$$

with $N_j/N_{j-1} \cong \Delta^{\lambda_j}$; and $\lambda \in \Lambda_n$ appears (exactly once) in the list of λ_j, if, and only if, $\lambda \nearrow \mu$. Now we inductively build bases of the N_j to obtain a basis of $N_s = \Delta^{\mu}$. The isomorphism $N_1 \cong \Delta^{\lambda_1}$ provides a basis of N_1. For $j \geqslant 2$, if we have a basis of N_{j-1}, then that basis together with any lift of a basis of $N_j/N_{j-1} \cong \Delta^{\lambda_j}$ gives a basis of N_j.

3. A framework for cellularity

In this section we describe our framework for cellularity of algebras related to the Jones basic construction.

3.1. Framework axioms

Let R be an integral domain with field of fractions F. We consider two sequences of R-algebras

$$A_0 \subseteq A_1 \subseteq A_2 \subseteq \cdots, \quad \text{and} \quad Q_0 \subseteq Q_1 \subseteq Q_2 \subseteq \cdots,$$

each with a common multiplicative identity element. We assume the following axioms:

(1) $(Q_n)_{n \geqslant 0}$ is a coherent tower of cellular algebras.
(2) There is an algebra involution i on $\cup_n A_n$ such that $i(A_n) = A_n$.
(3) $A_0 = Q_0 = R$, and $A_1 = Q_1$ (as algebras with involution).
(4) For all n, $A_n^F := A_n \otimes_R F$ is split semisimple.
(5) For $n \geqslant 2$, A_n contains an essential idempotent e_{n-1} such that $i(e_{n-1}) = e_{n-1}$ and $A_n/(A_n e_{n-1} A_n) \cong Q_n$, as algebras with involution.
(6) For $n \geqslant 1$, e_n commutes with A_{n-1} and $e_n A_n e_n \subseteq A_{n-1} e_n$.
(7) For $n \geqslant 1$, $A_{n+1} e_n = A_n e_n$, and the map $x \mapsto x e_n$ is injective from A_n to $A_n e_n$.
(8) For $n \geqslant 2$, $e_{n-1} \in A_{n+1} e_n A_{n+1}$.

Remark 3.1.

(1) Let $\Lambda_n^{(0)}$ denote the partially ordered set in the cell datum for Q_n. It follows from axioms (1) and (4) and Lemma 2.22 that $\Lambda_n^{(0)}$ can be identified with the n-th row of vertices of the branching diagram for $(Q_n^F)_{n \geqslant 0}$.
(2) Applying the involution in axiom (7), we also have $e_n A_{n+1} = e_n A_n$, and the map $x \mapsto e_n x$ is injective from A_n to $e_n A_n$.
(3) Since e_n is an essential idempotent, there is a non-zero $\delta_n \in R$ with $e_n^2 = \delta_n e_n$. Thus we have $e_n A_n e_n \supseteq e_n A_{n-1} e_n = A_{n-1} e_n^2 = \delta_n A_{n-1} e_n$. Combining this with axiom (6), we have $\delta_n A_{n-1} e_n \subseteq e_n A_n e_n \subseteq A_{n-1} e_n$. Hence $e_n A_n^F e_n = A_{n-1}^F e_n$.
(4) From axiom (6), we have for every $x \in A_n$, there is a $y \in A_{n-1}$ such that $e_n x e_n = y e_n$; but by axiom (7), y is uniquely determined, so we have a map $\mathrm{cl}_n : A_n \to A_{n-1}$ with $e_n x e_n = \mathrm{cl}_n(x) e_n$. It is easy to check that cl_n is an A_{n-1}–A_{n-1}-bimodule map, but it is not unital in general; if $e_{n-1}^2 = \delta_n e_{n-1}$, then $\mathrm{cl}_n(\mathbf{1}) = \delta_n \mathbf{1}$. If δ_n is invertible in R, then $\varepsilon_n = (1/\delta_n)\,\mathrm{cl}_n$ is a conditional expectation, i.e. a unital A_{n-1}–A_{n-1}-bimodule map.
(5) From axioms (4) and (5), we have $Q_n^F := Q_n \otimes_R F$ is split semisimple.
(6) In our examples, there is a single non-zero δ with $e_n^2 = \delta e_n$ for all n.

3.2. The main theorem

Theorem 3.2. *Let R be an integral domain with field of fractions F. Let $(Q_k)_{k \geqslant 0}$ and $(A_k)_{k \geqslant 0}$ be two towers of R-algebras satisfying the framework axioms of Section* 3.1. *Then*

(1) *$(A_k)_{k \geqslant 0}$ is a coherent tower of cellular algebras.*

(2) *For all k, the partially ordered set in the cell datum for A_k can be realized as*

$$\Lambda_k = \coprod_{\substack{i \leqslant k \\ k-i \text{ even}}} \Lambda_i^{(0)} \times \{k\},$$

with the following partial order: Let $\lambda \in \Lambda_i^{(0)}$ and $\mu \in \Lambda_j^{(0)}$, with i, j, and k all of the same parity. Then $(\lambda, k) > (\mu, k)$ if, and only if, $i < j$, or $i = j$ and $\lambda > \mu$ in $\Lambda_i^{(0)}$.

(3) *Suppose $k \geqslant 2$ and $(\lambda, k) \in \Lambda_i^{(0)} \times \{k\} \subseteq \Lambda_k$. Let $\Delta^{(\lambda,k)}$ be the corresponding cell module. If $i < k$, then $(A_k e_{k-1} A_k\, \Delta^{(\lambda,k)}) \otimes_R F = \Delta^{(\lambda,k)} \otimes_R F$, while if $i = k$ then $A_k e_{k-1} A_k\, \Delta^{(\lambda,k)} = 0$.*

(4) *The branching diagram $\mathfrak{B}$ for $(A_k^F)_{k \geqslant 0}$ is that obtained by reflections from the branching diagram $\mathfrak{B}_0$ for $(Q_k^F)_{n \geqslant 0}$.*

Remark 3.3. In most of our examples, the branching diagrams have no multiple edges. In this case, for all k and for all $(\lambda, k) \in \Lambda_k$, the index set $\mathscr{T}((\lambda, k))$ in the cell datum for A_k can be taken to be the set of paths on $\mathfrak{B}$ from $\emptyset$ to (λ, k). This follows from (1) and (4), using Lemma 2.23.

4. Proof of the main theorem

We will prove Theorem 3.2 in this section. Our strategy is to prove the following statement by induction on n:

Claim. *For all $n \geqslant 0$, the statements* (1)–(4) *of Theorem* 3.2 *hold for the finite tower* $(A_k)_{0 \leqslant k \leqslant n}$.

Of course, by statement (4) for the finite tower, we mean that the branching diagram for the finite tower $(A_k^F)_{0 \leqslant k \leqslant n}$ is that obtained by reflections from the branching diagram of the finite tower $(Q_k^F)_{0 \leqslant k \leqslant n}$.

The claim holds trivially for $n = 0$ and $n = 1$. We assume that the claim holds for some $n \geqslant 1$ and prove that it also holds for $n + 1$.

4.1. A_{n+1} is cellular

We will show that A_{n+1} is a cellular algebra.

Since $A_{n+1}/A_{n+1} e_n A_{n+1} \cong Q_{n+1}$ is cellular, to prove that A_{n+1} is cellular, it suffices to show that $A_{n+1} e_n A_{n+1}$ is a cellular ideal in A_{n+1}; see Remark 2.6.

Recall that Λ_k denotes the partially ordered set in the cell datum for A_k for each k, $0 \leqslant k \leqslant n$. Denote the elements of the cellular basis of A_k by $c_{u,v}^\lambda$ for $\lambda \in \Lambda_k$ and $u, v \in \mathscr{T}(\lambda)$.

For each order ideal Γ of Λ_{n-1}, recall that $A_{n-1}(\Gamma)$ is the span in A_{n-1} of all $c_{s,t}^\lambda$ with $\lambda \in \Gamma$. $A_{n-1}(\Gamma)$ is an i-invariant two sided ideal of A_{n-1}.

In the following, we will write $J(\Gamma) = A_{n-1}(\Gamma)$ and

$$\hat{J}(\Gamma) = A_n e_n J(\Gamma) A_n = A_{n+1} e_n J(\Gamma) A_{n+1},$$

which is a two sided ideal in A_{n+1}. Our goal is to show that $\Gamma \mapsto \hat{J}(\Gamma)$ is a Λ_{n-1}-cell net in $A_{n+1} e_n A_{n+1}$.

Lemma 4.1. *Let R be an integral domain and F its field of fractions. Suppose that A and B are R-algebras. Let P_A, ${}_A M_A$ and ${}_A Q$ be modules. Then*

$$P \otimes_A M \otimes_A Q \otimes_R F \cong P^F \otimes_{A^F} M^F \otimes_{A^F} Q^F$$

as F-vector spaces. The isomorphism

$$P \otimes_A M \otimes_A Q \otimes_R F \to P^F \otimes_{A^F} M^F \otimes_{A^F} Q^F$$

is determined by

$$x \otimes_A y \otimes_A z \otimes_R f \mapsto (x \otimes_R \mathbf{1}_F) \otimes_{A^F} (y \otimes_R \mathbf{1}_F) \otimes_{A^F} (z \otimes_R f).$$

If ${}_B P_A$ and ${}_A Q_B$ are bimodules, then the isomorphism is an isomorphism of B^F–B^F-bimodules.

Proof. By Lemma 2.19,

$$(P \otimes_A M) \otimes_A Q \otimes_R F \cong (P \otimes_A M)^F \otimes_{A^F} Q^F. \tag{4.1}$$

Applying Lemma 2.19 again, we have that

$$(P \otimes_A M)^F \cong P^F \otimes_{A^F} M^F \tag{4.2}$$

as right A^F-modules. Combining the two isomorphisms we have

$$P \otimes_A M \otimes_A Q \otimes_R F \cong P^F \otimes_{A^F} M^F \otimes_{A^F} Q^F. \tag{4.3}$$

If we track a simple tensor through these isomorphisms, we see that

$$\begin{aligned} x \otimes_A y \otimes_A z \otimes_R f &\mapsto (x \otimes_A y \otimes_R \mathbf{1}_F) \otimes_{A^F} (z \otimes_R f) \\ &\mapsto (x \otimes_R \mathbf{1}_F) \otimes_{A^F} (y \otimes_R \mathbf{1}_F) \otimes_{A^F} (z \otimes_R f). \end{aligned}$$

If ${}_B P_A$ and ${}_A Q_B$ are bimodules, then the isomorphism in (4.1) is an isomorphism of B^F–B^F-bimodules, and the isomorphism in (4.2) is an isomorphism of B^F–A^F-bimodules. Hence the final isomorphism (4.3) is an isomorphism of B^F–B^F-bimodules. □

Lemma 4.2. *Let K be a field and A a semisimple K-algebra. Suppose that $I \subseteq A$ is a two-sided ideal and M_A, ${}_A N$ are modules. Then the homomorphism $M \otimes_A I \otimes_A N \to M \otimes_A N$ defined by $x \otimes y \otimes z \mapsto x \otimes yz$ is injective.*

Proof. The semisimplicity of A implies that all A-modules are projective. Thus $N \otimes_A -$ and $- \otimes_A M$ are exact, and

$$N \otimes_A I \otimes_A M \to N \otimes_A A \otimes_A M \cong N \otimes_A M$$

is injective. □

Proposition 4.3. *For all order ideals Γ of Λ_{n-1}:*

(1) *The map*

$$\Phi_\Gamma : A_n e_n \otimes_{A_{n-1}} J(\Gamma) \otimes_{A_{n-1}} e_n A_n \to A_n e_n J(\Gamma) A_n$$

determined by

$$\Phi_\Gamma(a_1 e_n \otimes x \otimes e_n a_2) = a_1 e_n x a_2$$

is an isomorphism of A_{n+1}–A_{n+1}-bimodules.

(2) *$A_n e_n \otimes_{A_{n-1}} J(\Gamma) \otimes_{A_{n-1}} e_n A_n$ is a free R-module.*
(3) *Let Γ' be another order ideal containing Γ, such that $\Gamma' \setminus \Gamma$ is a singleton. Let ι denote the injection $J(\Gamma) \to J(\Gamma')$. Then*

$$\beta_{\Gamma,\Gamma'} := \mathrm{id} \otimes \iota \otimes \mathrm{id} : A_n e_n \otimes_{A_{n-1}} J(\Gamma) \otimes_{A_{n-1}} e_n A_n \to A_n e_n \otimes_{A_{n-1}} J\big(\Gamma'\big) \otimes_{A_{n-1}} e_n A_n$$

is injective.

We provide two lemmas on the way to proving Proposition 4.3.

Lemma 4.4. *Let $\Gamma \subseteq \Gamma'$ be two order ideals in Λ_{n-1} such that $\Gamma' \setminus \Gamma$ is a singleton. Suppose that Φ_Γ is an isomorphism and that $A_n e_n \otimes_{A_{n-1}} J(\Gamma) \otimes_{A_{n-1}} e_n A_n$ is a free R-module. Then $\beta_{\Gamma,\Gamma'}$ is injective and $A_n e_n \otimes_{A_{n-1}} J(\Gamma') \otimes_{A_{n-1}} e_n A_n$ is a free R-module.*

Proof. Let $\{\lambda\} = \Gamma' \setminus \Gamma$. Since Φ_Γ is assumed injective, it follows from considering the commutative diagram below that $\beta_{\Gamma,\Gamma'}$ is also injective:

$$\begin{tikzcd} A_n e_n \otimes_{A_{n-1}} J(\Gamma) \otimes_{A_{n-1}} e_n A_n \arrow[r, "\Phi_\Gamma"] \arrow[d, "\beta_{\Gamma,\Gamma'}"] & A_n e_n J(\Gamma) A_n \arrow[d] \\ A_n e_n \otimes_{A_{n-1}} J(\Gamma') \otimes_{A_{n-1}} e_n A_n \arrow[r, "\Phi_{\Gamma'}"] & A_n e_n J(\Gamma') A_n \end{tikzcd}$$

By the right exactness of tensor products, we have

$$\begin{aligned} &\big(A_n e_n \otimes_{A_{n-1}} J\big(\Gamma'\big) \otimes_{A_{n-1}} e_n A_n\big)/\beta_{\Gamma,\Gamma'}\big(A_n e_n \otimes_{A_{n-1}} J(\Gamma) \otimes_{A_{n-1}} e_n A_n\big) \\ &\quad\cong A_n e_n \otimes_{A_{n-1}} \big(J\big(\Gamma'\big)/J(\Gamma)\big) \otimes_{A_{n-1}} e_n A_n \\ &\quad\cong A_n e_n \otimes_{A_{n-1}} \Delta^\lambda \otimes_R i\big(\Delta^\lambda\big) \otimes_{A_{n-1}} e_n A_n. \end{aligned} \tag{4.4}$$

Consider $A_n e_n = A_{n+1} e_n$ (because of framework axiom (7)) as an A_{n+1}–A_{n-1}-bimodule. One can easily check that $i(A_n e_n) \cong e_n A_n$ as A_{n-1}–A_{n+1}-bimodules. Therefore,

$$i\big(\Delta^\lambda\big) \otimes_{A_{n-1}} e_n A_n \cong i\big(\Delta^\lambda\big) \otimes_{A_{n-1}} i(A_n e_n) \cong i\big(A_n e_n \otimes_{A_{n-1}} \Delta^\lambda\big), \tag{4.5}$$

using Lemma 2.1. By framework axioms (6) and (7), $A_n e_n \cong A_n$ as A_n–A_{n-1}-bimodules. Hence,

$$A_n e_n \otimes_{A_{n-1}} \Delta^\lambda \cong A_n \otimes_{A_{n-1}} \Delta^\lambda = \mathrm{Ind}_{A_{n-1}}^{A_n}\big(\Delta^\lambda\big), \tag{4.6}$$

as A_n modules. Combining (4.4), (4.5), and (4.6), we have

$$\begin{aligned} &\big(A_n e_n \otimes_{A_{n-1}} J\big(\Gamma'\big) \otimes_{A_{n-1}} e_n A_n\big)/\beta_{\Gamma,\Gamma'}\big(A_n e_n \otimes_{A_{n-1}} J(\Gamma) \otimes_{A_{n-1}} e_n A_n\big) \\ &\quad\cong \mathrm{Ind}_{A_{n-1}}^{A_n}\big(\Delta^\lambda\big) \otimes_R i\big(\mathrm{Ind}_{A_{n-1}}^{A_n}\big(\Delta^\lambda\big)\big), \end{aligned} \tag{4.7}$$

as A_n–A_n-bimodules.

By the induction assumption on n, $\mathrm{Ind}_{A_{n-1}}^{A_n}(\Delta^\lambda)$ has a filtration with subquotients isomorphic to cell modules for A_n, and in particular $\mathrm{Ind}_{A_{n-1}}^{A_n}(\Delta^\lambda)$ is a free R-module. By (4.7),

$$\big(A_n e_n \otimes_{A_{n-1}} J\big(\Gamma'\big) \otimes_{A_n} e_n A_n\big)/\beta_{\Gamma,\Gamma'}\big(A_n e_n \otimes_{A_{n-1}} J(\Gamma) \otimes_{A_{n-1}} e_n A_n\big)$$

is a free R-module. Since $A_n e_n \otimes_{A_{n-1}} J(\Gamma) \otimes_{A_{n-1}} e_n A_n$ is free by hypothesis, and $\beta_{\Gamma,\Gamma'}$ is injective,

$$\beta_{\Gamma,\Gamma'}\big(A_n e_n \otimes_{A_{n-1}} J(\Gamma) \otimes_{A_{n-1}} e_n A_n\big)$$

is a free R-module. Hence

$$A_n e_n \otimes_{A_{n-1}} J\big(\Gamma'\big) \otimes_{A_{n-1}} e_n A_n$$

is also a free R-module. $\square$

Lemma 4.5. *Let Γ be an order ideal in Λ_{n-1}. If $A_n e_n \otimes_{A_{n-1}} J(\Gamma) \otimes_{A_{n-1}} e_n A_n$ is a free R-module, then Φ_Γ is an isomorphism.*

Proof. Φ_Γ is surjective, so we only have to prove Φ_Γ is injective. Define

$$\alpha_1 : A_n e_n \otimes_{A_{n-1}} J(\Gamma) \otimes_{A_{n-1}} e_n A_n \to A_n e_n \otimes_{A_{n-1}} J(\Gamma) \otimes_{A_{n-1}} e_n A_n \otimes_R F$$

and

$$\alpha_2 : A_n e_n J(\Gamma) A_n \to A_n e_n J(\Gamma) A_n \otimes_R F$$

by $x \mapsto x \otimes \mathbf{1}_F$. Since $A_n e_n \otimes_{A_{n-1}} J(\Gamma) \otimes_{A_{n-1}} e_n A_n$ is a free R-module, by assumption, α_1 is injective, according to Lemma 2.20. Let

$$\tau : A_n e_n \otimes_{A_{n-1}} J(\Gamma) \otimes_{A_{n-1}} e_n A_n \otimes_R F \to A_n^F e_n \otimes_{A_{n-1}^F} J(\Gamma)^F \otimes_{A_{n-1}^F} e_n A_n^F$$

be the isomorphism from Lemma 4.1. (We are writing e_n for $e_n \otimes \mathbf{1}_F$.) Let

$$\Phi_\Gamma^F : A_n^F e_n \otimes_{A_{n-1}^F} J(\Gamma)^F \otimes_{A_{n-1}^F} e_n A_n^F \to A_n^F e_n J(\Gamma)^F A_n^F$$

be defined by $x e_n \otimes a \otimes e_n y \mapsto x e_n a y$.

Consider the following diagram

$$\begin{tikzcd}
A_n^F e_n \otimes_{A_{n-1}^F} J(\Gamma)^F \otimes_{A_{n-1}^F} e_n A_n^F \arrow[r, "\Phi_\Gamma^F"] & A_n^F e_n J(\Gamma)^F A_n^F \\
A_n e_n \otimes_{A_{n-1}} J(\Gamma) \otimes_{A_{n-1}} e_n A_n \otimes_R F \arrow[u, "\tau"] \arrow[r, "\Phi_\Gamma \otimes \mathrm{id}_F"] & A_n e_n J(\Gamma) A_n \otimes_R F \arrow[u, equal] \\
A_n e_n \otimes_{A_{n-1}} J(\Gamma) \otimes_{A_{n-1}} e_n A_n \arrow[u, "\alpha_1"] \arrow[r, "\Phi_\Gamma"] & A_n e_n J(\Gamma) A_n \arrow[u, "\alpha_2"']
\end{tikzcd}$$

It is straightforward to check that $\Phi_\Gamma^F \circ \tau \circ \alpha_1 = \alpha_2 \circ \Phi_\Gamma$. Thus, to prove that Φ_Γ is injective, it suffices to show that Φ_Γ^F is injective.

Define

$$\beta : A_n^F e_n \otimes_{A_{n-1}^F} J(\Gamma)^F \otimes_{A_{n-1}^F} e_n A_n^F \to A_n^F e_n \otimes_{A_{n-1}^F} e_n A_n^F$$

by $\beta(x \otimes y \otimes z) = x \otimes yz$. Observe that β is injective by Lemma 4.2. Define

$$\phi^F : A_n^F e_n \otimes_{A_{n-1}^F} e_n A_n^F \to A_n^F e_n A_n^F$$

by $\phi^F(xe_n \otimes e_n y) = xe_n y$. Observe that $\phi^F \circ \beta = \Phi_\Gamma^F$, so to prove that Φ_Γ^F is injective, it suffices to show that ϕ^F is injective.

Since A_{n+1}^F is split semisimple (by framework axiom (4)), the ideal $A_{n+1}^F e_n A_{n+1}^F$ (which equals $A_n^F e_n A_n^F$ by framework axiom (7)) is a unital algebra in its own right, and Morita equivalent to $e_n A_{n+1}^F e_n = e_n A_n^F e_n \cong A_{n-1}^F$. In fact, let

$$\psi^F : e_n A_n \otimes_{A_n^F e_n A_n^F} A_n^F e_n \to e_n A_n^F e_n$$

be given by $e_n x \otimes y e_n \mapsto (1/\delta_n) e_n x y e_n$, where $e_n^2 = \delta_n e_n$. Then

$$\left(e_n A_n^F e_n, A_n^F e_n A_n^F, A_n^F e_n, e_n A_n^F, \psi^F, \phi^F\right)$$

is a Morita context, in the sense of [30], Section 3.12, with surjective bimodule maps ψ^F and ϕ^F. It follows from Morita theory, for example [30], Morita Theorem I, p. 167, that ψ^F and ϕ^F are isomorphisms. $\square$

Proof of Proposition 4.3. Let Γ be an order ideal of Λ_{n-1}. There exists a chain of order ideals

$$\emptyset = \Gamma_0 \subseteq \Gamma_1 \subseteq \cdots \subseteq \Gamma_s = \Gamma,$$

such that the difference between any two successive order ideals is a singleton. Write β_j for $\beta_{\Gamma_j, \Gamma_{j+1}}$, for $0 \leqslant j < s$.

We prove by induction that for $0 \leqslant j \leqslant s$, Φ_{Γ_j} is an isomorphism and $A_n e_n \otimes_{A_{n-1}} J(\Gamma_j) \otimes_{A_{n-1}} e_n A_n$ is a free R-module; and that for $0 \leqslant j < s$, β_j is injective. For $j = 0$, these statements are trivial since $J(\emptyset) = 0$.

Fix j $(0 \leqslant j < s)$ and suppose that $A_n e_n \otimes_{A_{n-1}} J(\Gamma_j) \otimes_{A_{n-1}} e_n A_n$ is a free R-module, that Φ_{Γ_j} is an isomorphism. Then it follows from Lemma 4.4 that $A_n e_n \otimes_{A_{n-1}} J(\Gamma_{j+1}) \otimes_{A_{n-1}} e_n A_n$ is a free R-module. Next, it follows from Lemma 4.5 that $\Phi_{\Gamma_{j+1}}$ is an isomorphism.

We conclude that $A_n e_n \otimes_{A_{n-1}} J(\Gamma) \otimes_{A_{n-1}} e_n A_n$ is a free R-module and that Φ_Γ is an isomorphism. Applying Lemma 4.4 again gives statement (3) of the proposition. $\square$

We continue to work with the following assumptions: R is an integral domain with field of fractions F. $(Q_k)_{k \geqslant 0}$ and $(A_k)_{k \geqslant 0}$ are two towers of R-algebras satisfying the framework axioms of Section 3.1. The following induction assumption is in force: For some fixed $n \geqslant 1$, the conclusions (1)–(4) of Theorem 3.2 hold for the finite tower $(A_k)_{0 \leqslant k \leqslant n}$. We use the notation of the discussion preceding Lemma 4.1.

The following is a corollary of Proposition 4.3.

Corollary 4.6. *$A_n e_n \otimes_{A_{n-1}} e_n A_n \cong A_n e_n A_n$, as A_{n+1}–A_{n+1} bimodules, with the isomorphism determined by $xe_n \otimes e_n y \mapsto xe_n y$.*

Proof. In Proposition 4.3, take $\Gamma = \Lambda_{n-1}$, so $J(\Gamma) = A_{n-1}$. $\square$

Proposition 4.7.

(1) *$\Gamma \mapsto \hat{J}(\Gamma)$ is a Λ_{n-1}-cell net in $A_n e_n A_n$.*
(2) *$A_n e_n A_n$ is a cellular ideal in A_{n+1}.*

(3) *A_{n+1} is a cellular algebra. The partially ordered set in the cell datum for A_{n+1} can be realized as $\Lambda_{n+1} = \Lambda_{n-1} \cup \Lambda^{(0)}_{n+1}$, where $\Lambda^{(0)}_{n+1}$ is the partially ordered set in the cell datum for Q_{n+1}; moreover the partial order on Λ_{n+1} agrees with the original partial orders on Λ_{n-1} and $\Lambda^{(0)}_{n+1}$, and satisfies $\lambda > \mu$ if $\lambda \in \Lambda_{n-1}$ and $\mu \in \Lambda^{(0)}_{n+1}$.*

(4) *Let $\lambda \in \Lambda_{n-1}$, and let Δ^λ denote the corresponding cell module of A_{n-1}. The cell module of A_{n+1} corresponding to λ is isomorphic to $A_n e_n \otimes_{A_{n-1}} \Delta^\lambda$.*

Proof. It is evident that $\hat{J}(\emptyset) = \{0\}$, and that $\Gamma_1 \subseteq \Gamma_2$ implies $\hat{J}(\Gamma_1) \subseteq \hat{J}(\Gamma_2)$. Note that $J(\Gamma_{\geqslant\lambda}) = A^\lambda_{n-1}$, so $\hat{J}(\Gamma_{\geqslant\lambda}) = A_n e_n A^\lambda_{n-1} A_n$. Similarly, $\hat{J}(\Gamma_{>\lambda}) = A_n e_n \breve{A}^\lambda_{n-1} A_n$. It follows that $A_n e_n A_n = \operatorname{span}\{\hat{J}(\Gamma_{\geqslant\lambda})\colon \lambda \in \Lambda_{n-1}\}$ and that for all $\lambda \in \Lambda_{n-1}$, $\hat{J}(\Gamma_{>\lambda}) = \operatorname{span}\{\hat{J}(\Gamma_{\geqslant\mu})\colon \mu > \lambda\}$. We have shown that $\Gamma \mapsto \hat{J}(\Gamma)$ satisfies conditions (1) and (2) of Definition 2.12.

Next we show that $\Gamma \mapsto \hat{J}(\Gamma)$ satisfies condition (3) of Definition 2.12. Let $\Gamma \subseteq \Gamma'$ be two order ideals of Λ_{n-1}, with $\Gamma' \setminus \Gamma = \{\lambda\}$. From the proof of Proposition 4.3, we already have $\hat{J}(\Gamma')/\hat{J}(\Gamma) \cong M^\lambda \otimes_R i(M^\lambda)$, with $M^\lambda = A_n e_n \otimes_{A_{n-1}} \Delta^\lambda$. Let $\chi : \hat{J}(\Gamma')/\hat{J}(\Gamma) \to M^\lambda \otimes_R i(M^\lambda)$ denote the isomorphism. We have to check that $\chi \circ i = i \circ \chi$. The isomorphism Φ_Γ of Proposition 4.3 satisfies $i \circ \Phi_\Gamma = \Phi_\Gamma \circ i$. Moreover,

$$\beta_{\Gamma,\Gamma'}\big(A_n e_n \otimes_{A_{n-1}} J(\Gamma) \otimes_{A_{n-1}} e_n A_n\big) \subseteq A_n e_n \otimes_{A_{n-1}} J\big(\Gamma'\big) \otimes_{A_{n-1}} e_n A_n$$

and $\hat{J}(\Gamma) \subseteq \hat{J}(\Gamma')$ are i-invariant, so the induced isomorphism

$$\widetilde{\Phi}_\Gamma : A_n e_n \otimes_{A_{n-1}} J\big(\Gamma'\big) \otimes_{A_{n-1}} e_n A_n / \beta_{\Gamma,\Gamma'}\big(A_n e_n \otimes_{A_{n-1}} J(\Gamma) \otimes_{A_{n-1}} e_n A_n\big) \to \hat{J}\big(\Gamma'\big)/\hat{J}(\Gamma)$$

satisfies $i \circ \widetilde{\Phi}_\Gamma = \widetilde{\Phi}_\Gamma \circ i$. Next, the map

$$\pi : A_n e_n \otimes_{A_{n-1}} J\big(\Gamma'\big) \otimes_{A_{n-1}} e_n A_n \to A_n e_n \otimes_{A_{n-1}} J\big(\Gamma'\big)/J(\Gamma) \otimes_{A_{n-1}} e_n A_n$$

satisfies $i \circ \pi = \pi \circ i$, so the induced isomorphism

$$\begin{aligned} \tilde{\pi} : {}& A_n e_n \otimes_{A_{n-1}} J\big(\Gamma'\big) \otimes_{A_{n-1}} e_n A_n / \beta_{\Gamma,\Gamma'}\big(A_n e_n \otimes_{A_{n-1}} J(\Gamma) \otimes_{A_{n-1}} e_n A_n\big) \\ &\to A_n e_n \otimes_{A_{n-1}} J\big(\Gamma'\big)/J(\Gamma) \otimes_{A_{n-1}} e_n A_n \end{aligned}$$

satisfies $i \circ \tilde{\pi} = \tilde{\pi} \circ i$. Finally, we have an isomorphism $\alpha : J(\Gamma')/J(\Gamma) \to \Delta^\lambda \otimes_R i(\Delta^\lambda)$ satisfying $i \circ \alpha = \alpha \circ i$, so the map

$$\begin{aligned} \bar{\alpha} = \mathrm{id} \otimes \alpha \otimes \mathrm{id} : {}& A_n e_n \otimes_{A_{n-1}} J\big(\Gamma'\big)/J(\Gamma) \otimes_{A_{n-1}} e_n A_n \\ &\to A_n e_n \otimes_{A_{n-1}} \Delta^\lambda \otimes_R i\big(\Delta^\lambda\big) \otimes_{A_{n-1}} e_n A_n \end{aligned}$$

satisfies $i \circ \bar{\alpha} = \bar{\alpha} \circ i$. The map χ is $\bar{\alpha} \circ \tilde{\pi} \circ \widetilde{\Phi}_\Gamma^{-1}$, so we have $i \circ \chi = \chi \circ i$.

This completes the proof that $\Gamma \mapsto \hat{J}(\Gamma)$ is a Λ_{n-1}-cell net in $A_n e_n A_n$. By Proposition 2.13, $A_n e_n A_n$ has a cell datum with partially ordered set equal to Λ_{n-1}. Moreover, since the isomorphisms $\hat{J}(\Gamma')/\hat{J}(\Gamma) \cong M^\lambda \otimes_R i(M^\lambda)$ are actually isomorphisms of A_{n+1}–A_{n+1}-bimodules, the cellular basis $\widetilde{\mathscr{C}}$ of $A_n e_n A_n$ satisfies the property (2) of Definition 2.3 not only for $a \in A_n e_n A_n$ but also for $a \in A_{n+1}$; that is $A_n e_n A_n$ is a cellular ideal in A_{n+1}.

Statement (3) of the lemma follows from applying Remark 2.6. Statement (4) follows from the isomorphism $\hat{J}(\Gamma')/\hat{J}(\Gamma) \cong M^\lambda \otimes_R i(M^\lambda)$. $\square$

Corollary 4.8. *The description of the partially ordered set given in Theorem* 3.2*, point* (2)*, is valid for* $k = n+1$.

Proof. Combining point (3) of Proposition 4.7 with the induction assumption (specifically the description of Λ_{n-1} as the union of copies of $\Lambda_{n-1}^{(0)}$, $\Lambda_{n-3}^{(0)}$, etc.), we see that Λ_{n+1} is the union of copies of $\Lambda_{n+1}^{(0)}$, $\Lambda_{n-1}^{(0)}$, $\Lambda_{n-3}^{(0)}$, etc., with the following partial order: the partial order agrees with the original partial order on each $\Lambda_i^{(0)}$, and $\lambda > \mu$ if $\lambda \in \Lambda_i^{(0)}$, $\mu \in \Lambda_j^{(0)}$, and $i < j$. □

For the remainder of Section 4, we denote elements of Λ_k $(0 \leqslant k \leqslant n+1)$ by ordered pairs (λ, k), where it is understood that $\lambda \in \Lambda_i^{(0)}$ for some $i \leqslant k$ with $k - i$ even.

Corollary 4.9. *Point* (3) *of Theorem* 3.2 *holds for* $k = n+1$.

Proof. The cell modules of A_{n+1} are of two types: There are the cell modules $\Delta^{(\lambda,n+1)}$ with $\lambda \in \Lambda_{n+1}^{(0)}$, which are actually cell modules of $A_{n+1}/(A_n e_n A_n) \cong Q_{n+1}$. These satisfy

$$A_n e_n A_n \ \Delta^{(\lambda,n+1)} = 0.$$

On the other hand, there are the cell modules of the cellular ideal $A_n e_n A_n$, namely $\Delta^{(\lambda,n+1)} = A_n e_n \otimes_{A_{n-1}} \Delta^{(\lambda,n-1)}$, with $\lambda \in \Lambda_i^{(0)}$ for some $i < n+1$ with $n+1-i$ even. These satisfy

$$A_n e_n A_n \Delta^{(\lambda,n+1)} = A_n e_n A_n e_n \otimes_{A_{n-1}} \Delta^{(\lambda,n-1)}.$$

But

$$A_n e_n A_n e_n \otimes_R F = A_n^F A_{n-1}^F e_n = A_n^F e_n,$$

using framework axiom (6), so we have

$$A_n e_n A_n \ \Delta^{(\lambda,n+1)} \otimes_R F = \Delta^{(\lambda,n+1)} \otimes_R F,$$

by application of Lemma 2.19. □

4.2. Cell filtrations of restrictions and induced modules

Next we show that the restriction of a cell module from A_{n+1} to A_n, and the induction of a cell module from A_n to A_{n+1}, have cell filtrations.

Proposition 4.10. *Let* $(\lambda, n+1) \in \Lambda_{n+1}$*, and let* $\Delta = \Delta^{(\lambda,n+1)}$ *be the corresponding cell module of* A_{n+1}*. Then the restriction of* Δ *to* A_n *has a cell filtration.*

Proof. Write $\mathrm{Res}(\Delta)$ for the restriction to A_n.

If $A_{n+1} e_n A_{n+1} \Delta = 0$, then Δ is a Q_{n+1}-module; moreover, by framework axiom (8) from Section 3.1, $A_n e_{n-1} A_n \,\mathrm{Res}(\Delta) = 0$ as well, so $\mathrm{Res}(\Delta)$ is a Q_n-module. Then it follows from the assumption of coherence of $(Q_k)_{k \geqslant 0}$ that $\mathrm{Res}(\Delta)$ has a cell filtration as a Q_n-module, hence as an A_n-module.

If $A_{n+1} e_n A_{n+1} \Delta \neq 0$, then $\lambda \in \Lambda_i^{(0)}$ for some $i < n$, and

$$\Delta \cong A_n e_n \otimes_{A_{n-1}} \Delta^{(\lambda,n-1)}.$$

Since $A_n e_n \cong A_n$ as A_n–A_{n-1} bimodules, $\mathrm{Res}(\Delta) \cong \mathrm{Ind}_{A_{n-1}}^{A_n}(\Delta^{(\lambda,n-1)})$, which has a cell filtration by the induction assumption. □

Lemma 4.11. *Let R be an integral domain with field of fractions F. Let A be a unital R-algebra, P a right A-module, and $N_1 \subseteq N_2$ left A-modules, such that*

(1) *$A^F = A \otimes_R F$ is semisimple, and*
(2) *N_2 and $P \otimes_A N_1$ are free R-modules.*

Let $\iota : N_1 \to N_2$ denote the injection. Then

$$\mathrm{id}_P \otimes \iota : P \otimes_A N_1 \to P \otimes_A N_2$$

is injective.

Proof. First, $\iota \otimes \mathrm{id}_F : N_1 \otimes_R F \to N_2 \otimes_R F$ is injective by Lemma 2.21. Write $\beta = \mathrm{id}_P \otimes \iota$, and let

$$\beta^F = \mathrm{id}_{P^F} \otimes (\iota \otimes \mathrm{id}_F) : P^F \otimes_{A^F} N_1^F \to P^F \otimes_{A^F} N_2^F.$$

Since A^F is semisimple, P^F is projective; hence β^F is injective.

Consider the following diagram:

$$\begin{tikzcd}
P^F \otimes_{A^F} N_1^F \arrow[r, "\beta^F"] & P^F \otimes_{A^F} N_2^F \\
P \otimes_A N_1 \otimes_R F \arrow[u, "\tau_1"] \arrow[r, "\beta\otimes\mathrm{id}_F"] & P \otimes_A N_2 \otimes_R F \arrow[u, "\tau_2"'] \\
P \otimes_A N_1 \arrow[u, "\alpha_1"] \arrow[r, "\beta"] & P \otimes_A N_2 \arrow[u, "\alpha_2"']
\end{tikzcd}$$

where α_i is determined by $x \mapsto x \otimes 1_F$ and τ_i is the isomorphism of Lemma 2.19 ($i = 1, 2$). Note that α_1 is injective by Lemma 2.20, since $P \otimes_A N_1$ is assumed to be free over R. One can check that $\beta^F \circ \tau_1 \circ \alpha_1 = \tau_2 \circ \alpha_2 \circ \beta$. It follows that β is injective. □

Lemma 4.12. *Let M be an A_{n-1} module with a cell filtration*:

$$(0) = M_0 \subseteq M_1 \subseteq \cdots \subseteq M_t = M,$$

with $M_j/M_{j-1} \cong \Delta^{(\lambda_j,n-1)}$ for $1 \leqslant j \leqslant t$. Then for $1 \leqslant j \leqslant t$,

(1) *$A_n e_n \otimes_{A_{n-1}} M_j$ is a free R-module,*
(2) *$A_n e_n \otimes_{A_{n-1}} M_{j-1}$ imbeds in $A_n e_n \otimes_{A_{n-1}} M_j$, and*
(3) *$(A_n e_n \otimes_{A_{n-1}} M_j)/(A_n e_n \otimes_{A_{n-1}} M_{j-1}) \cong A_n e_n \otimes_{A_{n-1}} \Delta^{(\lambda_j,n-1)}$.*

Thus, the A_{n+1}-module $A_n e_n \otimes_{A_{n-1}} M$ has a cell filtration with subquotients $\Delta^{(\lambda_j,n+1)} = A_n e_n \otimes_{A_{n-1}} \Delta^{(\lambda_j,n-1)}$ $(1 \leqslant j \leqslant t)$.

Proof. We have $M_1 \cong \Delta^{(\lambda_1,n-1)}$, so $A_n e_n \otimes_{A_{n-1}} M_1$ is a free R-module. Fix $j \geqslant 2$ and suppose that $A_n e_n \otimes_{A_{n-1}} M_{j-1}$ is a free R-module. Let $\iota : M_{j-1} \to M_j$ denote the injection and let

$$\beta = \mathrm{id}_{A_n e_n} \otimes \iota : A_n e_n \otimes_{A_{n-1}} M_{j-1} \to A_n e_n \otimes_{A_{n-1}} M_j.$$

Then β is injective by an application of Lemma 4.11, with $A = A_{n-1}$, $P = A_n e_n$, $N_1 = M_{j-1}$, and $N_2 = M_j$. The quotient

$$(A_n e_n \otimes_{A_{n-1}} M_j,)/\beta(A_n e_n \otimes_{A_{n-1}} M_{j-1})$$

is free over R, because

$$\begin{aligned}(A_n e_n \otimes_{A_{n-1}} M_j)/\beta(A_n e_n \otimes_{A_{n-1}} M_{j-1}) &\cong A_n e_n \otimes_{A_{n-1}} (M_j/M_{j-1}) \\ &\cong A_n e_n \otimes_{A_{n-1}} \Delta^{(\lambda_j,n-1)}.\end{aligned}$$

Consequently, $A_n e_n \otimes_{A_{n-1}} M_j$ is free over R. All the assertions of the lemma now follow by induction on j. $\square$

Lemma 4.13. *Let M be an A_n-module, and let $\mathrm{Res}(M)$ denote the restriction of M to A_{n-1}. We have*

$$A_n e_n A_n \otimes_{A_n} M \cong A_n e_n \otimes_{A_{n-1}} \mathrm{Res}(M),$$

as A_{n+1} modules.

Proof. By Corollary 4.6, we have $A_n e_n A_n \cong A_n e_n \otimes_{A_{n-1}} e_n A_n \cong A_n e_n \otimes_{A_{n-1}} A_n$ as A_{n+1}–A_n bimodules. Thus

$$A_n e_n A_n \otimes_{A_n} M \cong A_n e_n \otimes_{A_{n-1}} A_n \otimes_{A_n} M \cong A_n e_n \otimes_{A_{n-1}} \mathrm{Res}(M). \qquad \square$$

Proposition 4.14. *Let $(\mu, n) \in \Lambda_n$ and let $\Delta^{(\mu,n)}$ be the corresponding cell module of A_n.*

(1) *$A_n e_n A_n \otimes_{A_n} \Delta^{(\mu,n)}$ has cell filtration (as an A_{n+1}-module). In particular, $A_n e_n A_n \otimes_{A_n} \Delta^{(\mu,n)}$ is free as an R-module.*
(2) *$A_n e_n A_n \otimes_{A_n} \Delta^{(\mu,n)}$ imbeds in $\mathrm{Ind}_{A_n}^{A_{n+1}}(\Delta^{(\mu,n)})$, and*

$$\mathrm{Ind}_{A_n}^{A_{n+1}}\big(\Delta^{(\mu,n)}\big)/\big(A_n e_n A_n \otimes_{A_n} \Delta^{(\mu,n)}\big) \cong Q_{n+1} \otimes_{A_n} \Delta^{(\mu,n)}.$$

(3) *$Q_{n+1} \otimes_{A_n} \Delta^{(\mu,n)}$ has cell filtration (as a Q_{n+1}-module, hence as an A_{n+1}-module).*
(4) *$\mathrm{Ind}_{A_n}^{A_{n+1}}(\Delta^{(\mu,n)})$ has a cell filtration.*

Proof. For point (1), let $\mathrm{Res}(\Delta^{(\mu,n)})$ denote the restriction to A_{n-1}. By Lemma 4.13, we have $A_n e_n A_n \otimes_{A_n} \Delta^{(\mu,n)} \cong A_n e_n \otimes_{A_{n-1}} \mathrm{Res}(\Delta^{(\mu,n)})$, as A_{n+1} modules. By the induction assumption stated at the beginning of Section 4, $\mathrm{Res}(\Delta^{(\mu,n)})$ has cell filtration,

$$(0) = M_0 \subseteq M_1 \subseteq \cdots \subseteq M_t = \mathrm{Res}\big(\Delta^{(\mu,n)}\big),$$

with $M_j/M_{j-1} \cong \Delta^{(\lambda_j,n-1)}$ for some $(\lambda_j, n-1) \in \Lambda_{n-1}$. By Lemma 4.12, $A_n e_n \otimes_{A_{n-1}} \mathrm{Res}(\Delta^{(\mu,n)})$ has a cell filtration with subquotients $\Delta^{(\lambda_j,n+1)} = A_n e_n \otimes_{A_{n-1}} \Delta^{(\lambda_j,n-1)}$.

Point (2) follows from Lemma 4.11 (with left and right modules interchanged), taking $A = A_n$, $P = \Delta^{(\mu,n)}$, $N_1 = A_n e_n A_n$, and $N_2 = A_{n+1}$. Note that A_{n+1} is a free R-module by Proposition 4.7, and $A_n e_n A_n \otimes_{A_n} \Delta^{(\mu,n)}$ is a free R-module by point (1). The statement regarding the quotient follows from the right exactness of tensor products.

For $n = 1$, $A_1 = Q_1$, and $\Delta^{(\mu,n)}$ is a Q_1-cell module; statement (3) follows from the assumption of coherence of $(Q_k)_{k\geqslant 0}$. If $n \geqslant 2$, then by the induction assumption, either $A_n e_{n-1} A_n\ \Delta^{(\mu,n)} = \Delta^{(\mu,n)}$, or $A_n e_{n-1} A_n\ \Delta^{(\mu,n)} = (0)$. In the former case,

$$Q_{n+1} \otimes_{A_n} \Delta^{(\mu,n)} = Q_{n+1} \otimes_{A_n} A_n e_{n-1} A_n \, \Delta^{(\mu,n)}$$
$$= Q_{n+1} A_n e_{n-1} A_n \otimes_{A_n} \Delta^{(\mu,n)} = 0,$$

because $e_{n-1} \in A_{n+1} e_n A_{n+1}$, by the framework axiom (8). In the latter case, $A_n e_{n-1} A_n$ annihilates both Q_{n+1} and $\Delta^{(\mu,n)}$, so both are $A_n/(A_n e_{n-1} A_n) \cong Q_n$-modules. Thus $Q_{n+1} \otimes_{A_n} \Delta^{(\mu,n)} = Q_{n+1} \otimes_{Q_n} \Delta^{(\mu,n)}$, which has a Q_{n+1}-cell filtration by the assumption of coherence of $(Q_k)_{k \geqslant 0}$. This proves point (3).

Finally, we have an exact sequence

$$0 \to A_n e_n A_n \otimes_{A_n} \Delta^{(\mu,n)} \to \operatorname{Ind}_{A_n}^{A_{n+1}}\big(\Delta^{(\mu,n)}\big) \to Q_{n+1} \otimes_{A_n} \Delta^{(\mu,n)} \to 0,$$

where both $A_n e_n A_n \otimes_{A_n} \Delta^{(\mu,n)}$ and $Q_{n+1} \otimes_{A_n} \Delta^{(\mu,n)}$ have A_{n+1}-cell filtrations. Hence $\operatorname{Ind}_{A_n}^{A_{n+1}}(\Delta^{(\mu,n)})$ has an A_{n+1}-cell filtration. □

Corollary 4.15. *The finite tower $(A_k)_{0 \leqslant k \leqslant n+1}$ is a coherent tower of cellular algebras.*

Proof. Combine the induction hypothesis, Proposition 4.7, Proposition 4.10, and Proposition 4.14. □

Corollary 4.16. *The branching diagram for the finite tower $(A_k^F)_{0 \leqslant k \leqslant n+1}$ is that obtained by reflections from the branching diagram of the finite tower $(Q_k^F)_{0 \leqslant k \leqslant n+1}$.*

Proof. From the induction hypothesis, we already know that the branching diagram for $(A_k^F)_{0 \leqslant k \leqslant n}$ is obtained by reflections from the branching diagram of the finite tower $(Q_k^F)_{0 \leqslant k \leqslant n}$. So we have only to consider the branching diagram for $A_{n-1}^F \subseteq A_n^F \subseteq A_{n+1}^F$; specifically, we need to show that if $\lambda \in \Lambda_i^{(0)}$ with $i < n+1$ and $n+1-i$ even, and $(\mu, n) \in \Lambda_n$ is arbitrary, then

$$(\mu, n) \nearrow (\lambda, n+1) \quad \text{if and only if} \quad (\lambda, n-1) \nearrow (\mu, n),$$

in the branching diagram for $A_{n-1}^F \subseteq A_n^F \subseteq A_{n+1}^F$, and the number of edges connecting (μ, n) and $(\lambda, n+1)$ is the same as the number of edges connecting $(\lambda, n-1)$ and (μ, n). But this follows from Lemma 2.22 and the proof of either Proposition 4.10, or Proposition 4.14, point (1). □

Conclusion of the proof of Theorem 3.2. Under the assumption that statements (1)–(4) of the theorem are valid for the finite tower $(A_k)_{0 \leqslant k \leqslant n}$, for some fixed n, we had to show that they are also valid for the tower $(A_k)_{0 \leqslant k \leqslant n+1}$. This was verified in Corollary 4.15, Corollary 4.8, Corollary 4.9, and Corollary 4.16. □

5. Examples

5.1. Preliminaries on tangle diagrams

Several of our examples involve *tangle diagrams* in the rectangle $\mathscr{R} = [0,1] \times [0,1]$. Fix points $a_i \in [0,1]$, $i \geqslant 1$, with $0 < a_1 < a_2 < \cdots$. Write $\boldsymbol{i} = (a_i, 1)$ and $\bar{\boldsymbol{i}} = (a_i, 0)$.

Recall that a *knot diagram* means a collection of piecewise smooth closed curves in the plane which may have intersections and self-intersections, but only simple transverse intersections. At each intersection or crossing, one of the two strands (curves) which intersect is indicated as crossing over the other.

An *(n,n)-tangle diagram* is a piece of a knot diagram in $\mathscr{R}$ consisting of exactly n topological intervals and possibly some number of closed curves, such that: (1) the endpoints of the intervals are the points $\boldsymbol{1}, \ldots, \boldsymbol{n}, \bar{\boldsymbol{1}}, \ldots, \bar{\boldsymbol{n}}$, and these are the only points of intersection of the family of curves with the boundary of the rectangle, and (2) each interval intersects the boundary of the rectangle transversally.

An (n, n)-*Brauer diagram* is a "tangle" diagram containing no closed curves, in which information about over and under crossings is ignored. Two Brauer diagrams are identified if the pairs of boundary points joined by curves is the same in the two diagrams. By convention, there is a unique $(0, 0)$-Brauer diagram, the empty diagram with no curves. For $n \geqslant 1$, the number of (n, n)-Brauer diagrams is $(2n-1)!! = (2n-1)(2n-3)\cdots(3)(1)$.

A *Temperley–Lieb* diagram is a Brauer diagram without crossings. For $n \geqslant 0$, the number of (n, n)-Temperley–Lieb diagrams is the Catalan number $\frac{1}{n+1}\binom{2n}{n}$.

For any of these types of diagrams, we call $P = \{\mathbf{1}, \dots, \boldsymbol{n}, \bar{\mathbf{1}}, \dots, \bar{\boldsymbol{n}}\}$ the set of *vertices* of the diagram, $P^+ = \{\mathbf{1}, \dots, \boldsymbol{n}\}$ the set of *top vertices*, and $P^- = \{\bar{\mathbf{1}}, \dots, \bar{\boldsymbol{n}}\}$ the set of *bottom vertices*. A curve or *strand* in the diagram is called a *vertical* or *through* strand if it connects a top vertex and a bottom vertex, and a *horizontal* strand if it connects two top vertices or two bottom vertices.

5.2. *The Brauer algebras*

5.2.1. *Definition of the Brauer algebras*

Let S be a commutative ring with identity, with a distinguished element δ. The Brauer algebra $B_n(S, \delta)$ is the free S-module with basis the set of (n, n)-Brauer diagrams, and with multiplication defined as follows. The product of two Brauer diagrams is defined to be a certain multiple of another Brauer diagram. Namely, given two Brauer diagrams a, b, first "stack" b over a; the result is a planar tangle that may contain some number of closed curves. Let r denote the number of closed curves, and let c be the Brauer diagram obtained by removing all the closed curves. Then $ab = \delta^r c$.

Definition 5.1. For $n \geqslant 1$, the *Brauer algebra* $B_n(S, \delta)$ over S with parameter δ is the free S-module with basis the set of (n, n)-Brauer diagrams, with the bilinear product determined by the multiplication of Brauer diagrams. In particular, $B_0(S, \delta) = S$.

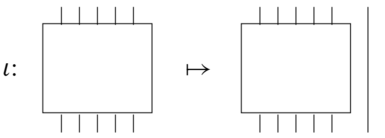
Note that the Brauer diagrams with only vertical strands are in bijection with permutations of $\{1, \dots, n\}$, and that the multiplication of two such diagrams coincides with the multiplication of permutations. Thus the Brauer algebra contains the group algebra $S\mathfrak{S}_n$ of the permutation group $\mathfrak{S}_n$. The identity element of the Brauer algebra is the diagram corresponding to the trivial permutation.

5.2.2. *Brief history of the Brauer algebras*

The Brauer algebras were introduced by Brauer [7] as a device for studying the invariant theory of orthogonal and symplectic groups. Wenzl [55] observed that generically, the sequence of Brauer algebras (over a field) is obtained by repeated Jones basic constructions from the symmetric group algebras; he used this to show that $B_n(k, \delta)$ is semisimple, when k is a field of characteristic zero and δ is not an integer. Graham and Lehrer [23] showed that the Brauer algebras are cellular, and classified the simple modules of $B_n(k, \delta)$ when k is a field and δ is arbitrary. Another illuminating proof of cellularity of the Brauer algebras was given by König and Xi [40]. Enyang's two proofs of cellularity for Birman–Wenzl algebras [14,15] also apply to the Brauer algebras.

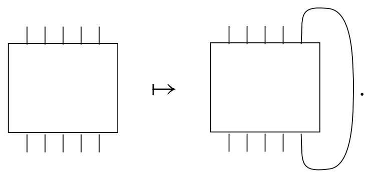
5.2.3. *Some properties of the Brauer algebras*

In this section, write B_n for $B_n(S, \delta)$. For $n \geqslant 1$, let ι denote the map from (n, n)-Brauer diagrams to $(n+1, n+1)$-Brauer diagrams that adds an additional strand to a diagram, connecting $\boldsymbol{n+1}$ to $\overline{\boldsymbol{n+1}}$.



The linear extension of ι to B_n is an injective unital homomorphism into B_{n+1}. Using ι, we identify B_n with its image in B_{n+1}.

For $n \geqslant 1$ define a map cl from (n,n)-Brauer diagrams into B_{n-1} as follows. First "partially close" a given (n,n)-Brauer diagram by adding an additional smooth curve connecting $\boldsymbol{n}$ to $\bar{\boldsymbol{n}}$,



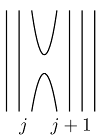
In case the resulting "tangle" contains a closed curve (which happens precisely when the original diagram already had a strand connecting $\boldsymbol{n}$ to $\bar{\boldsymbol{n}}$), remove this loop and replace it with a factor of δ. The linear extension of cl to B_n is a (non-unital) B_{n-1}–B_{n-1} bimodule map, and $\mathrm{cl} \circ \iota(x) = \delta\, x$ for $x \in B_n$.

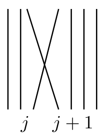
If δ is invertible in S, we can define $\varepsilon_n = (1/\delta)\,\mathrm{cl}$, which is a conditional expectation, that is, a unital B_{n-1}–B_{n-1} bimodule map. We have $\varepsilon_{n+1} \circ \iota(x) = x$ for $x \in B_n$. The map $\varepsilon = \varepsilon_1 \circ \cdots \circ \varepsilon_n : B_n \to B_0 \cong S$ is a normalized trace; that is, $\varepsilon(\mathbf{1}) = 1$ and $\varepsilon(ab) = \varepsilon(ba)$ for all a, b. The value of ε on a Brauer diagram d is obtained as follows: first close all the strands of d by introducing new curves joining $\boldsymbol{j}$ to $\bar{\boldsymbol{j}}$ for all j; let c be the number of components (closed loops) in the resulting $(0,0)$-tangle; then $\varepsilon(d) = \delta^{c-n}$ if $d \in B_n$. The trace and condition expectation play an essential role in Wenzl's treatment of the structure of the Brauer algebra over $\mathbb{Q}(\boldsymbol{\delta})$ [55], and thus implicitly in our verification of the framework axioms in Proposition 5.4.

The involution i on (n,n)-Brauer diagrams which reflects a diagram in the axis $y = 1/2$ extends linearly to an algebra involution of B_n. We have $\iota \circ i = i \circ \iota$ and $\mathrm{cl} \circ i = i \circ \mathrm{cl}$.

The products ab and ba of two Brauer diagrams have at most as many through strands as a. Consequently, the span of diagrams with at most r through strands ($r \leqslant n$ and $n - r$ even) is a two-sided ideal J_r in B_n. J_r is i-invariant.

Let e_j and s_j denote the (n,n)-Brauer diagrams:

$e_j =$ 

, $s_j =$ 

.

Note that $e_j^2 = \delta e_j$, so e_j is an essential idempotent if $\delta \neq 0$, and nilpotent if $\delta = 0$. We have $i(e_j) = e_j$ and $i(s_j) = s_j$. It is easy to see that $e_1, \ldots, e_{n-1}$ and $s_1, \ldots, s_{n-1}$ generate B_n as an algebra.

Let $r \leqslant n$ with $n - r$ even, and let $f_r = e_{r+1} e_{r+3} \cdots e_{n-1}$. Any Brauer diagram with exactly r through strands can be factored as $\pi_1 f_r \pi_2$, where π_i are permutation diagrams. Consequently, J_r is generated by f_r. In particular the ideal $J = J_{n-2}$ spanned by diagrams with fewer than n through strands is generated by e_{n-1}. We have $B_n / J \cong S\mathfrak{S}_n$, as algebras with involutions.

Lemma 5.2. *Write B_n for $B_n(S, \delta)$.*

(1) *For $n \geqslant 2$, $e_n B_n e_n = B_{n-1} e_n$.*
(2) $e_1 B_1 e_1 = \delta B_0 e_1$.
(3) *For $n \geqslant 2$, e_n commutes with B_{n-1}.*

Proof. For $n \geqslant 2$, if x is an (n,n)-Brauer diagram, then $e_n x e_n \in B_{n-1} e_n$. Thus, $e_n B_n e_n \subseteq B_{n-1} e_n$. On the other hand, for $x \in B_{n-1}$, we have $e_n x e_{n-1} e_n = x e_n$. Hence, $e_n B_n e_n \supseteq B_{n-1} e_n$. This proves (1). Points (2) and (3) are obvious. $\square$

Lemma 5.3. *Write B_n for $B_n(S,\delta)$. For $n \geqslant 1$, $B_{n+1}e_n = B_n e_n$. Moreover, $x \mapsto xe_n$ is injective from B_n to B_{n+1}.*

Proof. By [55], Proposition 2.1, any $(n+1,n+1)$-Brauer diagram is either already in B_n, or can be written in the form $a\chi_n b$, with $a, b \in B_n$ and $\chi_n \in \{e_n, s_n\}$. Applying this again to b, either $b \in B_{n-1}$, or b can be factored as $b_1\chi_{n-1}b_2$, with $b_i \in B_{n-1}$ and $\chi_{n-1} \in \{e_{n-1}, s_{n-1}\}$. Since $e_n^2 = \delta e_n$ and $s_n e_n = e_n$, it follows that if $b \in B_{n-1}$, then $a\chi_n b e_n = ab\chi_n e_n \in B_n e_n$. If $b = b_1\chi_{n-1}b_2$, then $a\chi_n b e_n = ab_1\chi_n\chi_{n-1}e_n b_2$. Now we can apply the following identities: $e_n\chi_{n-1}e_n = e_n$ for $\chi_{n-1} \in \{e_{n-1}, s_{n-1}\}$, $s_n e_{n-1} e_n = s_{n-1}e_n$, and $s_n s_{n-1} e_n = e_{n-1}e_n$ to conclude that $a\chi_n b e_n \in B_n e_n$. This shows that $B_{n+1}e_n = B_n e_n$.

For $x \in B_n$, we have $\mathrm{cl}(xe_n) = x$, so the map $x \mapsto xe_n$ is injective from B_n to $B_n e_n$. □

5.2.4. Verification of framework axioms for the Brauer algebras

We take $R = \mathbb{Z}[\boldsymbol{\delta}]$, where $\boldsymbol{\delta}$ is an indeterminant. Then R is the universal ground ring for the Brauer algebras; for any commutative ring S with distinguished element δ, we have $B_n(S,\delta) \cong B_n(R,\boldsymbol{\delta}) \otimes_R S$. Let $F = \mathbb{Q}(\boldsymbol{\delta})$ denote the field of fractions of R. Write $B_n = B_n(R,\boldsymbol{\delta})$.

Proposition 5.4. *The two sequence of R-algebras $(B_n)_{n\geqslant 0}$ and $(R\mathfrak{S}_n)_{n\geqslant 0}$ satisfy the framework axioms of Section* 3.1.

Proof. According to Example 2.16, $(R\mathfrak{S}_n)_{n\geqslant 0}$ is a coherent tower of cellular algebras, so axiom (1) holds. Framework axioms (2) and (3) are evident. B_n^F is split semisimple by [55], Theorem 3.2, so axiom (4) holds.

We take $e_{n-1} \in B_n$ to be the element defined in the previous section. Let us verify the axioms (5)–(8) involving e_{n-1}. As observed above, e_{n-1} is i-invariant, $J = B_n e_{n-1} B_n$ is the ideal spanned by diagrams with fewer than n through strands, and $B_n/J \cong R\mathfrak{S}_n$ as algebras with involution. This verifies axiom (5). Axiom (6) follows from Lemma 5.2 and axiom (7) from Lemma 5.3. Axiom (8) holds because $e_{n-1}e_n e_{n-1} = e_{n-1}$. □

Corollary 5.5. *For any commutative ring S and for any $\delta \in S$, the sequence of Brauer algebras $(B_n(S,\delta))_{n\geqslant 0}$ is a coherent tower of cellular algebras. $B_n(S,\delta)$ has cell modules indexed by all Young diagrams of size n, $n-2$, $n-4, \ldots$. The cell module labeled by a Young diagram λ has a basis labeled by up-down tableaux of length n and shape λ.*

5.3. The Jones–Temperley–Lieb algebras

5.3.1. Definition of the Jones–Temperley–Lieb algebras

Let S be a commutative ring with identity, with distinguished element δ. The Jones–Temperley–Lieb algebra $T_n(S,\delta)$ is the unital S-algebra with generators $e_1, \ldots, e_{n-1}$ satisfying the relation:

(1) $e_j^2 = \delta e_j$,
(2) $e_j e_{j\pm 1} e_j = e_j$,
(3) $e_j e_k = e_k e_j$, if $|j-k| \geqslant 2$,

whenever all indices involved are in the range from 1 to $n-1$.

5.3.2. Diagramatic realization of the Jones–Temperley–Lieb algebras

The S-span $\widetilde{T}_n(S,\delta)$ of Temperley–Lieb diagrams is a subalgebra of the Brauer algebra. We have an algebra map φ from $T_n(S,\delta)$ to $\widetilde{T}_n(S,\delta)$, determined by $e_j \mapsto e_j$ for $1 \leqslant j \leqslant n-1$. Kauffman shows [36, Theorem 4.3] that the map is an isomorphism. In fact, to show that φ is surjective, it suffices to show that any Temperley–Lieb diagram can be written as a product of e_j's. Kauffman indicates by example how this is to be done, and it is not difficult to invent a measure of complexity of Temperley–Lieb diagrams and to show this formally, by induction on complexity. For injectivity, Jones shows [32, p. 14] that $T_n(S,\delta)$ is spanned by a family $\mathbb{B}$ of $\frac{1}{n+1}\binom{2n}{n}$ reduced words in the e_j's. Since

φ is surjective and $\widetilde{T}_n(S,\delta)$ is a free S-module of rank $\frac{1}{n+1}\binom{2n}{n}$, it follows easily that $\mathbb{B}$ is a basis and φ is an isomorphism. Because of this, we will no longer distinguish between $T_n(S,\delta)$ and $\widetilde{T}_n(S,\delta)$.

5.3.3. *Brief history of the Jones–Temperley–Lieb algebras*

The Jones–Temperley–Lieb algebras were introduced by Jones in his study of subfactors [32] and then employed by him to define the Jones link invariant [33]. The name derives from the appearance of specific representations of the algebras in statistical mechanics that had been found some years earlier. By now, there is a huge literature related to these algebras because of their multiple roles in subfactor theory, invariants of links and 3-manifolds, statistical mechanics and quantum field theory. The Jones–Temperley–Lieb algebras were shown to be cellular in [23]. Several other proofs of cellularity are known, for example [57,24].

5.3.4. *Some properties of the Jones–Temperley–Lieb algebra*

The Brauer algebra maps ι, cl, ε_n (when δ is invertible), and i restrict to maps of the Jones–Temperley–Lieb algebras having similar properties. For example, i is an algebra involution on each $T_n(S,\delta)$ and $i \circ \iota = \iota \circ i$.

The span of Temperley–Lieb diagrams having at least one horizontal strand is an ideal J in $T_n(S,\delta)$, and $T_n(S,\delta)/J \cong S$. The proof of surjectivity of φ sketched above shows that any Temperley–Lieb diagram with at least one horizontal edge can be written as a non-trivial product of e_j's; so J is equal to the ideal generated by all of the e_j's. However, the identities $e_j e_{j+1} e_j = e_j$ imply that J is the ideal generated by e_{n-1}.

5.3.5. *Verification of the framework axioms for the Jones–Temperley–Lieb algebras*

We take $R = \mathbb{Z}[\boldsymbol{\delta}]$, where $\boldsymbol{\delta}$ is an indeterminant. Then R is the universal ground ring for the Jones–Temperley–Lieb algebras; for any integral domain S with distinguished element δ, we have $T_n(S,\delta) \cong T_n(R,\boldsymbol{\delta}) \otimes_R S$. Let $F = \mathbb{Q}(\boldsymbol{\delta})$ denote the field of fractions of R. Write $T_n = T_n(R,\boldsymbol{\delta})$.

Proposition 5.6. *The two sequences of R-algebras $(T_n)_{n\geqslant 0}$ and $(R)_{n\geqslant 0}$ satisfy the framework axioms of Section* 3.1.

Proof. Axioms (1), (2), and (3) are obvious. For semisimplicity of T_n^F, see [18], Theorem 2.8.5. This gives axiom (4). We checked axiom (5) in the previous section. The proof for axiom (6) is the same as for the Brauer algebras.

According to [32], Lemma 4.1.2, any $(n+1,n+1)$-Temperley–Lieb diagram is either already in T_n, or can be written in the form ae_nb, with $a, b \in T_n$. Given this, the verification of axiom (7) is the same as for the Brauer algebras; we have to use only the identity $e_n e_{n-1} e_n = e_n$ in place of several similar identities for the Brauer algebras.

As for the Brauer algebras, axiom (8) follows from the identity $e_{n-1} e_n e_{n-1} = e_{n-1}$. $\square$

Corollary 5.7. *For any ring S and $\delta \in S$, the sequence of Jones–Temperley–Lieb algebras $(T_n(S,\delta))_{n\geqslant 0}$ is a coherent tower of cellular algebras. The cell modules of $T_n(S,\delta)$ can be labeled by Young diagrams with one or two rows and size n, and the basis of the cell module labeled by λ by standard tableaux of shape λ.*

Proof. We only have to remark that the vertices on the n-th row of the branching diagram for $(T_k^F)_{k\geqslant 0}$ (see [18], Lemma 2.8.4) can be labeled by Young diagrams of size n with no more than 2 rows, and the paths on the branching diagram by standard tableaux. (Alternatively, the vertices on the n-th row of the branching diagram can be labeled by Young diagrams with one row and size n, $n-2$, $n-4, \ldots,$ and the paths on the branching diagram by up-down tableaux.) $\square$

5.4. The Birman–Wenzl–Murakami (BMW) algebras

5.4.1. Definition of the BMW algebras

The BMW algebras were first introduced by Birman and Wenzl [6] and independently by Murakami [49] as abstract algebras defined by generators and relations. The version of the presentation given here follows [48].

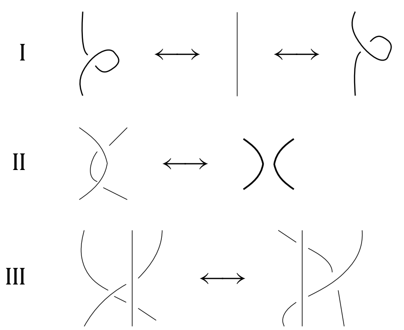
Definition 5.8. Let S be a commutative unital ring with invertible elements ρ and q and an element δ satisfying $\rho^{-1} - \rho = (q^{-1} - q)(\delta - 1)$. The *Birman–Wenzl–Murakami algebra* $W_n(S; \rho, q, \delta)$ is the unital S-algebra with generators $g_i^{\pm 1}$ and e_i ($1 \leqslant i \leqslant n-1$) and relations:

(1) (Inverses) $g_i g_i^{-1} = g_i^{-1} g_i = 1$.
(2) (Essential idempotent relation) $e_i^2 = \delta e_i$.
(3) (Braid relations) $g_i g_{i+1} g_i = g_{i+1} g_i g_{i+1}$ and $g_i g_j = g_j g_i$ if $|i-j| \geqslant 2$.
(4) (Commutation relations) $g_i e_j = e_j g_i$ and $e_i e_j = e_j e_i$ if $|i-j| \geqslant 2$.
(5) (Tangle relations) $e_i e_{i\pm1} e_i = e_i$, $g_i g_{i\pm1} e_i = e_{i\pm1} e_i$, and $e_i g_{i\pm1} g_i = e_i e_{i\pm1}$.
(6) (Kauffman skein relation) $g_i - g_i^{-1} = (q - q^{-1})(1 - e_i)$.
(7) (Untwisting relations) $g_i e_i = e_i g_i = \rho^{-1} e_i$, and $e_i g_{i\pm1} e_i = \rho e_i$.

5.4.2. Geometric realization of the BMW algebras

A geometric realization of the BMW algebra is as the algebra of framed (n, n)-tangles in the disc cross the interval, modulo certain skein relations. It is more convenient, at least for our purposes, to describe this geometric version in terms of tangle diagrams.

First, tangle diagrams can be multiplied by stacking, as for Brauer or Temperley–Lieb diagrams (but closed loops are allowed, and there is no reduction by removing closed loops after stacking). Recall that our convention is that the product ab of tangle diagrams is given by stacking b over a. This makes (n, n)-tangle diagrams into a monoid, the identity being the tangle diagram in which each top vertex $\boldsymbol{j}$ is connected to the bottom vertex $\bar{\boldsymbol{j}}$ by a vertical line segment, when $n \geqslant 1$. (The identity for the monoid of $(0, 0)$-tangle diagrams is the empty tangle.)

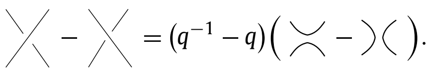


Reidemeister moves

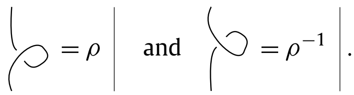
Two tangle diagrams are said to be *regularly isotopic* if they are related by a sequence of Reidemeister moves of types II and III, followed by an isotopy of $\mathscr{R}$ fixing the boundary. (Reidemeister moves of type I are not allowed.) See the figure above for the Reidemeister moves.

Stacking of tangle diagrams respects regular isotopy; thus one obtains a monoid structure on the regular isotopy classes of (n, n)-tangle diagrams. Let us denote this monoid by $\mathscr{U}_n$. Let S be a ring with elements ρ, q and δ as in the definition of the BMW algebras. The *Kauffman tangle algebra* $\mathrm{KT}_n(S; \rho, q, \delta)$ is the monoid algebra $S\,\mathscr{U}_n$ modulo the following skein relations:

(1) Crossing relation:

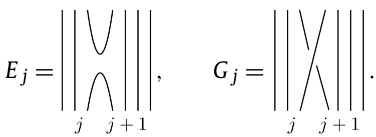


(2) Untwisting relation:



(3) Free loop relation: $T \cup \bigcirc = \delta T$, where $T \cup \bigcirc$ means the union of a tangle diagram T and a closed loop having no crossings with T.

Let E_j and G_j denote the following (n,n)-tangle diagrams:



Morton and Wassermann [48] showed that the assignments $e_j \mapsto E_j$ and $g_j \mapsto G_j$ determine an isomorphism from $W_n(S; \rho, q, \delta)$ to $\mathrm{KT}_n(S; \rho, q, \delta)$. Given this, we will no longer distinguish between the BMW algebras and the Kauffman tangle algebras. (However, we remark that it is possible to use our techniques to recover the theorem of Morton and Wasserman, using only results in the original paper of Birman and Wenzl; we prove the analogous isomorphism theorem for the cyclotomic BMW algebras in Section 5.5, and the result for the ordinary BMW algebras is a special case.)

5.4.3. *Brief history of the BMW algebras*

The origin of the BMW algebras was in knot theory. Kauffman defined [36] an invariant of regular isotopy for links in S^3, determined by skein relations. Birman and Wenzl [6] and Murakami [49] then defined the BMW algebras in order to give an algebraic setting for the Kauffman invariant. The BMW algebras were implicitly modeled on algebras of tangles. The definition of the Kauffman tangle algebra was made explicit by Morton and Traczyk [47], who also showed that $\mathrm{KT}_n(S; \rho, q, \delta)$ is free as an S-module of rank $(2n-1)!!$. Morton and Wassermann [48] showed that the BMW algebras and Kauffman tangle algebras are isomorphic.

Xi showed [62] that the tangle basis of Morton and Traczyk is a cellular basis. Enyang has exhibited two cellular bases of BMW algebras; the first [14] is a tangle type basis, and the second [15] is a basis indexed by up-down tableaux, which demonstrates the coherence of the cellular structures on $(W_n)_{n \geqslant 0}$.

5.4.4. *Some properties of the BMW algebras*

In the following, we write W_n for $W_n(S; \rho, q, \delta)$.

The BMW algebras have an algebra involution i uniquely determined by $i(e_j) = e_j$ and $i(g_j) = g_j$ for all j. The action of i on tangle diagrams is by the rotation through the axis $y = 1/2$. (It is by rotation rather than reflection, since the reflection would take $g_j \mapsto g_j^{-1}$.)

For $n \geqslant 0$, there is a unique homomorphism ι from W_n to W_{n+1} determined by $e_i \mapsto e_i$ and $g_i \mapsto g_i$ for $1 \leqslant i \leqslant n-1$. On the level of tangle diagrams, the map is given by adding a new vertical strand connecting $\boldsymbol{n+1}$ and $\overline{\boldsymbol{n+1}}$, as for the Brauer algebras.

For $n \geqslant 1$, a map cl from (n,n)-tangle diagrams to $(n-1,n-1)$-tangle diagrams can be defined as for Brauer diagrams. The linear extension of this map respects regular isotopy and the Kauffman skein relations, so determines a linear map from W_n to W_{n-1}. We have $i \circ \mathrm{cl} = \mathrm{cl} \circ i$ and $\mathrm{cl} \circ \iota = \delta x$. Moreover, for $x \in W_n$, we have $x = \mathrm{cl}(\iota(x)e_n)$, so it follows that $\iota : W_n \to W_{n+1}$ is injective. The involution i and inclusion ι satisfy $i \circ \iota = \iota \circ i$. Using ι, we identify W_n as a subalgebra of W_{n+1}.

If δ is invertible in S, we can define $\varepsilon_n = (1/\delta)\,\mathrm{cl}$, which is a conditional expectation, that is, an unital W_{n-1}–W_{n-1} bimodule map. We have $\varepsilon_{n+1} \circ \iota(x) = x$ for $x \in W_n$.

The ideal J in W_n generated by e_{n-1} contains e_j for all j because of the relations $e_j e_{j+1} e_j = e_j$. It follows from the BMW relations that W_n/J is isomorphic to the Hecke algebra $H_n(S; q^2)$ with the quadratic relation $g_j - g_j^{-1} = q - q^{-1}$, or $(g_j - q)(g_j + q^{-1}) = 0$.

Lemma 5.9.

(1) *For* $n \geqslant 2$, $e_n W_n e_n = W_{n-1} e_n$.
(2) $e_1 W_1 e_1 = \delta W_0 e_1$.
(3) *For* $n \geqslant 1$, e_n *commutes with* W_{n-1}.

Proof. The proof is the same as that of Lemma 5.2 for the Brauer algebras, using the tangle realization of the BMW algebras. □

Lemma 5.10. *For* $n \geqslant 1$, $W_{n+1} e_n = W_n e_n$. *Moreover,* $x \mapsto x e_n$ *is injective from* W_n *to* $W_n e_n$.

Proof. According to [6], Lemma 3.1, any $(n+1, n+1)$-tangle is already in W_n, or it can be written as a linear combination of elements $a \chi_n b$, with $a, b \in W_n$ and $\chi_n \in \{e_n, g_n\}$. Given this, the proof of the lemma is the same as the proof of Lemma 5.3 for the Brauer algebras, using the tangle relations and untwisting relations of Definition 5.8 in place of similar identities for the Brauer algebras. □

5.4.5. Verification of the framework axioms for the BMW algebras

The generic or universal ground ring for the BMW algebras is

$$R = \mathbb{Z}\big[\boldsymbol{\rho}^{\pm 1}, \boldsymbol{q}^{\pm 1}, \boldsymbol{\delta}\big] / \big\langle \boldsymbol{\rho}^{-1} - \boldsymbol{\rho} = \big(\boldsymbol{q}^{-1} - \boldsymbol{q}\big)(\boldsymbol{\delta} - 1)\big\rangle,$$

where $\boldsymbol{\rho}$, $\boldsymbol{q}$, and $\boldsymbol{\delta}$ are indeterminants over $\mathbb{Z}$. Suppose that S is an appropriate ground ring for the BMW algebras; that is, S is a commutative unital ring with invertible elements ρ and q and an element δ satisfying $\rho^{-1} - \rho = (q^{-1} - q)(\delta - 1)$. Then $W_n(S; \rho, q, \delta) \cong W_n(R; \boldsymbol{\rho}, \boldsymbol{q}, \boldsymbol{\delta}) \otimes_R S$.

R is an integral domain whose field of fractions is $F \cong \mathbb{Q}(\boldsymbol{\rho}, \boldsymbol{q})$ (with $\boldsymbol{\delta} = (\boldsymbol{\rho}^{-1} - \boldsymbol{\rho})/(\boldsymbol{q}^{-1} - \boldsymbol{q}) + 1$ in F.) We write W_n for $W_n(R; \boldsymbol{\rho}, \boldsymbol{q}, \boldsymbol{\delta})$ and H_n for $H_n(R; \boldsymbol{q}^2)$ in this section.

Proposition 5.11. *The two sequences of algebras* $(W_n)_{n \geqslant 0}$ *and* $(H_n)_{n \geqslant 0}$ *satisfy the framework axioms of Section* 3.1.

Proof. According to Example 2.16, $(H_n)_{n \geqslant 0}$ is a coherent tower of cellular algebras, so axiom (1) holds. Axioms (2) and (3) are evident. W_n^F is semisimple by [6], Theorem 3.7, or [56], Theorem 3.5. Thus axiom (4) holds.

We observed above that $W_n / W_n e_{n-1} W_n \cong H_n$; it is easy to check that the isomorphism respects the involutions. Thus axiom (5) holds. Axiom (6) follows from Lemma 5.9 and axiom (7) from Lemma 5.10. Finally, axiom (8) holds again because of the relation $e_{n-1} e_n e_{n-1} = e_{n-1}$. □

Corollary 5.12. *Let* S *be any ground ring for the BMW algebras, with parameters* ρ, q, *and* δ. *The sequence of BMW algebras* $(W_n(S; \rho, q, \delta))_{n \geqslant 0}$ *is a coherent tower of cellular algebras.* $W_n(S; \rho, q, \delta)$ *has cell modules indexed by all Young diagrams of size* $n, n-2, n-4, \ldots$. *The cell module labeled by a Young diagram* λ *has a basis labeled by up-down tableaux of length* n *and shape* λ.

5.5. The cyclotomic Birman–Wenzl–Murakami (BMW) algebras

5.5.1. Definition of the cyclotomic BMW algebras

In general, our notation will follow [22]. In order to simplify statements, we establish the following convention.

Definition 5.13. Fix an integer $r \geqslant 1$. A *ground ring* S is a commutative unital ring with parameters ρ, q, δ_j ($j \geqslant 0$), and $u_1, \ldots, u_r$, with ρ, q, and $u_1, \ldots, u_r$ invertible, and with $\rho^{-1} - \rho = (q^{-1} - q)(\delta_0 - 1)$.

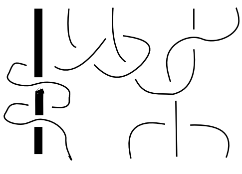
Definition 5.14. Let S be a ground ring with parameters ρ, q, δ_j ($j \geqslant 0$), and $u_1, \ldots, u_r$. The *cyclotomic BMW algebra* $W_{n,S,r}(u_1, \ldots, u_r)$ is the unital S-algebra with generators $y_1^{\pm 1}$, $g_i^{\pm 1}$ and e_i ($1 \leqslant i \leqslant n-1$) and relations:

(1) (Inverses) $g_i g_i^{-1} = g_i^{-1} g_i = 1$ and $y_1 y_1^{-1} = y_1^{-1} y_1 = 1$.
(2) (Idempotent relation) $e_i^2 = \delta_0 e_i$.
(3) (Affine braid relations)
 (a) $g_i g_{i+1} g_i = g_{i+1} g_i g_{i+1}$ and $g_i g_j = g_j g_i$ if $|i - j| \geqslant 2$,
 (b) $y_1 g_1 y_1 g_1 = g_1 y_1 g_1 y_1$ and $y_1 g_j = g_j y_1$ if $j \geqslant 2$.
(4) (Commutation relations)
 (a) $g_i e_j = e_j g_i$ and $e_i e_j = e_j e_i$ if $|i - j| \geqslant 2$,
 (b) $y_1 e_j = e_j y_1$ if $j \geqslant 2$.
(5) (Affine tangle relations)
 (a) $e_i e_{i\pm 1} e_i = e_i$,
 (b) $g_i g_{i\pm 1} e_i = e_{i\pm 1} e_i$ and $e_i g_{i\pm 1} g_i = e_i e_{i\pm 1}$,
 (c) for $j \geqslant 1$, $e_1 y_1^j e_1 = \delta_j e_1$.
(6) (Kauffman skein relation) $g_i - g_i^{-1} = (q - q^{-1})(1 - e_i)$.
(7) (Untwisting relations) $g_i e_i = e_i g_i = \rho^{-1} e_i$ and $e_i g_{i\pm 1} e_i = \rho e_i$.
(8) (Unwrapping relation) $e_1 y_1 g_1 y_1 = \rho e_1 = y_1 g_1 y_1 e_1$.
(9) (Cyclotomic relation) $(y_1 - u_1)(y_1 - u_2) \cdots (y_1 - u_r) = 0$.

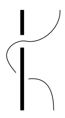
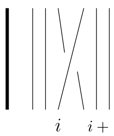
Thus, a cyclotomic BMW algebra is the quotient of the affine BMW algebra [20], by the cyclotomic relation $(y_1 - u_1)(y_1 - u_2) \cdots (y_1 - u_r) = 0$.

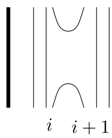
5.5.2. Geometric realization

We recall from [20] that the affine BMW algebra is isomorphic to the affine Kauffman tangle algebra, which is an algebra of "affine tangle diagrams," modulo Kauffman skein relations. An affine (n,n)-tangle diagram is just an ordinary $(n+1, n+1)$-tangle diagram with a fixed vertical strand connecting $\mathbf{1}$ and $\bar{\mathbf{1}}$, as in the following figure.



The affine Kauffman tangle algebra is generated by the following affine tangle diagrams:

$X_1 =$  , $G_i =$ 

, $E_i =$ 

.

One can also define a cyclotomic Kauffman tangle algebra $\mathrm{KT}_{n,S,r}(u_1, \dots, u_r)$ as the quotient of the affine Kauffman tangle algebra by a cyclotomic skein relation, which is a "local" version of the cyclotomic relation of Definition 5.14(9). See [21] for the precise definition. We denote the images of X_1, E_i and G_i in the cyclotomic Kauffman tangle algebra by the same letters. The assignments $e_i \mapsto E_i$, $g_i \mapsto G_i$ and $y_1 \mapsto \rho X_1$ defines a surjective homorphism from $\varphi : W_{n,S,r}(u_1, \dots, u_r) \to \mathrm{KT}_{n,S,r}(u_1, \dots, u_r)$, see [21], p. 1114.

It is shown in [21,22] and in [60] that the map φ is an isomorphism, assuming admissibility conditions on the ground ring (see Section 5.5.5). However, we are *not* going to assume this result here, but will give a new proof of the isomorphism.

5.5.3. Brief history of cyclotomic BMW algebras

Affine and cyclotomic BMW algebras were introduced by Häring–Oldenberg [27] and have recently been studied by three groups of mathematicians: Goodman and Hauschild Mosley [20–22,16], Rui, Xu, and Si [53,52], and Wilcox and Yu [58–60,63]. Under (slightly different) admissibility assumptions on the ground ring (see Section 5.5.5) all three groups have shown that the algebra $W_{n,S,r}$ is free over S of rank $r^n(2n-1)!!$ and in fact is cellular. (Wilcox and Yu produced cellular basis satisfying the strict equality $i(c^\lambda_{s,t}) = c^\lambda_{t,s}$, while the other groups only established cellularity in the weaker sense of Definition 2.3.) The cellular bases produced by all three groups are essentially tangle bases, i.e. cyclotomic analogues of the basis of Morton, Traczyk, and Wassermann for the ordinary BMW algebras. Goodman & Hauschild Mosley and Wilcox & Yu have shown that the algebras can be realized as algebras of tangles, when the ground ring is admissible. Rui et al. have achieved additional representation theoretic results. Further background on cyclotomic BMW algebras, motivation for the study of these algebras, relations to other mathematical topics (quantum groups, knot theory), and further literature citations can be found in [21] and in the other papers cited above.

5.5.4. Advantages of our approach to cellularity

Our proof of cellularity is more direct than the previous proofs cited above, in that it bypasses the lengthy proof (in [21], Proposition 3.7, or [60], Theorem 3.2) that these algebras have a finite spanning set of the appropriate cardinality. Our method does not depend on the isomorphism of the cyclotomic BMW algebras and cyclotomic Kauffman tangle algebras [21,22] or [60]; in fact, we can give a new proof of this isomorphism.

One might say that the difficulty in our proof has been displaced, because instead of the finite spanning set result cited above, we require Mathas' recent theorem on coherence of cellular structures for cyclotomic Hecke algebras [46].

5.5.5. Admissibility conditions on the ground ring.

The cyclotomic BMW algebras can be defined over arbitrary ground rings. However, it is necessary to impose conditions on the parameters in order to get a satisfactory theory.

One can see by a simple computation why one has to expect conditions on the parameters. First, one can show that there are elements δ_{-j} in the ground ring S for $j \geqslant 1$ such that $e_1 y_1^{-j} e_1 = \delta_{-j} e_1$; moreover, δ_{-j} is a polynomial in ρ^{-1}, $q - q^{-1}$, and $\delta_0, \delta_1, \dots, \delta_j$; see [22], Lemma 2.5. If one now multiplies the cyclotomic relation, Definition 5.14(9), by y_1^a and pre- and post-multiplies by e_1, one gets $(\sum_{k=0}^r a_k \delta_{k+a}) e_1 = 0$, for $a \in \mathbb{Z}$, where the a_k are signed elementary symmetric polynomials in $u_1, \dots, u_r$. Therefore, either e_1 is a torsion element over S, or the following *weak admissibility conditions* hold:

$$\sum_{k=0}^{r} a_k \delta_{k+a} = 0, \quad \text{for } a \in \mathbb{Z}.$$

If S is a field and the weak admissibility conditions do not hold, then $e_1 = 0$; it follows that all the e_i are zero, and the algebra reduces to the cyclotomic Hecke algebra over S with parameters q^2 and $u_1, \dots, u_r$.

The weak admissibility conditions are complicated and not strong enough to give satisfactory results on the representation theory of the algebras. Therefore, one wishes to find conditions that are both simpler and stronger. Two apparently different conditions have been proposed, one by Wilcox and Yu [58], and another by Rui and Xu [53]. It has been shown in [17] that the two conditions are equivalent in the case of greatest interest, when S is an integral domain with $q - q^{-1} \neq 0$. We consider only this case from now on.

Definition 5.15. Let S be an integral ground ring with parameters ρ, q, δ_j $(j \geqslant 0)$ and $u_1, \dots, u_r$, with $q - q^{-1} \neq 0$. One says that S is *admissible* (or that the parameters are *admissible*) if $\{e_1, y_1 e_1, \dots, y_1^{r-1} e_1\} \subseteq W_{2,S,r}$ is linearly independent over S.

It is shown in [58] that admissibility is equivalent to finitely many (explicit) polynomial relations on the parameters. Moreover, these relations give ρ and $(q - q^{-1})\delta_j$ as Laurent polynomials in the remaining parameters $q, u_1, \dots, u_r$; see [58] and [22] for details.

5.5.6. Morphisms of ground rings and a universal admissible ground ring

We consider what are the appropriate morphisms between ground rings for cyclotomic BMW algebras. The obvious notion would be that of a ring homomorphism taking parameters to parameters; that is, if S is a ground ring with parameters ρ, q, etc., and S' another ground ring with parameters ρ', q', etc., then a morphism $\varphi : S \to S'$ would be required to map $\rho \mapsto \rho'$, $q \mapsto q'$, etc.

However, it is better to require less, for the following reason: The parameter q enters into the cyclotomic BMW relations only in the expression $q^{-1} - q$, and the transformation $q \mapsto -q^{-1}$ leaves this expression invariant. Moreover, the transformation $g_i \mapsto -g_i$, $\rho \mapsto -\rho$, $q \mapsto -q$ (with all other generators and parameters unchanged) leaves the cyclotomic BMW relations unchanged.

Taking this into account, we arrive at the following notion:

Definition 5.16. Let S be a ground ring with parameters ρ, q, δ_j $(j \geqslant 0)$, and $u_1, \dots, u_r$. Let S' be another ground ring with parameters ρ', q', etc.

A unital ring homomorphism $\varphi : S \to S'$ is a *morphism of ground rings* if it maps

$$\begin{cases} \rho \mapsto \rho', \quad \text{and} \\ q \mapsto q' \text{ or } q \mapsto -q'^{-1}, \end{cases}$$

or

$$\begin{cases} \rho \mapsto -\rho', \quad \text{and} \\ q \mapsto -q' \text{ or } q \mapsto q'^{-1}, \end{cases}$$

and strictly preserves all other parameters.

Suppose there is a morphism of ground rings $\psi : S \to S'$. Then ψ extends to a homomorphism from $W_{n,S,r}$ to $W_{n,S',r}$. Moreover, $W_{n,S,r} \otimes_S S' \cong W_{n,S',r}$ as S'-algebras. These statements are discussed in [22], Section 2.4.

Let S be a ground ring with admissible parameters ρ, q, δ_j $(j \geqslant 0)$, and $u_1, \dots, u_r$. Then

$$\rho, -q^{-1}, \delta_j \quad (j \geqslant 0), \quad \text{and} \quad u_1, \dots, u_r$$

and

$$-\rho, -q, \delta_j \quad (j \geqslant 0), \quad \text{and} \quad u_1, \dots, u_r$$

are also sets of admissible parameters. Suppose that S is an integral ground ring with admissible parameters, with $q - q^{-1} \neq 0$, and that S' is another integral ground ring; if $\varphi : S \to S'$ is a morphism of ground rings such that $\varphi(q - q^{-1}) \neq 0$, then S' is also admissible.

It is easy to show (see [22], Theorem 3.19) that there is a universal integral admissible ground ring R, with parameters $\boldsymbol{\rho}$, $\boldsymbol{q}$, $\boldsymbol{\delta}_j$ ($j \geqslant 0$), and $\boldsymbol{u}_1, \ldots, \boldsymbol{u}_r$, with the following properties:

(1) The parameters $\boldsymbol{q}$, $\boldsymbol{u}_1$, …, $\boldsymbol{u}_r$ of R are algebraically independent over $\mathbb{Z}$.
(2) R is generated as a ring by $\boldsymbol{q}^{\pm 1}$, $\boldsymbol{\rho}^{\pm 1}$, $\boldsymbol{\delta}_0$, $\boldsymbol{\delta}_1, \ldots, \boldsymbol{\delta}_{r-1}$, and $\boldsymbol{u}_1^{\pm 1}, \ldots, \boldsymbol{u}_r^{\pm 1}$.
(3) Whenever S is an integral ground ring with admissible parameters, with $q - q^{-1} \neq 0$, there exists a morphism of ground rings from R to S; thus $W_{n,S,r} \cong W_{n,R,r} \otimes_R S$.
(4) The field of fractions of R is $\mathbb{Q}(\boldsymbol{q}, \boldsymbol{u}_1, \ldots, \boldsymbol{u}_r)$.
(5) Let $\boldsymbol{p} = \prod_{j=1}^r \boldsymbol{u}_j$. Then one has $\boldsymbol{\rho} = \boldsymbol{p}$ if r is even and $\boldsymbol{\rho} = \boldsymbol{q}^{-1}\boldsymbol{p}$ if r is odd. Since $\boldsymbol{\rho}^{-1} - \boldsymbol{\rho} = (\boldsymbol{q}^{-1} - \boldsymbol{q})(\boldsymbol{\delta}_0 - 1)$, and $\boldsymbol{q}$, $\boldsymbol{u}_1$, …, $\boldsymbol{u}_r$ are algebraically independent, one has $\boldsymbol{\delta}_0 \neq 0$.

5.5.7. Some properties of cyclotomic BMW and Kauffman tangle algebras

We restrict attention to the case of an integral admissible ground ring S with $q - q^{-1} \neq 0$. We write W_n for $W_{n,S,r}(u_1, \ldots, u_r)$ and KT_n for $\mathrm{KT}_{n,S,r}(u_1, \ldots, u_r)$.

The cyclotomic BMW algebras have an algebra involution i uniquely determined by $i(e_j) = e_j$ and $i(g_j) = g_j$ for all j, and $i(y_1) = y_1$. Likewise, the cyclotomic Kauffman tangle algebras have an algebra involution i, whose action on affine tangle diagrams is by the rotation through the axis $y = 1/2$. The surjective homomorphism $\varphi : W_n \to \mathrm{KT}_n$ respects the involutions.

For $n \geqslant 0$, there is a homomorphism (of involutive algebras) ι from W_n to W_{n+1} determined by $e_i \mapsto e_i$ and $g_i \mapsto g_i$ for $1 \leqslant i \leqslant n-1$, and $y_1 \mapsto y_1$; it is not clear *a priori* that ι is injective.

Likewise, there is a homomorphism (of involutive algebras) ι from KT_n to KT_{n+1}. On the level of affine tangle diagrams, the map is given by adding a new vertical strand connecting $\boldsymbol{n+1}$ and $\overline{\boldsymbol{n+1}}$, as for the Brauer algebras. This map is injective, as we will now explain.

For $n \geqslant 1$, a map cl from affine (n,n)-tangle diagrams to affine $(n-1,n-1)$-tangle diagrams can be defined as for Brauer diagrams and ordinary tangle diagrams. The linear extension of this map respects regular isotopy and all the skein relations defining the cyclotomic Kauffman tangle algebras, so determines a linear map from KT_n to KT_{n-1}. (See [20], Section 2.7, and [21], Section 3.3 for details.) The map cl respects the involutions, $i \circ \mathrm{cl} = \mathrm{cl} \circ i$. Moreover, for $x \in \mathrm{KT}_n$, we have $x = \mathrm{cl}(\iota(x)e_n)$, so it follows that $\iota : \mathrm{KT}_n \to \mathrm{KT}_{n+1}$ is injective. Using ι, we identify KT_n as a subalgebra of KT_{n+1}.

If δ_0 is invertible in S, we can define $\varepsilon_n = (1/\delta_0)\,\mathrm{cl}$, which is a conditional expectation, that is, an unital KT_{n-1}–KT_{n-1} bimodule map. We have $\varepsilon_{n+1} \circ \iota(x) = x$ for $x \in W_n$.

5.5.8. The cyclotomic Hecke algebra

We recall the definition of the affine and cyclotomic Hecke algebras, see [1].

Definition 5.17. Let S be a commutative unital ring with an invertible element q. The *affine Hecke algebra* $\widehat{H}_{n,S}(q^2)$ over S is the S-algebra with generators $t_1, g_1, \ldots, g_{n-1}$, with relations:

(1) The generators g_i are invertible, satisfy the braid relations, and $g_i - g_i^{-1} = (q - q^{-1})$.
(2) The generator t_1 is invertible, $t_1 g_1 t_1 g_1 = g_1 t_1 g_1 t_1$ and t_1 commutes with g_j for $j \geqslant 2$.

Let $u_1, \ldots, u_r$ be additional elements in S. The *cyclotomic Hecke algebra* $H_{n,S,r}(q^2; u_1, \ldots, u_r)$ is the quotient of the affine Hecke algebra $\widehat{H}_{n,S}(q^2)$ by the polynomial relation $(t_1 - u_1) \cdots (t_1 - u_r) = 0$.

We remark that since the generator t_1 can be rescaled by an arbitrary invertible element of S, only the ratios of the parameters u_i have invariant significance in the definition of the cyclotomic Hecke algebra. The affine and cyclotomic Hecke algebras have unique algebra involutions determined by $g_i \to g_i$ and $t_1 \to t_1$.

Now let S be a ground ring with parameters ρ, q, δ_j, and $u_1, \ldots, u_r$. For each n, let I_n be the two sided ideal in $W_{n,S,r}$ generated by e_{n-1}. Because of the relations $e_j e_{j\pm 1} e_j = e_j$, the ideal I_n is generated by any e_i ($1 \leqslant i \leqslant n-1$) or by all of them. It is easy to check that the quotient of $W_{n,S,r}$ by I_n is isomorphic (as involutive algebras) to the cyclotomic Hecke algebra $H_{n,S,r}(q^2; u_1, \ldots, u_r)$.

Let $\boldsymbol{\lambda} = (\lambda^{(1)}, \dots, \lambda^{(r)})$ be an r-tuple of Young diagrams. The total size of $\boldsymbol{\lambda}$ is $|\boldsymbol{\lambda}| = \sum_i |\lambda^{(i)}|$. If $\boldsymbol{\mu}$ and $\boldsymbol{\lambda}$ are r-tuples of Young diagrams of total size $f-1$ and f respectively, we write $\boldsymbol{\mu} \subset \boldsymbol{\lambda}$ if $\boldsymbol{\mu}$ is obtained from $\boldsymbol{\lambda}$ by removing one box from one component of $\boldsymbol{\lambda}$.

Theorem 5.18. *(See [1].) Let F be a field. The cyclotomic Hecke algebra $H_{n,F,r}(q^2; u_1, \dots, u_r)$ is split semisimple for all n as long as q is not a proper root of unity and, for all $i \neq j$, u_i/u_j is not an integer power of q. In this case, the simple components of $H_{n,F,r}(q; u_1, \dots, u_r)$ are labeled by r-tuples of Young diagrams of total size n, and a simple $H_{n,F,r}$ module $V_{\boldsymbol{\lambda}}$ decomposes as a $H_{n-1,F,r}$ module as the direct sum of all $V_{\boldsymbol{\mu}}$ with $\boldsymbol{\mu} \subset \boldsymbol{\lambda}$.*

Let us call the branching diagram for the cyclotomic Hecke algebras, as described in the theorem, the *r-Young lattice*. Note that, as for the usual Young's lattice, the r-Young lattice has no multiple edges.

Theorem 5.19 *(Ariki, Koike, Dipper, James, Mathas). The sequence of cyclotomic Hecke algebras $(H_{n,S,r}(q^2; u_1, \dots, u_r))_{n \geqslant 0}$ is a coherent tower of cellular algebras.*

Proof. Write H_n for $H_{n,S,r}(q^2; u_1, \dots, u_r)$. Ariki and Koike showed that the cyclotomic Hecke algebras are free as S modules [2], which implies that H_n imbeds naturally in H_{n+1}. Moreover, the algebras H_n have involutions that are consistent with the inclusions. Dipper, James and Mathas [12] constructed a cellular basis of the cyclotomic Hecke algebras, generalizing the Murphy basis of ordinary Hecke algebras. Ariki and Mathas showed [3], Proposition 1.9, that restrictions of cell modules from H_{n+1} to H_n have cell filtrations. Finally, Mathas has shown [46] that the module obtained from inducing a cell module from H_n to H_{n+1} has a cell filtration. □

5.5.9. Verification of the framework axioms for the cyclotomic BMW algebras

Let R be the generic admissible integral ground ring, with parameters $\boldsymbol{\rho}$, $\boldsymbol{q}$, $\boldsymbol{\delta}_j$ $(j \geqslant 0)$, and $\boldsymbol{u}_1, \dots, \boldsymbol{u}_r$, as introduced at the end of Section 5.5.6. In this section, we write W_n for $W_{n,R,r}(\boldsymbol{u}_1, \dots, \boldsymbol{u}_r)$, KT_n for $\mathrm{KT}_{n,R,r}(\boldsymbol{u}_1, \dots, \boldsymbol{u}_r)$, and H_n for $H_{n,R,r}(\boldsymbol{q}^2; \boldsymbol{u}_1, \dots, \boldsymbol{u}_r)$. Recall that the field of fractions of R is $F = \mathbb{Q}(\boldsymbol{q}, \boldsymbol{u}_1, \dots, \boldsymbol{u}_r)$. Let $W_n^F = W_n \otimes_R F$, and similarly for the other algebras.

If we would assume the isomorphism of W_n and KT_n, then we could verify the framework axioms for the pair of sequences $(W_n)_{n \geqslant 0}$ and $(H_n)_{n \geqslant 0}$ without difficulty, using elementary observations and some deeper results from the literature, and consequently apply Theorem 3.2 to the cyclotomic BMW algebras. However, we wish to give an independent proof of the isomorphism. Consequently, we have to verify the framework axioms and prove the isomorphism $W_n \cong \mathrm{KT}_n$ inductively, in tandem with the inductive step in the proof of Theorem 3.2.

Lemma 5.20. $W_0 \cong \mathrm{KT}_0 \cong R$.

Proof. Wilcox and Yu [60], Proposition 6.2, show that KT_0 is a free R module with basis $\{\emptyset\}$, where $\emptyset$ denotes the empty affine tangle diagram, which is also the identity element of KT_0. □

Lemma 5.21. *If for some n and for some admissible ground ring S, we have $\varphi : W_n^S \to \mathrm{KT}_n^S$ is an isomorphism, then $\iota : W_n^S \to W_{n+1}^S$ is injective.*

Proof. $\varphi \circ \iota = \iota \circ \varphi : W_n^S \to \mathrm{KT}_{n+1}^S$ is injective, because $\varphi : W_n^S \to \mathrm{KT}_n^S$ and $\iota : \mathrm{KT}_n^S \to \mathrm{KT}_{n+1}^S$ are injective. Thus $\iota : W_n^S \to W_{n+1}^S$ is injective. □

Lemma 5.22. *For all $n \geqslant 0$, $W_n^F \cong \mathrm{KT}_n^F$, W_n^F is split semisimple of dimension $r^n(2n-1)!!$, and $\iota : W_n^F \to W_{n+1}^F$ is injective.*

Proof. This is proved in [22], Theorem 4.8. We stress that the result is independent of the finite spanning set theorem, [21], Proposition 3.7. One thing that is not made clear in the proof of [22], Theorem 4.8 is why $\iota : W_n^F \to W_{n+1}^F$ is injective. But if one assumes inductively that the conclusions of the theorem hold for W_f^F, $f \leqslant n$, for some fixed n, and in particular that $\varphi : W_n^F \to \mathrm{KT}_n^F$ is an isomorphism, then $\iota : W_n^F \to W_{n+1}^F$ is injective by Lemma 5.21. One can then continue with the proof of the inductive step of [22], Theorem 4.8. $\square$

Lemma 5.23. *If for some n, W_n is a free R-module, then its rank is $r^n(2n-1)!!$.*

Proof. $x \mapsto x \otimes 1$ takes an R-basis of W_n to an F-basis of $W_n \otimes_R F = W_n^F$. $\square$

Lemma 5.24. *If for some n, W_n has a spanning set A of cardinality $r^n(2n-1)!!$, then $\varphi : W_n \to \mathrm{KT}_n$ is an isomorphism, and A is an R-basis of W_n.*

Proof. Say W_n has a spanning set A of cardinality $r^n(2n-1)!!$. To prove both conclusions, it suffices to show that $\varphi(A)$ is linearly independent in KT_n. But

$$\{\varphi(a) \otimes 1\colon a \in A\} \subseteq \mathrm{KT}_n \otimes_R F = \mathrm{KT}_n^F$$

is a spanning set of cardinality $r^n(2n-1)!!$, which is the dimension of KT_n^F, according to Lemma 5.22. Therefore $\{\varphi(a) \otimes 1 : a \in A\}$ is linearly independent in KT_n^F, and hence $\varphi(A)$ is linearly independent in KT_n. $\square$

Lemma 5.25. *$W_1 \cong \mathrm{KT}_1 \cong H_1$, W_1 is a free R-module of rank r, and both $\iota : W_0 \to W_1$ and $\iota : W_1 \to W_2$ are injective.*

Proof. By definition, $W_1 \cong H_1 \cong R[X]/((X-u_1)\cdots(X-u_r))$, and these algebras are free R-modules of rank r. Hence $\varphi : W_1 \to \mathrm{KT}_1$ is an isomorphism by Lemma 5.24. The injectivity statements follow from Lemma 5.21. $\square$

Lemma 5.26. *Suppose that for some $n \geqslant 1$ one has $W_k \cong \mathrm{KT}_k$ for $0 \leqslant k \leqslant n$. Then the maps $\iota : W_k \to W_{k+1}$ are injective for $0 \leqslant k \leqslant n$. Using the maps ι, regard W_k as a subalgebra of W_{k+1} for $0 \leqslant k \leqslant n$. One has:*

(1) *$\delta_0 Re_1 \subseteq e_1 W_1 e_1 \subseteq Re_1$.*
(2) *For $2 \leqslant k \leqslant n$, $e_k W_k e_k = W_{k-1} e_k$.*
(3) *For $1 \leqslant k \leqslant n$, e_k commutes with W_{k-1}.*
(4) *For $1 \leqslant k \leqslant n$, $W_{k+1} e_k = W_k e_k$. Moreover, $x \mapsto x e_k$ is injective from W_k to $W_k e_k$.*

Proof. The statement about injectivity of the maps ι follows from Lemma 5.21.

Point (1) follows from the relations $e_1 y_1^j e_1 = \delta_j e_1$ for $j \geqslant 0$. Point (2) and the first part of point (4) follows from the corresponding facts for the affine BMW algebras, [20], Proposition 3.17, and Proposition 3.20. Point (3) follows from the defining relations for the cyclotomic BMW algebras. For the injectivity statement in point (4), note that for $x \in W_k$,

$$\mathrm{cl}\big(\varphi(x e_k)\big) = \mathrm{cl}\big(\varphi(x) E_k\big) = \varphi(x).$$

Since $\varphi : W_k \to \mathrm{KT}_k$ is injective, so is $x \mapsto x e_k$. $\square$

Theorem 5.27.

(1) *The two sequences of algebras $(W_k)_{k \geqslant 0}$ and $(H_k)_{k \geqslant 0}$ satisfy the framework axioms of Section 3.1.*
(2) *For all $k \geqslant 0$, $\varphi : W_k \to \mathrm{KT}_k$ is an isomorphism, and $\iota : W_k \to W_{k+1}$ is injective.*
(3) *The conclusions of Theorem 3.2 are valid for the sequence $(W_k)_{k \geqslant 0}$.*

Proof. According to Proposition 5.19, $(H_k)_{k\geqslant 0}$ is a coherent tower of cellular algebras, so axiom (1) of the framework axioms holds. Axiom (3) holds by Lemmas 5.20 and 5.25. Axiom (4) holds by Lemma 5.22. We observed above that $W_k/W_k e_{k-1} W_k \cong H_k$ as involutive algebras; thus axiom (5) holds. Axiom (8) holds because of the relation $e_{k-1}e_k e_{k-1} = e_{k-1}$.

Suppose that for some $n \geqslant 0$, it is known that the maps $\varphi : W_k \to \mathrm{KT}_k$ are isomorphisms for $0 \leqslant k \leqslant n$. Then, from Lemma 5.26, we have the following versions of framework axioms (2), (6) and (7):

(2′) W_k is an i-invariant subalgebra of W_{k+1} for $0 \leqslant k \leqslant n$.
(6′) For $1 \leqslant k \leqslant n$, e_k commutes with W_{k-1} and $e_k W_k e_k \subseteq W_{k-1} e_k$.
(7′) For $1 \leqslant k \leqslant n$, $W_{k+1} e_k = W_k e_k$, and the map $x \mapsto x e_k$ is injective from W_k to $W_k e_k$.

Now we consider the following:

Claim. *For all $n \geqslant 0$,*

(a) *for $0 \leqslant k \leqslant n$, the maps $\varphi : W_k \to \mathrm{KT}_k$ are isomorphisms, and therefore W_k may be regarded as an i-invariant subalgebra of W_{k+1}, and*
(b) *the statements* (1)–(4) *of Theorem* 3.2 *hold for the finite tower $(W_k)_{0\leqslant k\leqslant n}$.*

For $n = 0$ and $n = 1$, the claim follows from Lemmas 5.20 and 5.25. We assume the claim holds for some $n \geqslant 1$ and show that it also holds for $n+1$. Then, by the discussion above, the framework axioms hold for the finite tower $(W_k)_{0\leqslant k\leqslant n}$ with axioms (2), (6) and (7) replaced by the finite versions (2′), (6′), and (7′). Now the inductive step in the proof of Theorem 3.2 goes through without change and yields part (b) of the claim for the tower $(W_k)_{0\leqslant k\leqslant n+1}$. In particular, W_{n+1} is a cellular algebra; the cardinality of its cellular basis is $r^{n+1}(2n+1)!!$, by Lemma 5.23. But then Lemma 5.24 gives that $\varphi : W_{n+1} \to \mathrm{KT}_{n+1}$ is an isomorphism, so part (a) of the claim also holds for $n+1$. □

Corollary 5.28. *Let S be any admissible integral ground ring with $q - q^{-1} \neq 0$.*

(1) *The sequence of cyclotomic BMW algebras $(W_{n,S,r})_{n\geqslant 0}$ is a coherent tower of cellular algebras. $W_{n,S,r}$ has cell modules indexed by all r-tuples of Young diagrams of total size n, $n-2$, $n-4, \ldots$. The cell module labeled by an r-tuple of Young diagrams $\boldsymbol{\lambda}$ has a basis labeled by up-down tableaux of length n and shape $\boldsymbol{\lambda}$.*
(2) *$W_{n,S,r} \cong \mathrm{KT}_{n,S,r}$ for all $n \geqslant 0$.*

Remark 5.29. It is possible to combine our results with the results of Wilcox and Yu [59] to obtain Murphy type bases of the cyclotomic BMW algebras that are strictly cellular, i.e. $i(c^{\lambda}_{s,t}) = c^{\lambda}_{t,s}$ for all λ, s, t. To do this, all we need, according to Remark 2.14, is an i-invariant R-module complement to the ideal $\breve{W}_n^{(\boldsymbol{\lambda},n)}$ in ${W_n}^{(\boldsymbol{\lambda},n)}$. However, one can check that the ideals $\breve{W}_n^{(\boldsymbol{\lambda},n)}$ and $W_n^{(\boldsymbol{\lambda},n)}$ for our cellular structure are the same as for the cellular structure of Wilcox and Yu, and therefore, since their cellular basis satisfies the strict equality $i(c^{\lambda}_{s,t}) = c^{\lambda}_{t,s}$ for all λ, s, t, the desired i-invariant R-module complement exists.

Remark 5.30. Our framework also applies to the degenerate cyclotomic BMW algebras (cyclotomic Nazarov Wenzl algebras) studied in [4]. For the details, see [19].

5.6. The walled Brauer algebras

5.6.1. Definition of the walled Brauer algebras

Let S be a commutative ring with identity, with a distinguished element δ. The walled (or rational) Brauer algebra $B_{r,s}(S,\delta)$ is a unital subalgebra of the Brauer algebra $B_{r+s}(S,\delta)$ spanned by certain Brauer diagrams. Divide the $r+s$ top vertices into a left cluster consisting of the leftmost r vertices and a right cluster consisting of the remaining s vertices, and similarly for the bottom vertices. The walled Brauer diagrams are those in which no vertical strand connects a left vertex and a right vertex, and every horizontal strand connects a left vertex and a right vertex. (If we draw a vertical line—the wall—separating left and right vertices, then vertical strands are forbidden to cross the wall, and horizontal strands are required to cross the wall.) One can easily check that the span of walled Brauer diagrams is a unital subalgebra of $B_{r+s}(S, \delta)$.

5.6.2. Brief history of the walled Brauer algebras

The walled Brauer algebras were introduced by Turaev [54] and by Koike [37], and studied by Benkart et al. [5] and by Nikitin [51]. The walled Brauer algebras arise in connection with the invariant theory of the general linear group acting on mixed tensors. Cellularity of walled Brauer algebras was proved by Green and Martin [24] and by Cox et al. [9]; the latter authors show that walled Brauer algebras can be arranged into coherent cellular towers.

5.6.3. Some properties of the walled Brauer algebras

The walled Brauer algebra $B_{r,s}$ is invariant under the involution i of the Brauer algebra B_{r+s}. Moreover, the inclusion map $\iota : B_{r+s} \to B_{r+s+1}$ maps $B_{r,s}$ to $B_{r,s+1}$, and the closure map $\mathrm{cl} : B_{r+s} \to B_{r+s-1}$ maps $B_{r,s}$ to $B_{r,s-1}$, when $s \geqslant 1$. If δ is invertible, $\varepsilon_{r,s} = (1/\delta)\,\mathrm{cl} : B_{r,s} \to B_{r,s-1}$ is a conditional expectation, and, of course, the trace ε on B_{r+s} restricts to a trace on $B_{r,s}$.

The Brauer algebras have an involutive inner automorphism ρ which maps each Brauer diagram to its reflection in the vertical line $x = 1/2$. (We might as well take the vertical line to coincide with our wall.) It is clear that ρ restricts to an isomorphism from $B_{r,s}$ to $B_{s,r}$. Given this, we can define "left versions" of ι, cl and $\varepsilon_{r,s}$ by $\iota' = \rho \circ \iota \circ \rho : B_{r,s} \to B_{r+1,s}$, $\mathrm{cl}' = \rho \circ \mathrm{cl} \circ \rho : B_{r,s} \to B_{r-1,s}$, and $\varepsilon' = \rho \circ \varepsilon \circ \rho : B_{r,s} \to B_{r-1,s}$. Note that ι' adds a vertical strand on the left, and cl' partially closes diagrams on the left.

Let $e_{a,b}$ be the Brauer diagram with horizontal strands connecting $\boldsymbol{a}$ to $\boldsymbol{b}$ and $\bar{\boldsymbol{a}}$ to $\bar{\boldsymbol{b}}$ and vertical strands connecting $\boldsymbol{j}$ to $\bar{\boldsymbol{j}}$ for all $j \neq a, b$. One can easily check the following properties:

Lemma 5.31.

(1) $e_{a,b}^2 = \delta e_{a,b}$.
(2) $e_{a,b}e_{a,b\pm1}e_{a,b} = e_{a,b}$ *and* $e_{a,b}e_{a\pm1,b}e_{a,b} = e_{a,b}$.
(3) *For* $e_{a,b} \in B_{r,s}$, $\iota(e_{a,b}) = e_{a,b}$ *and* $\iota'(e_{a,b}) = e_{a+1,b+1}$.
(4) *For* $x \in B_{r,s+1}$, *we have* $e_{1,r+s+2}\iota'(x)e_{1,r+s+2} = \iota' \circ \iota \circ \mathrm{cl}(x)e_{1,r+s+2}$.
(5) *For* $x \in B_{r+1,s}$, *we have* $e_{1,r+s+2}\iota(x)e_{1,r+s+2} = \iota' \circ \iota \circ \mathrm{cl}'(x)e_{1,r+s+2}$.
(6) $e_{1,r+s+2}$ *commutes with* $\iota' \circ \iota(x)$ *for all* $x \in B_{r,s}$.

The following statement is also easy to check:

Lemma 5.32. *The ideal J in $B_{r,s}(S, \delta)$ generated by $e_{1,r+s}$ is the ideal spanned by diagrams with fewer than $r + s$ through strands, and $B_{r,s}(S, \delta)/J \cong S(\mathfrak{S}_r \times \mathfrak{S}_s)$.*

Lemma 5.33.

(1) $B_{r,s+1}e_{1,r+s+1} = \iota(B_{r,s})e_{1,r+s+1}$.
(2) $B_{r+1,s}e_{1,r+s+1} = \iota'(B_{r,s})e_{1,r+s+1}$.

Proof. To prove part (1), we have to show that if d is a diagram in $B_{r,s+1}$, then there is a diagram $d' \in \iota(B_{r,s})$ such that $de_{1,r+s+1} = d'e_{1,r+s+1}$. We can suppose that d is not already in $\iota(B_{r,s})$; therefore, the vertex $\overline{\boldsymbol{r+s+1}}$ in d is connected to some vertex v other than $\mathbf{1}$ and $\boldsymbol{r+s+1}$. There are two cases to consider.

The first is that the vertices $\mathbf{1}$ and $\boldsymbol{r+s+1}$ are not connected to each other in d; let a and b be the vertices connected to $\mathbf{1}$ and $\boldsymbol{r+s+1}$. Now let d' be the diagram in which a and b are connected to each other; $\boldsymbol{r+s+1}$ is connected to $\overline{\boldsymbol{r+s+1}}$; $\mathbf{1}$ is connected to v; and all other strands are as in d. Then we have $de_{1,r+s+1} = d'e_{1,r+s+1}$. The case that the vertices $\mathbf{1}$ and $\boldsymbol{r+s+1}$ are connected to each other is similar and will be omitted.

Part (2) is proved by applying the map ρ to both sides of the equality in part (1) and then interchanging the roles of r and s. □

A unital trace ε on an S-algebra A is *non-degenerate* if for every non-zero $x \in A$ there exists a $y \in A$ such that $\varepsilon(xy) \neq 0$.

Lemma 5.34.

(1) *The trace ε on $B_n(\mathbb{Q}(\boldsymbol{\delta}), \boldsymbol{\delta})$ is non-degenerate, for any n.*
(2) *The trace ε on $B_{r,s}(\mathbb{Q}(\boldsymbol{\delta}), \boldsymbol{\delta})$ is non-degenerate, for any r, s.*

Sketch of proof. The argument for part (1) is from [47]. It suffices to show that the determinant of the Gram matrix $\varepsilon(dd')_{d,d'}$, where d, d' run over the list of all Brauer diagrams (in some order), is non-zero. Recall that $\varepsilon(dd')$ is $\boldsymbol{q}^{c(dd')-n}$, where $c(dd')$ is the number of components in the tangle obtained by closing all the strands of dd'. One can check that $c(di(d)) = n$ and $c(dd') < n$ for all diagrams other than $i(d)$. Therefore, each row and column of the Gram matrix has exactly one entry equal to 1 and all other entries have the form $\boldsymbol{q}^{-k}$ for some $k > 0$.

The argument for part (2) is identical. □

5.6.4. Verification of the framework axioms for the walled Brauer algebras

To fit the walled Brauer algebras to our framework, we have to reduce the double sequence of algebras to a single sequence. We adopt the following scheme, as in [51], or [9]: Fix some integer $t \geqslant 0$. For any S and $\delta \in S$, we consider the sequence of walled Brauer algebras $A_n = A_n(S, \delta)$, where $A_{2k}(S, \delta) = B_{k,k+t}(S, \delta)$, and $A_{2k+1}(S, \delta) = B_{k,k+t+1}(S, \delta)$, with the inclusions

$$A_{2k} \xrightarrow{\iota} A_{2k+1} \xrightarrow{\iota'} A_{2k+2}.$$

We put $f_{2k-1} = e_{1,2k+t} \in A_{2k}$ and $f_{2k} = e_{1,2k+t+1} \in A_{2k+1}$. We identify A_n as a subalgebra of A_{n+1} via these embeddings. With these conventions, Lemma 5.31, points (2) and (3) give $f_n f_{n\pm1} f_n = f_n$. Moreover, if we write $\mathrm{cl}_n = \mathrm{cl}$ when n is even and $\mathrm{cl}_n = \mathrm{cl}'$ when n is odd, then we have $f_{n-1} x f_{n-1} = \mathrm{cl}_{n-1}(x) f_{n-1}$ for $x \in A_{n-1}$, by Lemma 5.31, points (4) and (5). Point (6) of the lemma says that f_{n-1} commutes with A_{n-2}.

If J is the ideal in A_n generated by f_{n-1}, then we have $A_{2k}/J \cong S(\mathfrak{S}_k \times \mathfrak{S}_{k+t})$, and $A_{2k+1}/J \cong S(\mathfrak{S}_k \times \mathfrak{S}_{k+t+1})$. So we set $Q_{2k}(S) = S(\mathfrak{S}_k \times \mathfrak{S}_{k+t})$ and $Q_{2k+1}(S) = S(\mathfrak{S}_k \times \mathfrak{S}_{k+t+1})$, with the natural embeddings.

Since $A_0 = B_{0,t} \cong S\mathfrak{S}_t$, and $A_1 = B_{0,t+1} \cong S\mathfrak{S}_{t+1}$, we cannot hope to satisfy our framework axiom (3). However, we can replace axiom (3) with the weaker

(3′) $A_0 \cong Q_0$, and $A_1 \cong Q_1$.

We also have to drop our usual convention (see Definition 2.17) regarding branching diagrams that the 0-th row of the branching diagram has a single vertex. Our conclusions will have to be modified, but not severely.

We now take $R = \mathbb{Z}[\boldsymbol{\delta}]$ and $\delta = \boldsymbol{\delta}$. R is the generic ground ring for walled Brauer algebras; if S is any commutative unital ring with parameter δ, then $B_{r,s}(S, q) = B_{r,s}(R, \boldsymbol{q}) \otimes_R S$. Let $F = \mathbb{Q}(\boldsymbol{\delta})$. In the remainder of this section, we write $A_n = A_n(R, \boldsymbol{\delta})$ and $Q_n = Q_n(R)$. (Recall that $Q_n(R) = R(\mathfrak{S}_k \times \mathfrak{S}_{k+t})$ if $n = 2k$, and $Q_n(R) = R(\mathfrak{S}_k \times \mathfrak{S}_{k+t+1})$ if $n = 2k+1$.)

Lemma 5.35. *The walled Brauer algebra $B_{r,s}(\mathbb{Q}(\boldsymbol{\delta}), \boldsymbol{\delta})$ is split semisimple.*

Sketch of proof. It suffices to show that (for any t) the algebras in the sequence A_n are split semisimple. This was proved by Nikitin in [51], following Wenzl's method for the Brauer algebra in [55]. Nikitin's proof involves obtaining the weights of the trace ε, but little detail is given. For our purposes, we can bypass this issue, and use Lemma 5.34 instead. Then the method of proof of Theorem 3.2 from [55] applies. □

Proposition 5.36. *The two sequence of R-algebras $(A_n)_{n\geqslant 0}$ and $(Q_n)_{n\geqslant 0}$ satisfy the framework axioms of Section* 3.1, *with axiom* (3) *replaced by* (3′), *specified above, and with the elements f_n taking the role of the elements e_n in the list of framework axioms.*

Proof. The sequence $(Q_n)_{n\geqslant 0}$ is clearly a coherent tower of cellular algebras, so axiom (1) holds. Axiom (2) is evident, and we have remarked about substituting axiom (3′) for axiom (3). A_n^F is split semisimple by Lemma 5.35. Thus axiom (4) holds.

We have f_{n-1} is an essential idempotent with $i(f_{n-1}) = f_{n-1}$. We have $A_n/(A_n f_{n-1} A_n) \cong Q_n$ by Lemma 5.32, which gives axiom (5).

We have seen that f_{n-1} commutes with A_{n-2} and $f_{n-1}A_{n-1}f_{n-1} \subseteq A_{n-2}f_{n-1}$. Moreover, if $x \in A_{n-2}$, then $f_{n-1}xf_{n-1} = \boldsymbol{\delta} x f_{n-1}$, so $f_{n-1}A_{n-1}f_{n-1} \supseteq \boldsymbol{\delta} A_{n-2}f_{n-1}$. Therefore, $f_{n-1}A^F_{n-1}f_{n-1} = A^F_{n-2}f_{n-1}$, so axiom (6) holds.

Axiom (7) results from Lemma 5.33, and axiom (8) from $f_{n-1}f_nf_{n-1} = f_{n-1}$. □

Remark 5.37. The branching diagram for the sequence (Q_n^F) is the following: Each row has vertices labeled by pairs of Young diagrams; on an even row $2k$, the the first Young diagram in a pair has k boxes and the second $k+t$ boxes; on an odd row $2k+1$, the first Young diagram has k boxes and the second $k+t+1$ boxes; finally, there is an edge between pairs of Young diagrams in successive rows that differ by exactly one box.

Corollary 5.38. *Let S be any commutative unital ring with parameter δ.*

(1) *The walled Brauer algebras $B_{r,s}(S,\delta)$ are cellular algebras.*
(2) *The family is coherent in the sense that the restriction of a cell module from $B_{r,s}(S,\delta)$ to $B_{r-1,s}(S,\delta)$ or to $B_{r,s-1}(S,\delta)$ and induction of a cell module from $B_{r,s}(S,\delta)$ to $B_{r+1,s}(S,\delta)$ or to $B_{r,s+1}(S,\delta)$ have filtrations by cell modules.*
(3) *The cell modules of $B_{r,s}(S,\delta)$ are labeled by pairs of Young diagrams $(\lambda^{(1)},\lambda^{(2)})$, where $|\lambda^{(2)}| - |\lambda^{(1)}| = s - r$ and $|\lambda^{(2)}| + |\lambda^{(1)}| \leqslant s + r$.*

A basis for any cell module for $B_{r,s}$ can be labeled by paths on a certain branching diagram. Suppose without loss of generality that $t = s - r \geqslant 0$. Let $(A_n)_{n\geqslant 0}$ and $(Q_n)_{n\geqslant 0}$ be the two sequences of algebras defined above, depending on t, so in particular, $B_{r,s} = A_{2r}$. Let $\mathfrak{B}_0$ be the branching diagram for $(Q_n^F)_{n\geqslant 0}$, which was described above, and let $\mathfrak{B}$ be that obtained by reflections from $\mathfrak{B}_0$. On the 0-th row, $\mathfrak{B}$ has vertices labeled by all pairs $(\emptyset,\lambda)$, where λ is a Young diagram of size t. Finally, augment $\mathfrak{B}$ with a copy of Young's lattice up to the $(t-1)$-st level, with vertices labeled by pairs $(0,\mu)$ with $0 \leqslant |\mu| \leqslant t-1$. The pairs of Young diagrams labeling the cell modules of $B_{r,s}$ are located on the $r+s$-th row of the augmented branching diagram, and a basis of any cell module can be labeled by paths on the augmented branching diagram from $(\emptyset,\emptyset)$ to the pair in question.

We note that several of the results of Section 3 of [9] follow from the application of our method to the walled Brauer algebras.

5.7. Partition algebras

5.7.1. Definition of the partition algebras

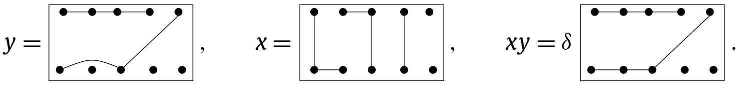
Let n be an integer, $n \geqslant 1$. Let $[\boldsymbol{n}] = \{\boldsymbol{1},\dots,\boldsymbol{n}\}$ and $[\bar{\boldsymbol{n}}] = \{\bar{\boldsymbol{1}},\dots,\bar{\boldsymbol{n}}\}$ be disjoint sets of size n, and let X_n be the family of all set partitions of $[\boldsymbol{n}] \cup [\bar{\boldsymbol{n}}]$.

We can represent an element x of X_n by any graph with vertex set equal to $[\boldsymbol{n}] \cup [\bar{\boldsymbol{n}}]$ whose connected components are the blocks or classes of the partition x. We picture such a graph as a diagram in the rectangle $\mathscr{R}$, with the vertices in $[\boldsymbol{n}]$ arranged on the top edge and those in $[\bar{\boldsymbol{n}}]$ arranged on the bottom edge of $\mathscr{R}$, as in the tangle diagrams discussed in Section 5.1.

Let S be any commutative ring with identity, with a distinguished element δ. We define a product on X_n as follows: Let x and y be elements of X_n. Realize y as a set partition of $[\boldsymbol{n}] \cup [\boldsymbol{n}']$ (with $[\boldsymbol{n}']$ the set of bottom vertices). Realize x as a set partition of $[\boldsymbol{n}'] \cup [\bar{\boldsymbol{n}}]$ (with $[\boldsymbol{n}']$ the set of top vertices). Let E_x and E_y be the corresponding equivalence relations, regarded as equivalence relations on $[\boldsymbol{n}] \cup [\boldsymbol{n}'] \cup [\bar{\boldsymbol{n}}]$. Let E be the smallest equivalence relation on $[\boldsymbol{n}] \cup [\boldsymbol{n}'] \cup [\bar{\boldsymbol{n}}]$ containing $E_x \cup E_y$. Let r be the number of equivalence classes of E contained in $[\boldsymbol{n}']$. Let E_{xy} be the equivalence relation obtained by restricting E to $[\boldsymbol{n}] \cup [\bar{\boldsymbol{n}}]$, and let z be the corresponding set partition of $[\boldsymbol{n}] \cup [\bar{\boldsymbol{n}}]$. Then xy is defined to be $\delta^r z$.

Here is an example of two set partitions represented by graphs and their product.

$y =$, $x =$, $xy = \delta$.



We let $A_{2n}(S, \delta)$ be the free S module with basis X_n. We give $A_{2n}(S, \delta)$ the bilinear product extending the product defined on X_n. One can check the multiplication is associative. Note that $A_0(S, \delta) \cong S$. For $n \geqslant 1$, the multiplicative identity of $A_{2n}(S, \delta)$ is the partition with blocks $\{\boldsymbol{i}, \bar{\boldsymbol{i}}\}$ for $1 \leqslant i \leqslant n$.

For $n \geqslant 1$, let $X_n' \subset X_n$ be the family of set partitions with $\boldsymbol{n}$ and $\bar{\boldsymbol{n}}$ in the same block. The S-span of X_n' is a unital subalgebra of $A_{2n}(S, \delta)$, which we denote by $A_{2n-1}(S, \delta)$.

The algebras $A_k(S, \delta)$ for $k \geqslant 0$ are called the *partition algebras*.

Note that the set partitions $x \in X_n$ each of whose blocks has size 2 can be identified with Brauer diagrams on $2n$ vertices, and the product of such diagrams in the Brauer algebra $B_n(S, \delta)$ agrees with the product in A_{2n}. Thus $B_n(S, \delta)$ can be identified with a unital subalgebra of $A_{2n}(S, \delta)$.

5.7.2. *Brief history of the partition algebras*

The partition algebras A_{2n} were introduced independently by Martin [41,42] and Jones [34]. Partition algebras arise as centralizer algebras for the symmetric group $\mathfrak{S}_k$ acting as a subgroup of $\mathrm{GL}(k, \mathbb{C})$ on tensor powers of $\mathbb{C}^k$ [34,43]. The algebras A_{2n+1} have been used as an auxiliary device for studying the partition algebras, by Martin and others. Halverson and Ram [26] emphasized putting the even and odd algebras on an equal footing, which reveals the role played by the basic construction. Cellularity of the partition algebras was proved in [61,13,57]. For further literature citations, see the review article [26].

5.7.3. *Some properties of the partition algebras*

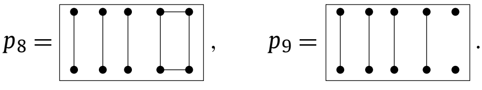
Fix a ground ring S and $\delta \in S$. In this section write A_k for $A_k(S, \delta)$.

For $n \geqslant 1$, A_{2n-1} is defined as a subalgebra of A_{2n}. The map $\iota : X_n \to X'_{n+1}$ which adds the additional block $\{\boldsymbol{n+1}, \overline{\boldsymbol{n+1}}\}$ to $x \in X_n$ is an imbedding; the linear extension of ι to A_{2n} is a unital algebra monomorphism into A_{2n+1}. Using ι, we identify A_{2n} with its image in A_{2n+1}.

For $n \geqslant 1$, let $p_{2n-1} \in A_{2n}$ be the set partition of $[\boldsymbol{n}] \cup [\bar{\boldsymbol{n}}]$ with blocks $\{\boldsymbol{n}\}$, $\{\bar{\boldsymbol{n}}\}$, and $\{\boldsymbol{i}, \bar{\boldsymbol{i}}\}$ for $1 \leqslant i \leqslant n-1$. The element p_{2n-1} satisfies $p_{2n-1}^2 = \delta p_{2n-1}$. Let $p_{2n} \in A_{2n+1}$ be the set partition of $[\boldsymbol{n+1}] \cup [\overline{\boldsymbol{n+1}}]$ with blocks $\{\boldsymbol{n}, \boldsymbol{n+1}, \bar{\boldsymbol{n}}, \overline{\boldsymbol{n+1}}\}$ and $\{\boldsymbol{i}, \bar{\boldsymbol{i}}\}$ for $1 \leqslant i \leqslant n-1$. Then p_{2n} is an idempotent.

Here are graphs representing the p_k for k even and odd:

$p_8 =$, $p_9 =$.



One can check that

$$p_k p_{k\pm1} p_k = p_k \quad \text{for all } k. \tag{5.1}$$

Define an involution i on X_n by interchanging $\boldsymbol{j}$ with $\bar{\boldsymbol{j}}$ for each j. The map i reflects a graph $d(x)$ representing $x \in X_n$ in the line $y = 1/2$. The linear extension of i to A_n is an algebra involution. Note that X'_n and A_{2n-1} are invariant under i. The embeddings of A_k in A_{k+1} commute with the involutions. The elements p_k are invariant under i.

Define a map $\mathrm{cl} : X_n \to X'_n$ by merging the blocks containing $\boldsymbol{n}$ and $\bar{\boldsymbol{n}}$, and define $\mathrm{cl} : A_{2n} \to A_{2n-1}$ as the linear extension of the map $\mathrm{cl} : X_n \to X'_n$.

Define a map $\mathrm{cl} : X'_n \to A_{2n-2}$ as follows: For $x \in X'_n$, if $\{\boldsymbol{n}, \bar{\boldsymbol{n}}\}$ is a block of x, then $\mathrm{cl}(x) = \delta x'$, where $x' \in X_{n-1}$ is obtained by removing the block $\{\boldsymbol{n}, \bar{\boldsymbol{n}}\}$. Otherwise, $\mathrm{cl}(x) \in X_{n-1}$ is obtained by intersecting each block of x with $[\boldsymbol{n-1}] \cup [\overline{\boldsymbol{n-1}}]$. Define $\mathrm{cl} : A_{2n-1} \to A_{2n-2}$ as the linear extension of the map $\mathrm{cl} : X'_n \to A_{2n-2}$.

One can check that for all k, $\mathrm{cl} : A_k \to A_{k-1}$ is a non-unital A_{k-1}–A_{k-1} bimodule map. Moreover, $\mathrm{tr} = \mathrm{cl} \circ \mathrm{cl} \circ \cdots \circ \mathrm{cl} : A_k \to A_0 \cong S$ is a non-unital trace. The trace tr can be computed as follows: given $x \in X_n$, let $d(x)$ be any graph representing x and let $d'(x)$ be the graph augmented by drawing edges between each pair of vertices $\{\boldsymbol{j}, \bar{\boldsymbol{j}}\}$; then $\mathrm{tr}(x) = \delta^r$, where r is the number of components of $d'(x)$.

The maps cl commute with the algebra involutions i, and $\mathrm{tr}(a) = \mathrm{tr}(i(a))$. Moreover,

$$p_k x p_k = \mathrm{cl}(x) p_k \quad \text{for all } x \in A_k,\ k \geqslant 1. \tag{5.2}$$

If δ is invertible, define $\varepsilon_{2n} : A_{2n} \to A_{2n-1}$ by $\varepsilon_{2n} = \mathrm{cl}$, and $\varepsilon_{2n-1} : A_{2n-1} \to A_{2n-2}$ by $\varepsilon_{2n-1} = \delta^{-1}\,\mathrm{cl}$. Then the maps ε_k are unital conditional expectations, and the map $\varepsilon = \varepsilon_1 \circ \cdots \circ \varepsilon_k : A_k \to A_0 \cong S$ is a unital trace.

Let $x \in X_n$. Call a block of x a *through block* if the block has non-empty intersection with both $[\boldsymbol{n}]$ and $[\bar{\boldsymbol{n}}]$. The number of through blocks of x is called the propagating number of x, denoted $\mathrm{pn}(x)$. Clearly, $\mathrm{pn}(x) \leqslant n$ for all $x \in X_n$. The only $x \in X_n$ with propagating number equal to n are Brauer diagrams with only vertical strands, i.e. permutation diagrams.

If $x, y \in X_n$ and $xy = \delta^r z$, then $\mathrm{pn}(z) \leqslant \min\{\mathrm{pn}(x), \mathrm{pn}(y)\}$. Hence the span of the set of $x \in X_n$ with $\mathrm{pn}(x) < n$ is an ideal $J_{2n} \subset A_{2n}$. Moreover, $J_{2n-1} := J_{2n} \cap A_{2n-1}$ is the span of $x \in X'_n$ with $\mathrm{pn}(x) < n$.

Lemma 5.39. *For $n \geqslant 1$, $A_{2n}/J_{2n} \cong S\mathfrak{S}_n$, and $A_{2n-1}/J_{2n-1} \cong S\mathfrak{S}_{n-1}$, as algebras with involution.*

Proof. The span of permutation diagrams is a linear complement to J_{2n}, and is an i-invariant subalgebra of A_{2n} isomorphic to $S\mathfrak{S}_n$; hence, $A_{2n}/J_{2n} \cong S\mathfrak{S}_n$. The span of permutation diagrams π with $\pi(n) = n$ is a linear complement to J_{2n-1} in A_{2n-1}; hence $A_{2n-1}/J_{2n-1} \cong S\mathfrak{S}_{n-1}$. □

Lemma 5.40. *For $k \geqslant 2$, $J_k = A_{k-1} p_{k-1} A_{k-1}$.*

Proof. It is straightforward to check that if $x \in X_n$ has propagating number strictly less than n, then x can be factored as $x = x' p_{2n-1} x''$, with $x', x'' \in X'_n$. Likewise, if $n \geqslant 2$ and $x \in X'_n$ has propagating number strictly less than n, then x can be factored as $x = x' p_{2n-2} x''$ with $x', x'' \in X'_{n-1}$. □

Lemma 5.41.

(1) *For $k \geqslant 3$, $p_{k-1} A_{k-1} p_{k-1} = A_{k-2} p_{k-1}$.*
(2) $p_1 A_1 p_1 = \delta A_0 p_1$.
(3) *For $k \geqslant 2$, p_{k-1} commutes with A_{k-2}.*

Proof. Let $x \in A_{2n}$ with $n \geqslant 1$. Then $p_{2n-1} x p_{2n-1}$ is contained in the span of $y \in X_n$ such that $\{\boldsymbol{n}\}$ and $\{\bar{\boldsymbol{n}}\}$ are blocks of y, and any such y can be written as $y = z p_{2n-1}$, where $z \in A_{2n-2}$.

Now consider $x \in A_{2n+1}$ with $n \geqslant 1$. Then $p_{2n}xp_{2n}$ is contained in the span of $y \in X'_{n+1}$ such that $\{\boldsymbol{n}, \boldsymbol{n+1}, \bar{\boldsymbol{n}}, \overline{\boldsymbol{n+1}}\}$ is contained in one block of y. Any such y can be written as $y = zp_{2n}$ where $z \in A_{2n-1}$.

This shows that $p_{k-1}A_kp_{k-1} \subseteq A_{k-2}p_{k-1}$ for all $k \geqslant 3$. On the other hand, if $x \in A_{k-2}$ then $xp_{k-1} = xp_{k-1}p_{k-2}p_{k-1} = p_{k-1}xp_{k-2}p_{k-1} \in p_{k-1}A_kp_{k-1}$, so $p_{k-1}A_kp_{k-1} \supseteq A_{k-2}p_{k-1}$. This proves (1).

Points (2) and (3) are easy to check. $\square$

Lemma 5.42. *For $k \geqslant 2$, $A_kp_{k-1} = A_{k-1}p_{k-1}$. Moreover, $x \mapsto xe_{k-1}$ is injective from A_{k-1} to A_k.*

Proof. For $k = 2$, we have $A_2p_1 = Sp_1 = A_1p_1$. For $k \geqslant 3$, we have

$$\begin{aligned} A_kp_{k-1} &= A_kp_{k-1}p_{k-2}p_{k-1} \\ &\subseteq J_kp_{k-1} = A_{k-1}p_{k-1}A_{k-1}p_{k-1} \\ &\subseteq A_{k-1}A_{k-2}p_{k-1} = A_{k-1}p_{k-1}. \end{aligned}$$

Checking k odd and even separately, one can check that $x = \mathrm{cl}(xp_{k-1})$ for $k \geqslant 2$ and $x \in A_{k-1}$. $\square$

Lemma 5.43. *The trace ε on $A_k(\mathbb{Q}(\boldsymbol{\delta}), \boldsymbol{\delta})$ is non-degenerate.*

Proof. For any set partition $x \in X_n$, let $r(x)$ be the number of blocks of x. Let E_x be the equivalence relation on $[\boldsymbol{n}] \cup [\bar{\boldsymbol{n}}]$ whose equivalence classes are the blocks of x.

For any $x, y \in X_n$, define an integer $r(x, y)$ as follows: Let $E(x, y)$ be the smallest equivalence relation on $[\boldsymbol{n}] \cup [\bar{\boldsymbol{n}}]$ containing $E_x \cup E_{i(y)}$ and let $r(x, y)$ be the number of equivalence classes of $E(x, y)$. Clearly, $r(x, y) \leqslant \min\{r(x), r(y)\}$. Moreover, if $r(x) = r(y)$, then $r(x, y) < r(x)$ unless $y = i(x)$, and $r(x, i(x)) = r(x)$.

It is not hard to see that $\mathrm{tr}(xy) = \boldsymbol{\delta}^{r(x,y)}$, so $\varepsilon(x, y) = \boldsymbol{\delta}^{r(x,y)-n}$. It follows that the Gram determinant $\det(\varepsilon(xy))_{x,y}$ is a Laurent polynomial that has a unique term of highest degree namely $\pm \prod_x \varepsilon(xi(x))$. In particular the Gram determinant is non-zero. This shows that the trace on $A_{2n}(\mathbb{Q}(\boldsymbol{\delta}), \boldsymbol{\delta})$ is non-degenerate, and the same method shows that the restriction of the trace to $A_{2n-1}(\mathbb{Q}(\boldsymbol{\delta}), \boldsymbol{\delta})$ is non-degenerate. $\square$

Lemma 5.44. *$A_k(\mathbb{Q}(\boldsymbol{\delta}), \boldsymbol{\delta})$ is split semisimple. The branching diagram for $(A_k(\mathbb{Q}(\boldsymbol{\delta}), \boldsymbol{\delta}))_{k\geqslant 0}$ has vertices on levels $2n$ and $2n+1$ labeled by all Young diagrams of size j, $0 \leqslant j \leqslant n$. There is an edge connecting λ on level $2n$ and μ on level $2n \pm 1$ if, and only if, $\lambda = \mu$ or μ is obtained by removing one box from λ.*

Proof. This is proved by Martin [41]. It can also be proved using the method of Wenzl from [55], using Lemma 5.43. $\square$

5.7.4. Verification of framework axioms for the partition algebras

We take $R = \mathbb{Z}[\boldsymbol{\delta}]$, where $\boldsymbol{\delta}$ is an indeterminant. Then R is the universal ground ring for the partition algebras; for any commutative ring S with distinguished element δ, we have $A_k(S, \delta) \cong A_k(R, \boldsymbol{\delta}) \otimes_R S$. Let $F = \mathbb{Q}(\boldsymbol{\delta})$ denote the field of fractions of R. Write $A_k = A_k(R, \boldsymbol{\delta})$. Define $Q_{2n} = Q_{2n+1} = R\mathfrak{S}_n$.

Proposition 5.45. *The two sequence of R-algebras $(A_k)_{k\geqslant 0}$ and $(Q_k)_{k\geqslant 0}$ satisfy the framework axioms of Section* 3.1.

Proof. According to Example 2.16, $(Q_k)_{k\geqslant 0}$ is a coherent tower of cellular algebras, so axiom (1) holds. Framework axioms (2) and (3) are evident. A_k^F is split semisimple by Lemma 5.44. This verifies axiom (4).

We take $p_{k-1} \in A_k$ to be the element defined in the previous section. Then p_{k-1} is an i-invariant essential idempotent. With $J_k = A_k p_{k-1} A_k$, we have $A_k/J_k \cong Q_k$ as algebras with involution by Lemma 5.39. This verifies axiom (5).

Axiom (6) follows from Lemma 5.41, and axiom (7) from Lemma 5.42. Axiom (8) holds because $p_{n-1} p_n p_{n-1} = p_{n-1}$. $\square$

Corollary 5.46. *For any commutative ring S and for any $\delta \in S$, the sequence of partition algebras $(A_n(S,\delta))_{n\geqslant 0}$ is a coherent tower of cellular algebras. $A_n(S,\delta)$ has cell modules indexed by all Young diagrams of size j, $0 \leqslant j \leqslant n$. The cell module labeled by a Young diagram λ has a basis labeled by paths on the branching diagram for $(A_k(\mathbb{Q}(\boldsymbol{\delta}),\boldsymbol{\delta}))_{k\geqslant 0}$, described in Lemma* 5.44.

5.8. Contour algebras

We define generalizations of the *contour algebras* of Cox et al. [8], which in turn include several sorts of diagram algebras. The algebras are obtained as a sort of wreath product of the Jones–Temperley–Lieb algebras with some other algebra A with involution; varying A gives a wide variety of examples.

5.8.1. Definition of contour algebras

Let S be a commutative ring with distinguished element δ. Let A be an S-algebra with involution i and with a unital S-valued trace ε. We first define the A-Temperley–Lieb algebras $T_n(A)$ and then the contour algebras $C_n^d(A)$ as subalgebras of $T_n(A)$. In case we need to emphasize the ground ring S and parameter δ, we write $C_n^d(A,S,\delta)$.

An A-Temperley–Lieb diagram is a Temperley–Lieb (TL) diagram with strands labeled by elements of A. For convenience, we adopt the convention that an unlabeled strand is the same as a strand labeled with the identity of A.

We will define the product of two A-Temperley–Lieb diagrams. First we note that ordinary TL diagrams have an inherent orientation. Label the top vertices of a TL diagram by $\mathbf{1},\dots,\boldsymbol{n}$ and the bottom vertices by $\bar{\mathbf{1}},\dots,\bar{\boldsymbol{n}}$. Place a small arrow pointing down at each odd numbered vertex (top or bottom) and a small arrow pointing up at each even numbered vertex. Then because of the planarity of TL diagrams, each strand of a TL diagram must connect one arrow pointing into the rectangle $\mathscr{R}$ of the diagram with one arrow pointing out of $\mathscr{R}$; the strand can be thought of as oriented from the inward pointing arrow to the outward pointing arrow. When two TL diagrams are multiplied by stacking, the orientation of composed strands agrees.

Now consider two A-Temperley–Lieb diagrams X and Y. To form the product XY, stack Y over X as for tangles, forming a composite diagram $X \circ Y$. Label each non-closed composite strand with the product of the labels of its component strands from X and Y, taken in the order of their occurrence as the strand is traversed according to its orientation. For each closed strand s in $X \circ Y$, let $\varepsilon(s)$ be the trace of the product of the labels of its component strands; the product is unique up to cyclic permutation of the factors, so the trace is uniquely determined. Let r be the number of closed strands and let Z be the labeled diagram obtained by removing all the closed strands. Then $XY = \delta^r(\prod_s \varepsilon(s))Z$.

As an S-module, $T_n(A)$ is $A^{\otimes n} \otimes T_n(S,\delta) = \bigoplus_x (A^{\otimes n} \otimes x)$, where the sum is over ordinary Temperley–Lieb diagrams x. We identify a simple tensor $a_1 \otimes \dots \otimes a_n \otimes x$ with a labeling of x with the labels $a_1,\dots,a_n$. We have to specify how to place the labels. We fix an ordering of the vertices, for example $\mathbf{1} < \dots < \boldsymbol{n} < \bar{\mathbf{1}} < \dots < \bar{\boldsymbol{n}}$, and then order the strands of x according to the order of the initial vertex of each (oriented) strand. The simple tensor $a_1 \otimes \dots \otimes a_n \otimes x$ is identified with the diagram with underlying TL diagram x, with the j-th strand of x labeled by a_j for each j.

Fix TL diagrams x and y. The product of A-Temperley–Lieb diagrams with underlying TL diagrams x and y, defined above, determines a multilinear map $A^{2n} \to A^{\otimes n} \otimes xy$, and hence a bilinear map $(A^{\otimes n} \otimes x) \times (A^{\otimes n} \otimes y) \to A^{\otimes n} \otimes xy$. This product extends to a bilinear product on $T_n(A)$, which one can check to be associative.

Next we define an involution on $T_n(A)$. Define i on an A-labeled TL diagram by flipping the diagram over the line $y = 1/2$ and applying the involution in A to the label of each strand. For a fixed TL diagram x, this gives a multilinear map from A^n to $A^{\otimes n} \otimes i(x)$, and hence a linear map from $A^{\otimes n} \otimes x$ to $A^{\otimes n} \otimes i(x)$. Now i extends to a linear map on $T_n(A)$. One can check that i is an algebra involution.

This completes the definition of the A-Temperley–Lieb algebra, as an algebra with involution.

Next we define the A-contour algebras. We assign a depth to each strand in an ordinary TL diagram x, as follows: Draw a curve from a point on a given strand s to the western boundary of $\mathscr{R}$, having only transverse intersections with any strands of x. The depth of s is the minimum, over all such curves γ, of the number of points of intersection of γ with the strands of x (including s). The depth of an A-labeled TL diagram is the maximum depth of the strands with non-identity labels.

Fix $d \leqslant n$. As an S-module $C_n^d(A)$ is the span of those A-labeled TL diagrams of depth no greater than d. It is easy to check as in [8] Lemma 2.1 that $C_n^d(A)$ is an i-invariant subalgebra of $T_n(A)$.

For $a \in A$ and $1 \leqslant j \leqslant n$ let $a^{(j)}$ be the identity TL diagram in $T_n(A)$ with the j-th strand labeled with a (and the other strands unlabeled). We have $a^{(j)}$ and $b^{(k)}$ commute if $j \neq k$. Also $a^{(j)}$ commutes with e_k unless $j \in \{k, k+1\}$ and $e_k a^{(k)} = e_k a^{(k+1)}$, and, likewise, $a^{(k)} e_k = a^{(k+1)} e_k$. Note that $a \mapsto a^{(k)}$ is an algebra homomorphism if k is odd, but an algebra anti-homomorphism if k is even.

Lemma 5.47. *$C_n^d(A)$ is generated as an algebra by $e_1, \ldots, e_{n-1}$ and by $\{a^{(k)}\colon 1 \leqslant k \leqslant d\}$.*

Sketch. It is enough to show that if x is a Temperley–Lieb diagram and $X = xa^{(k)}$ has depth r, then X can be rewritten as a product of $a^{(r)}$ and TL diagrams. First one can check that X can be written as $x_1 x_2 a^{(k')} x_3$ where the x_i are TL diagrams, x_2 is a product of commuting e_i's, and the depth of $x_2 a^{(k')}$ is r. Finally, it suffices to show that $x_2 a^{(k')}$ can be written as a product of TL diagrams with $a^{(r)}$. We give an example that captures the idea: $e_1 e_3 a^{(6)}$ has depth 2. We have

$$\begin{aligned} e_1 e_3 a^{(6)} &= (e_1 e_3)(e_2 e_4)(e_1 e_3) a^{(6)} \\ &= (e_1 e_3)(e_2 e_4)(e_3 e_5)(e_2 e_4)(e_1 e_3) a^{(6)} \\ &= (e_1 e_3)(e_2 e_4)(e_3 e_5) a^{(2)} (e_2 e_4)(e_1 e_3), \end{aligned}$$

by repeated use of the relations listed before the statement of the lemma. $\square$

5.8.2. Brief history of contour algebras

The contour algebras introduced by Cox et al. [8] are the special case with A the group algebra of the cyclic group $\mathbb{Z}_m$. On the other hand, the A-Temperley–Lieb algebras $T_n(A)$ have been considered in [31], Example 2.2. The contour subalgebras of $T_n(A)$ were discussed in [25].

5.8.3. Some properties of A-Temperley–Lieb and contour algebras

We deal with the contour algebras and the A-Temperley–Lieb algebras together; regard $T_n(A)$ as $C_n^\infty(A)$.

We define maps $\iota : C_n^d(A) \to C_{n+1}^d(A)$ as for other classes of diagram or tangle algebras, and likewise maps $\mathrm{cl} : C_n^d(A) \to C_{n-1}^d(A)$; if closing the rightmost strand of an A-Temperley–Lieb diagram produces a closed loop, remove the loop and multiply the resulting diagram by δ times the trace of the product of labels along the loop. The map ι is injective, since $x = \mathrm{cl}(\iota(x) e_n)$ for $x \in C_n^d(A)$. The maps ι and cl commute with the involutions.

If δ is invertible in S, we can define $\varepsilon_n = (1/\delta)\,\mathrm{cl} : C_n^d(A) \to C_{n-1}^d(A)$, which is a unital conditional expectation. We have $\varepsilon_{n+1} \circ \iota(x) = x$ for $x \in C_n^d(A)$. The map $\varepsilon = \varepsilon_1 \circ \cdots \circ \varepsilon_n : C_n^d(A) \to C_0^d(A) \cong S$ is a normalized trace. The value of ε on an A-Temperley–Lieb diagram X with n strands is obtained as follows: first close all the strands of X by introducing new curves joining $\boldsymbol{j}$ to $\bar{\boldsymbol{j}}$ for all j; let r be the number of closed loops in the resulting diagram; then $\varepsilon(X) = \delta^{r-n} \prod_s \varepsilon(s)$, where the product is over the collection of closed loops s, and $\varepsilon(s)$ denotes the trace in A of the product of labels along the loop s.

The span J of A-Temperley–Lieb diagrams of depth $\leqslant d$ and with at least one horizontal strand is an ideal in $C_n^d(A)$. By Lemma 5.47, any A-Temperley–Lieb diagram with depth $\leqslant d$ can be written as a word in the e_i's and in elements $a^{(k)}$ with $k \leqslant d$; the diagram is in J if, and only if, some e_i appears in the word. Thus J is the ideal generated by the e_i's. Because of the relations $e_i e_{i\pm 1} e_i = e_i$, J is generated by e_{n-1}. The quotient $C_n^d(A)/J$ is isomorphic (as algebras with involution) to the subalgebra generated by the $a^{(k)}$ with $k \leqslant d$, and thus to $A^{\otimes d}$ if $n \geqslant d$ and $A^{\otimes n}$ if $n < d$.

Lemma 5.48.

(1) *For $n \geqslant 3$, $e_{n-1} C_{n-1}^d(A) e_{n-1} = C_{n-2}^d(A) e_{n-1}$.*
(2) *$e_1 C_1^d(A) e_1 = \delta S e_1$.*
(3) *For $n \geqslant 2$, e_{n-1} commutes with $C_{n-2}^d(A)$.*

Proof. The proof is the same as that of Lemma 5.2 for the Brauer algebras. □

Lemma 5.49. *For $n \geqslant 2$, $C_n^d(A) e_{n-1} = C_{n-1}^d(A) e_{n-1}$. Moreover, $x \mapsto x e_{n-1}$ is injective from $C_{n-1}^d(A)$ to $C_{n-1}^d(A) e_{n-1}$.*

Proof. Any A-TL diagram in $C_n^d(A)$ is either already in $C_{n-1}^d(A)$, or it can be written as $\alpha\chi\beta$, with $\alpha, \beta \in C_{n-1}^d(A)$, and $\chi \in \{e_{n-1}, a^{(n)}\}$ if $n \leqslant d$, or $\chi = e_{n-1}$ if $n > d$.

The remainder of the proof is the same as the proof of Lemma 5.3 for the Brauer algebras, using the identities: $a^{(n)} x e_{n-1} = x a^{(n-1)} e_{n-1}$, and $e_{n-1} x e_{n-1} = \mathrm{cl}(x) e_{n-1}$ for $x \in C_{n-1}^d(A)$. □

5.8.4. Hypotheses on the algebra A

We will suppose that the algebra A has a generic version defined over an integral domain R_0. Let F_0 be the field of fractions of R_0. We suppose that $A = A(R_0)$ satisfies the following hypotheses:

(1) $A = A(R_0)$ is cellular.
(2) $A(F_0) = A(R_0) \otimes_{R_0} F_0$ is split semisimple.
(3) The trace ε on $A(R_0)$ is non-degenerate.

We take $R = R_0[\boldsymbol{\delta}]$, where $\boldsymbol{\delta}$ is an indeterminant, and let $F = F_0(\boldsymbol{\delta})$ denote the field of fractions of R. We will show that $(C_n^d(A, R, \boldsymbol{\delta}))_{n \geqslant 0}$ is a coherent tower of cellular algebras.

5.8.5. Special instances

The cellular algebra A in Section 5.8.4 can be taken to be the generic version of any of the diagram or tangle algebras treated in this paper. A could be taken to be a generic Hecke algebra or cyclotomic Hecke algebra, or the group ring of a symmetric group over $R_0 = \mathbb{Z}$.

The contour algebras of Cox et al. [8] are recovered by taking $R_0 = \mathbb{Z}[\boldsymbol{\delta}_1, \dots, \boldsymbol{\delta}_{m-1}]$ and A the group algebra of $\mathbb{Z}_m$ over R_0. The trace on A is determined by $\varepsilon([k]) = \boldsymbol{\delta}_k$ for $[k] \neq [0]$ and $\varepsilon([0]) = 1$. The parameter δ_0 in [8] becomes identified with our δ.

5.8.6. Verification of the framework axioms for contour algebras

Adopt the hypotheses and notation of Section 5.8.4.

Lemma 5.50. *The trace ε on $C_n^d(A, F, \boldsymbol{\delta})$ is non-degenerate.*

Proof. We take any basis $\mathbb{A}$ of A over F_0 with $\mathbf{1} \in \mathbb{A}$. As a basis $\mathbb{B}$ of $C_n^d(A)$ over F we take all n-strand TL diagrams decorated up to depth d with elements of $\mathbb{A}$. We consider the modified Gram determinant $\det[\varepsilon(Xi(Y))]_{X,Y \in \mathbb{B}}$. If X and Y have different underlying TL diagrams, then $\varepsilon(Xi(Y)) \in \delta^{-1} F_0$.

Next consider matrix entries $\varepsilon(Xi(Y))$ where X and Y have the same underlying TL diagram, say x. Suppose x has ℓ strands at depth d or less and these strands are decorated by basis elements $a_1, \dots, a_\ell$ in X, respectively $b_1, \dots, b_\ell$ in Y. Then $\varepsilon(Xi(Y)) = \prod_{j=1}^{\ell} \varepsilon(a_j i(b_j))$. The determinant of the square submatrix of $[\varepsilon(Xi(Y))]$ consisting of those entries for which X and Y both have underlying TL diagram x is therefore D^ℓ, where D is the determinant of $[\varepsilon(ai(b))]_{a,b\in\mathbb{A}}$. It follows that $\det[\varepsilon(Xi(Y))]_{X,Y\in\mathbb{B}}$ is equal to a power of D modulo $\delta^{-1}R_0$, and is therefore non-zero. □

Consider

$$Q_n = C_n^d(A)/J \cong \begin{cases} A^{\otimes n} & \text{if } n < d, \\ A^{\otimes d} & \text{if } n \geqslant d. \end{cases}$$

By the assumptions in Section 5.8.4, $Q_n(R)$ is cellular and $Q_n(F)$ is split semisimple. Moreover, it is easy to see that $(Q_n)_{n\geqslant 0}$ is a coherent tower of cellular algebras.

Lemma 5.51. *$C_n^d(A, F, \boldsymbol{\delta})$ is split semisimple for all n.*

Proof. The method of Wenzl from [55] applies, using the non-degeneracy of the trace and the split semisimplicity of $Q_n(F)$ for all n. □

Proposition 5.52. *The pair of sequences $(C_n^d(A, R, \boldsymbol{\delta}))_{n\geqslant 0}$ and $(Q_n(R))_{n\geqslant 0}$ satisfy the framework axioms of Section* 3.1. *Hence, $(C_n^d(A, R, \boldsymbol{\delta}))_{n\geqslant 0}$ is a coherent tower of cellular algebras.*

Proof. We observed above that $(Q_k)_{k\geqslant 0}$ is a coherent tower of cellular algebras, so axiom (1) holds. Framework axioms (2) and (3) are evident. Framework axiom (4) follows from Lemma 5.51.

The elements e_k are i-invariant essential idempotents. With $J = C_k^d(A)e_{k-1}C_k^d(A)$, we have $C_k^d(A)/J \cong Q_k$ as algebras with involution. This verifies axiom (5). Axiom (6) follows from Lemma 5.48, and axiom (7) from Lemma 5.49. Axiom (8) holds because $e_{n-1}e_ne_{n-1} = e_{n-1}$. □

Acknowledgments

Part of this work was done while both authors were visiting MSRI in 2008. We are grateful to the organizers of the program in Combinatorial Representation Theory and to the staff at MSRI for a pleasant and stimulating visit. We thank the referees for helpful suggestions which resulted in several improvements.

References

[1] Susumu Ariki, Representations of Quantum Algebras and Combinatorics of Young Tableaux, Univ. Lecture Ser., vol. 26, American Mathematical Society, Providence, RI, 2002, translated from the 2000 Japanese edition and revised by the author.
[2] Susumu Ariki, Kazuhiko Koike, A Hecke algebra of $(\boldsymbol{Z}/r\boldsymbol{Z}) \wr \mathfrak{S}_n$ and construction of its irreducible representations, Adv. Math. 106 (1994) 216–243.
[3] Susumu Ariki, Andrew Mathas, The number of simple modules of the Hecke algebras of type $G(r,1,n)$, Math. Z. 233 (2000) 601–623.
[4] Susumu Ariki, Andrew Mathas, Rui Hebing, Cyclotomic Nazarov–Wenzl algebras, Nagoya Math. J. 182 (2006) 47–134.
[5] Georgia Benkart, Manish Chakrabarti, Thomas Halverson, Robert Leduc, Chanyoung Lee, Jeffrey Stroomer, Tensor product representations of general linear groups and their connections with Brauer algebras, J. Algebra 166 (1994) 529–567.
[6] Joan S. Birman, Hans Wenzl, Braids, link polynomials and a new algebra, Trans. Amer. Math. Soc. 313 (1989) 249–273.
[7] Richard Brauer, On algebras which are connected with the semisimple continuous groups, Ann. of Math. (2) 38 (1937) 857–872.
[8] Anton Cox, Paul Martin, Alison Parker, Changchang Xi, Representation theory of towers of recollement: theory, notes, and examples, J. Algebra 302 (2006) 340–360.
[9] Anton Cox, Maud De Visscher, Stephen Doty, Paul Martin, On the blocks of the walled Brauer algebra, J. Algebra 320 (2008) 169–212.
[10] Richard Dipper, Gordon James, Representations of Hecke algebras of general linear groups, Proc. Lond. Math. Soc. (3) 52 (1986) 20–52.
[11] Richard Dipper, Gordon James, Blocks and idempotents of Hecke algebras of general linear groups, Proc. Lond. Math. Soc. (3) 54 (1987) 57–82.
[12] Richard Dipper, Gordon James, Andrew Mathas, Cyclotomic q-Schur algebras, Math. Z. 229 (1998) 385–416.

[13] William F. Doran IV, David B. Wales, Philip J. Hanlon, On the semisimplicity of the Brauer centralizer algebras, J. Algebra 211 (1999) 647–685.
[14] John Enyang, Cellular bases for the Brauer and Birman–Murakami–Wenzl algebras, J. Algebra 281 (2004) 413–449.
[15] John Enyang, Specht modules and semisimplicity criteria for Brauer and Birman–Murakami–Wenzl algebras, J. Algebraic Combin. 26 (2007) 291–341.
[16] Frederick M. Goodman, Cellularity of cyclotomic Birman–Wenzl–Murakami algebras, J. Algebra 321 (2009) 3299–3320, Special issue in honor of Gus Lehrer.
[17] Frederick M. Goodman, Comparison of admissibility conditions for cyclotomic Birman–Wenzl–Murakami algebras, J. Pure Appl. Algebra 214 (11) (2010) 2009–2016.
[18] Frederick M. Goodman, Pierre de la Harpe, Vaughan F.R. Jones, Coxeter Graphs and Towers of Algebras, Math. Sci. Res. Inst. Publ., vol. 14, Springer-Verlag, New York, 1989.
[19] Frederick M. Goodman, John Graber, On cellular algebras with Jucys–Murphy elements, preprint, 2009.
[20] Frederick M. Goodman, Holly Hauschild, Affine Birman–Wenzl–Murakami algebras and tangles in the solid torus, Fund. Math. 190 (2006) 77–137.
[21] Frederick M. Goodman, Holly Hauschild Mosley, Cyclotomic Birman–Wenzl–Murakami algebras, I, Freeness and realization as tangle algebras, J. Knot Theory Ramifications 18 (2009) 1089–1127.
[22] Frederick M. Goodman, Holly Hauschild Mosley, Cyclotomic Birman–Wenzl–Murakami algebras, II, Admissibility relations and freeness, Algebr. Represent. Theory, doi:10.1007/s10468-009-9173-2, in press.
[23] J.J. Graham, G.I. Lehrer, Cellular algebras, Invent. Math. 123 (1996) 1–34.
[24] Richard M. Green, Paul Martin, Constructing cell data for diagram algebras, J. Pure Appl. Algebra 209 (2007) 551–569.
[25] Richard M. Green, Paul Martin, Constructing cell data for diagram algebras, J. Pure Appl. Algebra 209 (2007) 551–569.
[26] Tom Halverson, Arun Ram, Partition algebras, European J. Combin. 26 (2005) 869–921.
[27] Reinhard Häring-Oldenburg, Cyclotomic Birman–Murakami–Wenzl algebras, J. Pure Appl. Algebra 161 (2001) 113–144.
[28] Robert Hartmann, Rowena Paget, Young modules and filtration multiplicities for Brauer algebras, Math. Z. 254 (2006) 333–357.
[29] David J. Hemmer, Daniel K. Nakano, Specht filtrations for Hecke algebras of type A, J. Lond. Math. Soc. (2) 69 (2004) 623–638.
[30] Nathan Jacobson, Basic Algebra, II, second ed., W.H. Freeman and Company, New York, 1989.
[31] Vaughan F.R. Jones, Planar algebras, I, unpublished manuscript, arXiv:math/9909027.
[32] Vaughan F.R. Jones, Index for subfactors, Invent. Math. 72 (1983) 1–25.
[33] Vaughan F.R. Jones, A polynomial invariant for knots via von Neumann algebras, Bull. Amer. Math. Soc. (N.S.) 12 (1985) 103–111.
[34] Vaughan F.R. Jones, The Potts model and the symmetric group, in: Subfactors, Kyuzeso, 1993, World Sci. Publ., River Edge, NJ, 1994, pp. 259–267.
[35] Thomas Jost, Morita equivalence for blocks of Hecke algebras of symmetric groups, J. Algebra 194 (1997) 201–223.
[36] Louis H. Kauffman, An invariant of regular isotopy, Trans. Amer. Math. Soc. 318 (1990) 417–471.
[37] Kazuhiko Koike, On the decomposition of tensor products of the representations of the classical groups: by means of the universal characters, Adv. Math. 74 (1989) 57–86.
[38] Steffen König, Changchang Xi, On the structure of cellular algebras, in: Algebras and Modules, II, Geiranger, 1996, in: CMS Conf. Proc., vol. 24, American Mathematical Society, Providence, RI, 1998, pp. 365–386.
[39] Steffen König, Changchang Xi, Cellular algebras: inflations and Morita equivalences, J. Lond. Math. Soc. (2) 60 (1999) 700–722.
[40] Steffen König, Changchang Xi, A characteristic free approach to Brauer algebras, Trans. Amer. Math. Soc. 353 (2001) 1489–1505 (electronic).
[41] Paul Martin, Temperley–Lieb algebras for nonplanar statistical mechanics—the partition algebra construction, J. Knot Theory Ramifications 3 (1994) 51–82.
[42] Paul Martin, The structure of the partition algebras, J. Algebra 183 (1996) 319–358.
[43] Paul Martin, The partition algebra and the Potts model transfer matrix spectrum in high dimensions, J. Phys. A 33 (2000) 3669–3695.
[44] Andrew Mathas, Iwahori–Hecke Algebras and Schur Algebras of the Symmetric Group, Univ. Lecture Ser., vol. 15, American Mathematical Society, Providence, RI, 1999.
[45] Andrew Mathas, Seminormal forms and Gram determinants for cellular algebras, J. Reine Angew. Math. 619 (2008) 141–173, with an appendix by Marcos Soriano.
[46] Andrew Mathas, A Specht filtration of an induced Specht module, J. Algebra 322 (2009) 893–902, Special issue in honor of John Cannon and Derek Holt.
[47] Hugh Morton, Paweł Traczyk, Knots and algebras, in: E. Martin-Peindador, A. Rodez Usan (Eds.), Contribuciones Matematicas en homenaje al profesor D. Antonio Plans Sanz de Bremond, University of Zaragoza, Zaragoza, 1990, pp. 201–220.
[48] Hugh Morton, Antony Wassermann, A basis for the Birman-Wenzl algebra, unpublished manuscript, 1989, revised 2000, pp. 1–29.
[49] Jun Murakami, The Kauffman polynomial of links and representation theory, Osaka J. Math. 24 (1987) 745–758.
[50] G.E. Murphy, The representations of Hecke algebras of type A_n, J. Algebra 173 (1995) 97–121.
[51] P.P. Nikitin, A description of the commutant of the action of the group $\mathrm{GL}_n(\mathbb{C})$ in mixed tensors, Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI) 331 (2006), Teor. Predst. Din. Sist. Komb. i Algoritm. Metody 14, pp. 170–198, 224; translation in J. Math. Sci. (N. Y.) 141 (4) (2007) 1479–1493.
[52] Hebing Rui, Mei Si, The representation theory of cyclotomic BMW algebras, II, Algebr. Represent. Theory, in press.
[53] Hebing Rui, Jie Xu, The representations of cyclotomic BMW algebras, J. Pure Appl. Algebra 213 (2009) 2262–2288.

[54] V.G. Turaev, Operator invariants of tangles, and R-matrices, Izv. Akad. Nauk SSSR Ser. Mat. 53 (1989) 1073–1107, 1135; translation in Math. USSR-Izv. 35 (2) (1990) 411–444.
[55] Hans Wenzl, On the structure of Brauer's centralizer algebras, Ann. of Math. (2) 128 (1988) 173–193.
[56] Hans Wenzl, Quantum groups and subfactors of type B, C, and D, Comm. Math. Phys. 133 (1990) 383–432.
[57] Stewart Wilcox, Cellularity of diagram algebras as twisted semigroup algebras, J. Algebra 309 (2007) 10–31.
[58] Stewart Wilcox, Shona Yu, The cyclotomic BMW algebra associated with the two string type B braid group, Comm. Algebra, in press.
[59] Stewart Wilcox, Shona Yu, On the cellularity of the cyclotomic Birman–Murakami–Wenzl algebras, J. London Math. Soc., in press.
[60] Stewart Wilcox, Shona Yu, On the freeness of the cyclotomic BMW algebras: admissibility and an isomorphism with the cyclotomic Kauffman tangle algebras, preprint, 2009.
[61] Changchang Xi, Partition algebras are cellular, Compos. Math. 119 (1999) 99–109.
[62] Changchang Xi, On the quasi-heredity of Birman–Wenzl algebras, Adv. Math. 154 (2000) 280–298.
[63] Shona Yu, The cyclotomic Birman–Murakami–Wenzl algebras, PhD thesis, University of Sydney, 2007.