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On Cantor sets and doubling measures

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ABSTRACT

For a large class of Cantor sets on the real-line, we find sufficient and necessary conditions implying that a set has positive (resp. null) measure for all doubling measures of the real-line. We also discuss same type of questions for atomic doubling measures defined on certain midpoint Cantor sets.

1. Introduction and notation

Our main goal in this paper is to study the size of Cantor sets on the real-line $\mathbb{R}$ from the point of view of doubling measures. Recall that a measure μ on a metric space X is called *doubling* if there is a constant $c < \infty$ such that

$$0 < \mu(B(x, 2r)) \leq c\mu(B(x, r)) < \infty$$

for all $x \in X$ and $r > 0$. Here $B(x, r)$ is the open ball with centre $x \in X$ and radius $r > 0$. We note that the collection of doubling measures on $\mathbb{R}$, and more generally, on any complete doubling metric space where isolated points are not dense, is rather rich. For instance, given $\varepsilon > 0$, there are doubling measures on $\mathbb{R}$ having full measure on a set of Hausdorff and packing dimension at most ε. See [1–4].

Let $\mathcal{D}(\mathbb{R})$ be the collection of all doubling measures on $\mathbb{R}$ and denote

$$\mathcal{T} = \{C \subset \mathbb{R} : \mu(C) = 0 \text{ for all } \mu \in \mathcal{D}(\mathbb{R})\},$$
$$\mathcal{F} = \{C \subset \mathbb{R} : \mu(C) > 0 \text{ for all } \mu \in \mathcal{D}(\mathbb{R})\}.$$

In the literature, the sets in $\mathcal{F}$ have been called quasisymmetrically thick [1,5], thick for doubling measures [6], and very fat [7] and those in $\mathcal{T}$ have been termed quasisymmetrically null [1,5], null for doubling measures [6], and thin [7]. We call $C \subset \mathbb{R}$ *thin* if $C \in \mathcal{T}$ and *fat* if $C \in \mathcal{F}$.

In this paper, we address the problems of finding sufficient and/or necessary conditions for a Cantor set $C \subset \mathbb{R}$ to be fat (resp. thin). These problems arise naturally from the study of compression and expansion properties of quasisymmetric maps $f : \mathbb{R} \to \mathbb{R}$; see [1, 13.20]. A related problem is to characterise those subsets $U \subset \mathbb{R}$ which carry nontrivial doubling measures [8, Open problem 1.18]; if $C \subset \mathbb{R}$ is a fat Cantor set, then it is easy to see that $U = \mathbb{R} \setminus C$ does not carry nontrivial doubling measures. For if it did, then one could extend any doubling measure μ on U to $\mathbb{R}$ by letting $\mu(C) = 0$, and this would contradict C being fat.

We begin by discussing thinness and fatness for the middle interval Cantor sets $C(\alpha_n)$ determined via sequences $(\alpha_n)_{n=1}^{\infty}$, $0 < \alpha_n < 1$, as follows. We first remove an open interval of length α_1 from the middle of $I_{1,1} = [0, 1]$ and denote the

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remaining two intervals by $I_{2,1}$ and $I_{2,2}$. At the kth step, $k \geq 2$, we have 2^{k-1} intervals $I_{k,1}, \ldots, I_{k,2^{k-1}}$ of length $\ell_k = 2^{-k+1}\prod_{n=1}^{k-1}(1-\alpha_n)$ and we remove an interval of length $\alpha_k \ell_k$ from the middle of each $I_{k,i}$. Finally, the *middle interval Cantor set* $C = C(\alpha_n)$ is defined by

$$C = \bigcap_{k\in\mathbb{N}} \bigcup_{i=1}^{2^k} I_{k,i}.$$

The theorem below follows by combining results of Wu [9, Theorem 1], Staples and Ward [5, Theorem 1.4], and Buckley et al. [7, Theorem 0.3]. For $0 < p < \infty$, we denote by ℓ^p the set of all sequences $(\alpha_n)_{n=1}^{\infty}$, $0 < \alpha_n < 1$, for which $\sum_{n=1}^{\infty} \alpha_n^p < \infty$.

Theorem 1.1. *Let $C = C(\alpha_n)$. Then*

(1) *C is thin if and only if $(\alpha_n) \notin \bigcup_{0<p<\infty} \ell^p$.*
(2) *C is fat if and only if $(\alpha_n) \in \bigcap_{0<p<\infty} \ell^p$.*

In a recent paper, Han et al. [6] generalised Theorem 1.1 for a broader collection of (still very symmetric) Cantor sets. Related results on thin and fat sets may be found in [3,5,7,9–12].

The known proofs for Theorem 1.1 and its generalisation in [6] rely heavily on the symmetries of the sets $C(\alpha_n)$. In this paper, we wish to consider analogues of Theorem 1.1 for Cantor sets with much less symmetry. To be more precise, we introduce the following notation. Suppose that for each $n \in \mathbb{N}$, we have a collection of closed intervals $\mathcal{I}_n = \{I_{n,i}\}_i$ with mutually disjoint interiors and open intervals $\mathcal{J}_n = \{J_{n,i} \subset I_{n,i}\}$ such that each $I_{n+1,i}$ is a subset of some $I_{n,j}$, $\bigcup \mathcal{I}_{n+1} = \bigcup \mathcal{I}_n \setminus \bigcup \mathcal{J}_n$ and that $\sup_j |I_{n,j}| \to 0$ as $n \to \infty$. We also assume that $\bigcup \mathcal{I}_1$ is bounded. We refer to $\{\mathcal{I}_n, \mathcal{J}_n\}_n$ as a *Cantor construction*. The resulting *Cantor set* is given by

$$C = C_{\{\mathcal{I}_n,\mathcal{J}_n\}} = \bigcap_n \bigcup_i I_{n,i}.$$

Given the collections $\mathcal{I}_n$ and $\mathcal{J}_n$ as above, we also denote $\mathcal{I} = \bigcup_n \mathcal{I}_n$ and $\mathcal{J} = \bigcup_n \mathcal{J}_n$. If there exists $0 < c < 1$ so that $cI_{n,i} \bigcap J_{n,i} \neq \emptyset$ for all $I_{n,i}$, we say that our Cantor construction (and set) is *nice*.[1]Here $cI_{n,i}$ denotes the interval concentric with $I_{n,i}$ and with length $c|I_{n,i}|$. Furthermore, given a sequence $0 < \alpha_n < 1$, we say that the Cantor set $C = C_{\{\mathcal{I}_n,\mathcal{J}_n\}_n}$ is *(α_n)-porous* if $|J_{n,i}| \geq \alpha_n |I_{n,i}|$ for all $I_{n,i} \in \mathcal{I}_n$ and *(α_n)-thick*, if $|J_{n,i}| \leq \alpha_n |I_{n,i}|$ for all $I_{n,i}$. Finally, C is called *(α_n)-regular* if $\lambda\alpha_n|I_{n,i}| \leq |J_{n,i}| \leq \Lambda\alpha_n|I_{n,i}|$ for all $I_{n,i}$ (here $0 < \lambda \leq \Lambda < \infty$ are constants that do not depend on n nor i). We underline that these definitions do not refer only to the set C but also to the construction of C via $\{\mathcal{I}_n, \mathcal{J}_n\}_n$.

Remarks 1.2. (a) Using our notation, it is possible that a Cantor set C contains isolated points as some of the intervals $I_{n,i}$ could be degenerated. We allow this for technical reasons although in most interesting cases, e.g. if C is nice, the set C is a true Cantor set in the sense that it has no isolated points.

(b) Observe that in our definitions, we do not impose any conditions on the number or relative size of the intervals $I_{n+1,j} \subset I_{n,i}$. Note also that $I_{n+1,i} \in \mathcal{I}_{n+1}$ does not have to be a component of any $I_{n,j} \setminus J_{n,j}$.

(c) We formulate our results for Cantor sets, but it is reasonable to speak about (α_n)-porosity and (α_n)-thickness for general subsets of $\mathbb{R}$ and not only for the ones obtained from Cantor constructions. Roughly speaking, $A \subset \mathbb{R}$ is (α_n)-porous if it is contained in an (α_n)-porous Cantor set and (α_n)-thick, if it contains an (α_n)-thick Cantor sets. See [3,5] for more details. In Section 4 we provide a notion of (α_n)-porosity which is useful in any metric space.

Our main result concerning doubling measures and Cantor sets is the following theorem.

Theorem 1.3. *Suppose that $C = C_{\{\mathcal{I}_n,\mathcal{J}_n\}}$ is a nice Cantor set. Then, for each $0 < p < \infty$, there is $\mu \in \mathcal{D}(\mathbb{R})$ and $0 < \lambda \leq \Lambda < \infty$ so that*

$$\lambda \left(\frac{|J_{n,i}|}{|I_{n,i}|}\right)^p \leq \frac{\mu(J_{n,i})}{\mu(I_{n,i})} \leq \Lambda \left(\frac{|J_{n,i}|}{|I_{n,i}|}\right)^p \tag{1.1}$$

for each $I_{n,i}$.

Remark 1.4. This result is interesting already for the middle interval Cantor sets $C(\alpha_n)$. After the submission of this paper, we were informed that for uniform Cantor sets, the result has been proved independently by Peng and Wen. See [13] for the precise formulation of their result.

Let us now discuss what can be said about the validity of Theorem 1.1 for the general Cantor sets $C_{\{\mathcal{I}_n,\mathcal{J}_n\}}$. Observe that Theorem 1.1 includes the following four statements:

[1] Geometrically, this only means that if the removed holes $J_{n,i}$ are small, then they cannot lie too close to the boundary of $I_{n,i}$.

(I) If $(\alpha_n) \notin \bigcup_{0<p<\infty} \ell^p$, then $\mu(C) = 0$ for all $\mu \in \mathcal{D}(\mathbb{R})$.
(II) If $(\alpha_n) \in \bigcup_{0<p<\infty} \ell^p$, then there is $\mu \in \mathcal{D}(\mathbb{R})$ with $\mu(C) > 0$.
(III) If $(\alpha_n) \in \bigcap_{0<p<\infty} \ell^p$, then $\mu(C) > 0$ for all $\mu \in \mathcal{D}(\mathbb{R})$.
(IV) If $(\alpha_n) \notin \bigcap_{0<p<\infty} \ell^p$, then there is $\mu \in \mathcal{D}(\mathbb{R})$ so that $\mu(C) = 0$.

The Claim (I) holds for general (α_n)-porous sets $A \subset \mathbb{R}$ as shown by Wu [3, Theorem 1]. In fact, her result remains true in all metric spaces. We provide a simple proof in Lemma 4.1. The Claim (III) is a special case of a more general result of Staples and Ward [5, Theorem 1.4]. They proved that if $C \subset \mathbb{R}$ is (α_n)-thick for some $(\alpha_n) \in \bigcap_{0<p<\infty} \ell^p$, then C is fat.

Our new results in Section 2 deal with the Claims (IV) and (II). These are the claims whose earlier proofs rely on the symmetries of $C(\alpha_n)$. We show that if $\sum_{n=1}^{\infty} \alpha_n^p = \infty$ for some $p > 0$, and C is a nice (α_n)-porous Cantor set, then $C \notin \mathcal{F}$. On the other hand, if there is $p < \infty$ with $\sum_{n=1}^{\infty} \alpha_n^p < \infty$, and if C is a nice (α_n)-thick Cantor set, then $C \notin \mathcal{T}$. Putting all these results together, we arrive at a complete analogue of Theorem 1.1 for nice (α_n)-regular Cantor sets $C \subset \mathbb{R}$. The proofs of our results in Section 2 are all based on the Theorem 1.3.

In the last part of the paper in Section 5, we discuss *purely atomic* doubling measures. Recall that a measure μ on a metric space X is called *purely atomic*, if there is a countable set $F \subset X$ so that $\mu(X \setminus F) = 0$. Purely atomic doubling measures have reached some attention recently, see e.g. [3,10,11,14].

Denote by F_X the set of isolated points of a metric space X and let $E_X = X \setminus F_X$. If E_X is nowhere dense, it is reasonable to ask if there are purely atomic doubling measures on X and on the other hand, what conditions guarantee that all doubling measures on X are purely atomic. We will treat these questions for a class of metric spaces obtained by adding the midpoints of the intervals $J \in \mathcal{J}$ to the Cantor sets $C = C_{\{\mathcal{I}_n, \mathcal{J}_n\}}$. If C is (α_n)-regular, we will classify in terms of the sequence (α_n), which of the corresponding midpoint sets carry purely atomic doubling measures. We find a characterisation of the same nature for all doubling measures being purely atomic. The result, Theorem 5.1, is analogous to Theorem 1.1. We will also answer two questions on atomic doubling measures posed by Kaufman and Wu [10], and Lou et al. [14].

We finish this section with some notation. By a measure on (a metric space) X, we always mean a Borel regular outer measure, defined on all subsets of X. If $A \subset X$, we denote by $\mu|_A$ the restriction of μ to A given by $\mu|_A(B) = \mu(A \cap B)$ for $B \subset X$. For an interval $I \subset \mathbb{R}$, we denote by ∂I the set of its endpoints. We adopt the convention that $0 < c < \infty$ always denotes a constant that only depends on parameters which should be clear from the context. Sometimes we write $c = c(a, \ldots, b)$ to emphasise that c depends only on the values of $a, \ldots, b$. For notational convenience, the exact value of c may vary even inside a given chain of inequalities. Given a family of numbers $0 < A_\alpha, B_\alpha < \infty$, parametrised by α, we denote $A_\alpha \lesssim B_\alpha$ if there is a constant c so that $A_\alpha \le cB_\alpha$ for all α. By $A_\alpha \approx B_\alpha$ we mean that $A_\alpha \lesssim B_\alpha$ and $B_\alpha \lesssim A_\alpha$.

2. Results for (α_n)-porous and (α_n)-thick sets

Our new results concerning (α_n)-porous and (α_n)-thick Cantor sets are based on Theorem 1.3.

Theorem 2.1. *Suppose that $C = C_{\{\mathcal{I}_n, \mathcal{J}_n\}}$ is nice and (α_n)-porous for some $(\alpha_n) \notin \bigcap_{0<p<\infty} \ell^p$. Then there is $\mu \in \mathcal{D}(\mathbb{R})$ with $\mu(C) = 0$.*

Proof. We may assume that $C \subset [0, 1]$. Choose $p > 0$ such that $\sum_{n=1}^{\infty} \alpha_n^p = \infty$. Let μ be a doubling measure given by Theorem 1.3. Then $\mu(J_{n,i}) \approx \left(|J_{n,i}|/|I_{n,i}|\right)^p \mu(I_{n,i})$. As $|J_{n,i}| \ge \alpha_n |I_{n,i}|$, we get

$$\mu(J_{n,i}) \gtrsim \alpha_n^p \mu(I_{n,i}). \tag{2.1}$$

This gives, for some $c > 0$,

$$\mu\left([0, 1] \setminus \left(\cup\mathcal{J}_1 \bigcup \cdots \bigcup \cup\mathcal{J}_n\right)\right) \le (1 - c\alpha_n^p)\mu\left([0, 1] \setminus \left(\cup\mathcal{J}_1 \bigcup \cdots \bigcup \cup\mathcal{J}_{n-1}\right)\right)$$

for all $n \in \mathbb{N}$ and consequently,

$$\mu(C) = \mu\left([0, 1] \setminus \bigcup_{n=1}^{\infty} \mathcal{J}_n\right) \le \mu[0, 1] \prod_{n=1}^{\infty}(1 - c\alpha_n^p) = 0,$$

as $\sum_{n=1}^{\infty} \alpha_n^p = \infty$. □

Theorem 2.2. *Suppose that $C = C_{\{\mathcal{I}_n, \mathcal{J}_n\}} \subset \mathbb{R}$ is nice and (α_n)-thick for some $(\alpha_n) \in \bigcup_{0<p<\infty} \ell^p$. Then there is $\mu \in \mathcal{D}(\mathbb{R})$ with $\mu(C) > 0$.*

Proof. The proof is very similar to the proof of Theorem 2.1 and thus we skip the details. The estimate (2.1) gets replaced by $\mu(J_{n,i}) \lesssim \alpha_n^p \mu(I_{n,i})$ and this leads to $\mu(C) > 0$ when $\sum_{n=1}^{\infty} \alpha_n^p < \infty$. □

Putting together Theorems 2.1 and 2.2, and the results of Wu [3, Theorem 1], and Staples and Ward [5, Theorem 1.4] mentioned earlier, we get the following classification for the thinness and fatness of nice (α_n)-regular Cantor sets.

Corollary 2.3. *If $C \subset \mathbb{R}$ is a nice (α_n)-regular Cantor set, then*

(1) *C is thin if and only if* $(\alpha_n) \notin \bigcup_{0<p<\infty} \ell^p$.
(2) *C is fat if and only if* $(\alpha_n) \in \bigcap_{0<p<\infty} \ell^p$.

3. Proof of Theorem 1.3

We begin with the following simple lemma.

Lemma 3.1. *For all $0 < p < \infty$ there is $c = c(p) < \infty$ and a doubling measure μ on $[0, 1]$ such that*

$$c^{-1}t^p \le \mu[0, t] = \mu[1 - t, 1] \le ct^p$$

for all $0 < t < 1$.

Proof. We obey the following construction. Let m be an integer so large that $2^{-mp+1} < 1$. Define

$$\mu[0, 2^{-m}] = \mu[1 - 2^{-m}, 1] = 2^{-mp}$$

and let μ be uniformly distributed on $[2^{-m}, 1 - 2^{-m}]$ with total measure $1 - 2^{-mp+1}$. For each integer $k \ge 2$, put

$$\mu[0, 2^{-km}] = \mu[1 - 2^{-km}] = 2^{-kmp}$$

and let μ be uniformly distributed on the interval $[2^{-km}, 2^{-(k-1)m}]$ (resp. $[1 - 2^{-(k-1)m}, 1 - 2^{-km}]$) with total measure $2^{-(k-1)mp} - 2^{-kmp}$. It is now easy to see that μ has the required properties. □

We now start to prove Theorem 1.3. We assume without loss of generality that $\inf C = 0$ and $\sup C = 1$. Fix $0 < p < \infty$ and let $\widetilde{c} < 1$ be a constant so that

$$\widetilde{c} I_{n,i} \cap J_{n,i} \ne \emptyset \tag{3.1}$$

for all $I_{n,i} \in \mathcal{I}$ (such a constant exists since the Cantor construction is nice). From now on, in this proof, all constants of comparability will only depend on p and $\widetilde{c}$.

Let $\eta > 0$ be a small constant so that $\eta^p < \frac{1}{4}$. We start by dividing the interval $[0, 1]$ into *construction intervals*[2] of level 1 and *gaps* of level 1 as follows. For all integers $k \ge 2$, we choose gaps $J, J' \in \mathcal{J}$ so that $J \cap [2^{-k}, 2^{-k+1}] \ne \emptyset$ and $J' \cap [1 - 2^{-k+1}, 1 - 2^{-k}] \ne \emptyset$. Denote the union of all these gaps by $\mathcal{G}_1^i$. Let also $\mathcal{G}_1^b = \{J_{n,i} : I_{n,i} \cap \{0, 1\} \ne \emptyset\}$ and $\mathcal{G}_1 = \mathcal{G}_1^b \cup \mathcal{G}_1^i$. Call the elements of $\mathcal{G}_1$ gaps of level one and their complementary intervals the construction intervals of level one. Denote the collection of all construction intervals of level one by $\mathcal{C}_1$.

Next we describe how the total measure $\mu[0, 1] = 1$ is distributed among the construction intervals and gaps of level 1. Denote by G_l^1 the rightmost gap for which $\operatorname{dist}(G_l^1, 0) < \eta$ and by G_r^1 the leftmost gap so that $\operatorname{dist}(G_r^1, 1) < \eta$. Let $G_1, \ldots, G_n$ be the gaps between G_l^1 and G_r^1 and $K_1, \ldots, K_{n+1}$ the complementary intervals in between $G_l^1, G_1, \ldots, G_n, G_r^1$. It is possible that $G_l^1 = G_r^1$ (if there is a huge gap in the middle) and in this case, the collection $\{G_1, \ldots, G_n, K_1, \ldots, K_{n+1}\}$ is considered to be empty.

Claim 1. $n \le c$.

Proof of Claim. Clearly, there are at most $-c\log(\eta)$ gaps in $\mathcal{G}_1^i$ whose distance to the boundary of $[0, 1]$ is at least η (here and in what follows $\log = \log_2$). Taking (3.1) into account, we observe that a similar estimate applies also to the number of elements in $\mathcal{G}_1^b$ whose distance to the boundary of $[0, 1]$ is greater than η. □

Let $K_l^1 = [0, \operatorname{dist}(0, G_l^1)]$ and $K_r^1 = [1 - \operatorname{dist}(1, G_r^1), 1]$ and define

$$\mu(K_l^1) = |K_l^1|^p, \qquad \mu(K_r^1) = |K_r^1|^p \quad \text{and}$$

$$\mu(U) = \gamma|U|^p, \quad \text{for } U \in \{G_l^1, G_r^1, G_1, \ldots, G_n, K_1, \ldots, K_{n+1}\} \text{ where}$$

$$\gamma = \frac{1 - |K_l^1|^p - |K_r^1|^p}{|G_l^1|^p + |G_r^1|^p + \sum_{i=1}^{n}|G_i|^p + \sum_{i=1}^{n+1}|K_i|^p}.$$

(In case $G_l = G_r$, we simply let $\mu(G_l) = 1 - |K_l|^p - |K_r|^p$.) It follows from the Claim 1 and the choice of η that $\frac{1}{c} \le \gamma \le c$ for some $c > 1$ and thus $\mu(U) \approx |U|^p$. We continue distributing the mass inside K_l^1 (and K_r^1). Denote by G_l^2 the rightmost gap inside K_l^1 with $\operatorname{dist}(G_l^2, 0) < \eta|K_l^1|$. Define $K_l^2 = [0, \operatorname{dist}(0, G_l^2)]$ and $\mu(K_l^2) = (|K_l^2|/|K_l^1|)^p\mu(K_l^1) = |K_l^2|^p$. If U is one of the

[2] This refers to the construction of the measure rather than construction of the set C.

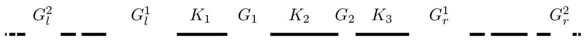

Fig. 1. The gaps and construction intervals of level one.

gaps of level one between G_l^2 and G_l^1 (resp. G_r^2 and G_r^1) or one of the complementary intervals of level one in between these gaps, we put $\mu(U) = \gamma|U|^p$, where γ is a constant defined so that the total measure of K_l^1 (resp K_r^1) remains unchanged. A similar argument as in the proof of Claim 1 implies again that $\gamma \approx 1$. We continue the construction inductively inside K_l^2 (and K_r^2) by letting G_l^3 be the rightmost gap inside K_l^2 for which $\operatorname{dist}(0, G_l^3) < \eta|K_l^2|$, $K_l^3 = [0, \operatorname{dist}(0, G_l^3)]$, $\mu(K_l^3) = (|K_l^3|/|K_l^2|)^p\mu(K_l^2) = |K_l^3|^p$ and so on. Continuing in this manner, we eventually get to define the measure of each gap and construction interval of level one. See Fig. 1.

We proceed with the mass distribution process inside the construction intervals of level one. For such an interval I, we consider gaps $\mathcal{G}_I^b = \{J_{n,i} \subset I : I_{n,i} \cap \partial I \neq \emptyset\}$ and also let $\mathcal{G}_I^i$ consist of a dyadic sequence of gaps defined similarly as $\mathcal{G}_{[0,1]}^i = \mathcal{G}_1^i$ was defined for $I = [0, 1]$. More precisely, if $I = [a, b]$, for each $k \geq 2$ we choose gaps $J, J' \in \mathcal{J}$ so that $J \cap [a + 2^{-k}(b-a), a + 2^{-k+1}(b-a)] \neq \emptyset$ and $J' \cap [b - 2^{-k+1}(b-a), b - 2^{-k}(b-a)] \neq \emptyset$. Put $\mathcal{G}_I = \mathcal{G}_I^i \cup \mathcal{G}_I^b$. We call the elements of $\mathcal{G}_I$ the gaps of I. Their complementary intervals inside I are called the sub-construction intervals of I. The mass $\mu(I)$ is distributed for the gaps and construction intervals of level two inside I by the same procedure as the unit mass was distributed for the gaps and construction intervals of level one. The only difference is, that we replace $1 = \mu[0, 1]$ by $\mu(I)$. We repeat this process inductively for all construction intervals of all levels. We denote by $\mathcal{G}_n$ the set of all gaps of level n and by $\mathcal{C}_n$ the collection of construction intervals of level n. Observe that the construction intervals do not have to be covering intervals (i.e. members of $\mathcal{I}$). So most likely, $\mathcal{C}_n \neq \mathcal{I}_n$ and also $\mathcal{G}_n \neq \mathcal{J}_n$ even though $\bigcup_n \mathcal{G}_n = \bigcup_n \mathcal{J}_n = \mathcal{J}$. Let us further denote $\mathcal{C} = \bigcup_{n=1}^{\infty} \mathcal{C}_n$.

We have now defined the measure of all the gaps and construction intervals and we may use a standard mass distribution principle, see e.g. [15, Proposition 1.7], to define the measure $\mu|_C$. Inside the gaps the measure will be distributed in the following manner: Let $G =]a, b[\in \mathcal{J}$. Then we let $\mu|_G$ be a doubling measure on G given by a scaled version of Lemma 3.1 so that

$$\mu]a, a+t] = \mu[b-t, b[\approx \left(\frac{t}{|G|}\right)^p \mu(G) \tag{3.2}$$

for all $0 < t < b - a$. By the proof of Lemma 3.1, this may be done in such a way that the doubling constant of $\mu|_G$ is independent of $G \in \mathcal{J}$. This completes the construction of μ. To complete the proof of Theorem 1.3, we have to show that μ is doubling and satisfies (1.1).

Our next claim follows directly from the way μ is defined.

Claim 2. *Let $K \in \mathcal{C}_n$ and $I \subset K$, $I \in \mathcal{C}_{n+1} \cup \mathcal{G}_{n+1}$. Then*

$$\mu(I) \approx \left(\frac{|I|}{|K|}\right)^p \mu(K).$$

If $K = [a, b]$, then for all $0 < t < 1$,

$$\mu[a, a + t(b-a)] \approx t^p\mu(K) \approx \mu[b - t(b-a), b].$$

Denote $N = \{0, 1\} \cup \bigcup_{G \in \mathcal{J}} \partial G$.

Claim 3. *Suppose that $I \subset [0, 1]$ is an interval with $I \cap N \neq \emptyset$ and let $K \in \mathcal{C}$ be the shortest construction interval containing I. Then*

$$\mu(I) \approx \left(\frac{|I|}{|K|}\right)^p \mu(K).$$

Proof of Claim 3. Denote $K = [a, b]$ and let $c > 1$ be a constant from the Claim 2 so that

$$\frac{t^p\mu(K)}{c|K|^p} \leq \mu]a, a+t], \qquad \mu[b-t, b[\leq \frac{ct^p\mu(K)}{|K|^p} \tag{3.3}$$

for all $0 < t < |K|$. Fix $\varepsilon = \varepsilon(p) > 0$ so that $\varepsilon^p \leq 1/(2c^2)$.

Assume first, that $\operatorname{dist}(I, \{a, b\}) \geq \varepsilon|I|$. Let $K_1, \ldots, K_n$ be the sub-construction intervals of K intersecting I and $G_1, \ldots, G_m$ ($m \in \{n-1, n, n+1\}$) the gaps of K intersecting I. It may happen that $U \setminus I \neq \emptyset$ for some (but at most two) $U \in \{K_1, \ldots, K_n, G_1, \ldots, G_m\}$. In this case, we replace U by $U \cap I$ in the calculation below. As $\operatorname{dist}(I, \{a, b\}) \geq \varepsilon|I|$, it follows as in the proof of Claim 1, that $n, m \leq c$. Using Claim 2 and (3.2), it now follows that

$$\mu(I) = \sum_{i=1}^n \mu(K_i) + \sum_{j=1}^m \mu(G_j) \approx \sum_{i=1}^n \left(\frac{|K_i|}{|K|}\right)^p \mu(K) + \sum_{j=1}^m \left(\frac{|G_j|}{|K|}\right)^p \mu(K) \approx \left(\frac{|I|}{|K|}\right)^p \mu(K).$$

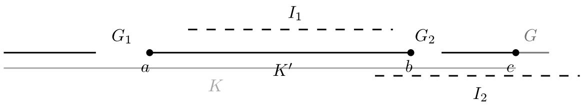

Fig. 2. Illustration for the proof of Claim 4.

Suppose then that $\delta = \text{dist}(I, \{a, b\}) < \varepsilon|I|$. We may assume by symmetry, that $\text{dist}(a, I) < \varepsilon|I|$. The claimed upper bound now follows from Claim 2 since

$$\mu(I) \leq \mu[a, a + 2|I|] \approx \left(\frac{2|I|}{|K|}\right)^p \mu(K).$$

For the lower bound, we use (3.3) to obtain

$$\mu(I) = \mu([a, a + \delta + |I|]) - \mu([a, a + \delta]) \geq \frac{\mu(K)}{|K|^p}\left(\frac{1}{c}(\delta + |I|)^p - c\delta^p\right)$$
$$\geq (|I|/|K|)^p \mu(K)\left(\frac{1}{c} - c\varepsilon^p\right) \gtrsim (|I|/|K|)^p \mu(K)$$

where the last estimate follows from the choice of ε. □

Now we are ready to verify (1.1). Fix $J = J_{n,i} \in \mathcal{J}$, and let K be the smallest construction interval containing $I = I_{n,i}$. By Claim 3, we have

$$\mu(I) \approx \left(\frac{|I|}{|K|}\right)^p \mu(K). \tag{3.4}$$

If J is a gap of K, it follows from Claim 2 that $\mu(J) \approx (|J|/|K|)^p \mu(K)$. Combining this with (3.4), we get $\mu(J)/\mu(I) \approx (|J|/|I|)^p$. If J is not a gap of K, we argue as follows: Since K is the smallest construction interval containing I, there is a gap of K intersecting I. Thus, if K' is the sub-construction interval of K containing J, we have $I \cap \partial K' \neq \emptyset$ and consequently $J \in \mathcal{G}^b_{K'}$. Now, using Claim 2, we obtain

$$\mu(J) \approx \left(\frac{|J|}{|K'|}\right)^p \mu(K') \approx \left(\frac{|J|}{|K'|}\right)^p \left(\frac{|K'|}{|K|}\right)^p \mu(K) = \left(\frac{|J|}{|K|}\right)^p \mu(K)$$

and it follows as above that $\mu(J)/\mu(I) \approx (|J|/|I|)^p$. Whence, (1.1) follows.

It remains to show that μ is doubling on $[0, 1]$. For this, it is clearly enough to show that

$$\mu(I_1) \approx \mu(I_2) \tag{3.5}$$

if I_1 and I_2 are closed sub-intervals of $[0, 1]$ with equal length and $I_1 \cap I_2 \neq \emptyset$. Let I_1 and I_2 be such intervals aligned from left to right. If $(I_1 \cup I_2) \cap N = \emptyset$, then $I_1 \cup I_2 \subset G$ for some $G \in \mathcal{G}$ and (3.5) follows from the way $\mu|_G$ was defined.

Suppose next that $I_1 \cap N \neq \emptyset \neq I_2 \cap N$. Let K_1 (resp. K_2) be the smallest construction interval containing I_1 (resp. I_2). Then $K_1 \subset K_2$ or $K_2 \subset K_1$ since $I_1 \cap I_2 \neq \emptyset$ and any two construction intervals are either disjoint or within each other. We may assume without loss of generality, that $K_1 \subset K_2$.

Claim 4. *If $K_1 \in \mathcal{C}_n$ and $K_2 \in \mathcal{C}_m$, then $n \leq m + 3$.*

Proof of Claim 4. If $K_1 = K_2$, we are done, so assume $K_2 \setminus K_1 \neq \emptyset$. Let G be the leftmost gap of K_2 that intersects I_2 and let $K \in \mathcal{C}_{m+1}$ be the construction interval next to G on the left-hand side. As $I_1 \subset K_2$ and $G \cap I_1 = \emptyset$ (otherwise $K_1 = K_2$), the intervals K and G are well defined. Moreover, we have $I_1 \subset K$. Consider the collection $\mathcal{G}_K$. If $I_1 \cap \cup\mathcal{G}_K \neq \emptyset$, it follows that $K_1 = K$ (i.e. $n = m + 1$) and we are done. Otherwise, there are two consecutive gaps $G_1, G_2 \in \mathcal{G}_K$ and a sub-construction interval of K denoted by $K' \in \mathcal{C}_{m+2}$ in between G_1 and G_2 so that $I_1 \subset K'$. Let us denote by a the right endpoint of G_1 (= left endpoint of K'), by b the left endpoint of G_2 (= right endpoint of K'), and by c the right endpoint of K (= left endpoint of G), see Fig. 2. From the way $\mathcal{G}^i_K$ is constructed, it follows that $|a - c| < 4|b - c|$ and so

$$|K'| = |a - b| < 3|b - c| \leq 3|I_2| = 3|I_1|.$$

We also know, by considering $\mathcal{G}^i_{K'}$ that all sub-construction intervals of K' have length at most $|K'|/2$ and similarly their sub-construction intervals are shorter than $|K'|/4 < \frac{3}{4}|I_1|$. Thus, $|I_1|$ cannot be contained in a construction interval of level $m + 4$ and the claim follows. □

The Claims 2 and 4 now imply that $\mu(K_1) \approx (|K_1|/|K_2|)^p \mu(K_2)$. On the other hand, by Claim 3, we have $\mu(I_1) \approx (|I_1|/|K_1|)^p \mu(K_1)$ as well as $\mu(I_2) \approx (|I_2|/|K_2|)^p \mu(K_2)$. Putting these estimates together implies

$$\mu(I_1) \approx \left(\frac{|I_1|}{|K_1|}\right)^p \mu(K_1) \approx \left(\frac{|I_1|}{|K_1|}\right)^p \left(\frac{|K_1|}{|K_2|}\right)^p \mu(K_2)$$
$$= \left(\frac{|I_2|}{|K_2|}\right)^p \mu(K_2) \approx \mu(I_2).$$

Suppose finally, that only one of the intervals I_1 or I_2, say I_2, hits N. Then I_1 is a subset of a gap $G =]a, b[$ and $\delta = \operatorname{dist}(I_1, b) \le |I_1|$. Letting $I_3 = I_1 + \delta$, we have $\mu(I_1) \approx \mu(I_3)$ as $\mu|_G$ is doubling. On the other hand, since $I_3 \cap N \ne \emptyset \ne I_2 \cap N$, and $I_2 \cap I_3 \ne \emptyset$, we already know that $\mu(I_3) \approx \mu(I_2)$. Combining these estimates, we get $\mu(I_1) \approx \mu(I_2)$. This completes the proof of Theorem 1.3. □

It is natural to ask if we could drop the word "nice" from the assumptions in Theorems 1.3, 2.1 and 2.2 and in Corollary 2.3. The example below shows that, at least, in Theorems 1.3 and 2.1 this is not possible. We do not know if one could remove this assumption from Theorem 2.2.

Example 3.2. If $E \subset \mathbb{R}$ is nowhere dense and $0 < p < 1$, then there is a Cantor set $C \supset E$ which is (α_n)-porous for some $(\alpha_n) \notin \ell^p$.

Proof. We construct inductively the required intervals $I_{n,i}$ and $J_{n,i}$ that satisfy $E \subset [0, 1] \setminus \bigcup_{n,i} J_{n,i}$.

Step 1: Pick any sub-interval $G \subset [0, 1] \setminus E$ of length $\le \frac{1}{2}$ so that $G \cap [\frac{1}{4}, \frac{3}{4}] \ne \emptyset$ and denote $r = |G|$. Choose a number $M_1 \in \mathbb{N}$ so that

$$M_1^{1-p}(r/2)^p \ge 1. \tag{3.6}$$

Let $J_1, \dots, J_{2M_1}$ be disjoint open sub-intervals of G with length $\delta = r/(2M_1)$, enumerated from left to right. Define $\alpha_1 = \alpha_2 = \cdots \alpha_{M_1} = \delta$. From (3.6), we get $\sum_{n=1}^{M_1} \alpha_n^p \ge 1$. If a is the centre point of G, define $\mathcal{I}_1 = \{[0, a], [a, 1]\}\mathcal{J}_1 = \{J_{M_1}, J_{M_1+1}\}$, $\mathcal{I}_2 = \{[0, a - \delta], [a + \delta, 1]\}$, $\mathcal{J}_2 = \{J_{M_1-1}, J_{M_1+2}\}, \dots, \mathcal{I}_{M_1} = \{[0, a - r/2 + \delta], [a + r/2 - \delta, 1]\}$, $\mathcal{J}_{M_1} = \{J_1, J_{2M_1}\}$.

Step m: Suppose that $M_1, \dots, M_{m-1} \in \mathbb{N}$ as well as the collections $\mathcal{I}_j$, $\mathcal{J}_j$ for $1 \le j \le \sum_{k=1}^{m-1} M_k$ have been defined. We now perform the step 1 construction inside each of the elements of $\mathcal{I}_{\sum_{k=1}^{m-1} M_k}$. The number M_m as well as α_n for $\sum_{k=1}^{m-1} M_k < n \le \sum_{k=1}^{m} M_k$ will be determined according to the smallest relative gap chosen inside the intervals $I \in \mathcal{I}_{\sum_{k=1}^{m-1} M_k}$, and we choose the number M_m so large, that

$$\sum_{n=\sum_{k=1}^{m-1} M_k}^{\sum_{k=1}^{m} M_k} \alpha_n^p \ge 1.$$

It is now evident from the construction, that $(\alpha_n) \notin \ell^p$ and that the set $C = \bigcap_{j=1}^{\infty} \bigcup \mathcal{I}_j$ is (α_n)-porous. □

Remark 3.3. To formally fulfil the requirement $\bigcup \mathcal{I}_{n+1} = \bigcup \mathcal{I}_n \setminus \bigcup \mathcal{J}_n$ we should add to each $\mathcal{I}_{n+1}$ the boundary points of the deleted intervals $J \in \mathcal{J}_n$ and also empty sets as their "holes" to $\mathcal{J}_{n+1}$. For those readers who consider this cheating, we suggest to modify the construction so that $\bigcup \mathcal{I}_{n+1} = \bigcup \mathcal{I}_n \setminus \bigcup \mathcal{J}_n$ holds and the resulting Cantor set $C = C_{\{\mathcal{I}_n, \mathcal{J}_n\}_n}$ contains no isolated points. It is also possible to modify the construction so that $(\alpha_n) \notin \bigcup_{0<q<1} \ell^q$.

4. A lemma on (α_n)-porous sets in metric spaces

For the purpose of proving results for midpoint Cantor sets in Section 5, we present here a metric space version of Wu's result on (α_n)-porous sets being null for all doubling measures if $(\alpha_n) \notin \bigcup_{0<p<\infty} \ell^p$. Her argument to prove the result in $\mathbb{R}$ readily works in much general situations once we find a reasonable definition of (α_n)-porosity to use. There are basically two options: If one wants that the covering collection consists of distinct elements, then one has to use more general covering objects than just balls or intervals. The second option, which is more useful for us, is to relax the disjointness condition a bit and still keep using coverings with balls. For an analogous result using the first mentioned option, see [16, Theorem 4.9].

We say that a subset $E \subset X$ of a metric space X is (α_n)-*porous* for a sequence $(\alpha_n)_{n=1}^{\infty}$, $0 < \alpha_n < 1$, if there is a constant $N \in \mathbb{N}$ and a sequence of (finite or countably infinite) coverings $\mathcal{B}_n = \{B_{n,j}\}_i$ of E by balls $B_{n,j} = B(x_{n,j}, r_{n,j})$ with the following properties:

(P1) Each $B_{n,j}$ contains a sub ball $B'_{n,j} = B(y_{n,j}, \alpha_n r_{n,j}) \subset B_{n,j} \setminus E$.
(P2) Each point $x \in X$ belongs to at most N different balls $B'_{n,j}$.

It is clear that if $C = C_{\{\mathcal{I}_n, \mathcal{J}_n\}} \subset \mathbb{R}$ is (α_n)-porous in the sense defined in the introduction, then it is also (α_n)-porous in the sense of the above definition.

Lemma 4.1. *Let X be a metric space. If $(\alpha_n) \notin \bigcup_{0<p<\infty} \ell^p$, and $E \subset X$ is (α_n)-porous, then $\mu(E) = 0$ for all doubling measures μ on X.*

Proof. Let $\mathcal{B}_n$ be the coverings that fulfil the (α_n)-porosity conditions (P1) and (P2) and let μ be a doubling measure on X with doubling constant $1 < c < \infty$. Without loss of generality, we may assume that E is bounded and that $B_{n,j} \subset B$ for some fixed ball $B \subset X$. For each n, let k_n be the smallest integer so that $k_n \geq -\log(\alpha_n) + 1$. Then $B_{n,i} \subset B(y_{n,j}, 2^{k_n}\alpha_n r_{n,j})$ for all $B_{n,i} \in \mathcal{B}_n$ and thus the doubling condition gives

$$\mu(E) \leq \sum_i \mu(B_{n,i}) \leq c^{-\log(\alpha_n)+1} \sum_i \mu(B'_{n,i}) = c\,\alpha_n^{-p} \sum_i \mu(B'_{n,i}),$$

where $p = \log c > 0$. Let $\varepsilon > 0$. To complete the proof it suffices to find $n \in \mathbb{N}$ so that $\sum_i \mu(B'_{n,i}) \leq \varepsilon \alpha_n^p$. But if this is not the case, then (P2) yields

$$\infty > \mu(B) \geq \frac{1}{N} \sum_n \sum_i \mu(B'_{n,i}) > \varepsilon \sum_{n=1}^{\infty} \alpha_n^p = \infty$$

giving a contradiction. □

5. Purely atomic doubling measures

5.1. On midpoint Cantor sets

In this subsection, we show how the theorems of Section 2 can be turned to theorems on atomic doubling measures for certain class of midpoint Cantor sets.

For each Cantor set $C = C_{\{\mathcal{I}_n, \mathcal{J}_n\}}$, we define a *midpoint Cantor set* $M = M_{\{\mathcal{I}_n, \mathcal{J}_n\}}$ by letting $M = C \cup_{J \in \mathcal{J}} \{x_J\}$, where x_J is the centre point of $J \in \mathcal{J}$. If C is a middle interval Cantor set $C = C(\alpha_n)$, we denote the corresponding midpoint Cantor set by $M(\alpha_n)$. We consider each such M as a metric space, with the inherited Euclidean metric.

For these midpoint Cantor sets, we verify the following results analogous to the results obtained for doubling measures on the real-line.

Theorem 5.1. *Suppose that $C = C_{\{\mathcal{I}_n, \mathcal{J}_n\}}$ is a Cantor set and let $M = M_{\{\mathcal{I}_n, \mathcal{J}_n\}}$. Then:*

(1) *If C is (α_n)-porous for some $(\alpha_n) \notin \bigcup_{0<p<\infty} \ell^p$, then all doubling measures on M are purely atomic.*

Suppose further that C is nice and let c be a constant so that $J_{n,i} \cap cI_{n,i} \neq \emptyset$ for all $I_{n,i}$. If

$$|J_{n,i}| < \frac{1-c}{3} |I_{n,i}| \quad \text{for all } I_{n,i} \tag{5.1}$$

then also the following holds:

(2) *If C is (α_n)-thick for some $(\alpha_n) \in \bigcup_{0<p<\infty} \ell^p$, then there are doubling measures μ on M with $\mu(C) > 0$.*
(3) *If C is (α_n)-thick for some $(\alpha_n) \in \bigcap_{0<p<\infty} \ell^p$, then there are no purely atomic doubling measures on M.*
(4) *If C is (α_n)-porous and $(\alpha_n) \notin \bigcap_{0<p<\infty} \ell^p$, then there are purely atomic doubling measures on M.*
(5) *Finally, suppose that C is nice and (α_n)-regular. Then all doubling measures on M are purely atomic if and only if $(\alpha_n) \notin \bigcup_{0<p<\infty} \ell^p$. There are no purely atomic doubling measures on M if and only if $(\alpha_n) \in \bigcap_{0<p<\infty} \ell^p$.*

Our main tool to prove Theorem 5.1 is the following lemma. We denote by δ_x the Dirac unit mass located at $x \in \mathbb{R}$.

Lemma 5.2. *Suppose that $C = C_{\{\mathcal{I}_n, \mathcal{J}_n\}}$ is a nice Cantor set and assume that (5.1) holds. Let $M = M_{\{\mathcal{I}_n, \mathcal{J}_n\}}$. If μ is a doubling measure on $[\inf C, \sup C]$, we may define a doubling measure ν on M by setting $\nu = \mu|_C + \sum_{J \in \mathcal{J}} \mu(J)\delta_{x_J}$. On the other hand, if ν is a doubling measure on M, there is a doubling measure μ on $[\inf C, \sup C]$ so that $\nu|_C = \mu|_C$ and $\mu(J) = \nu\{x_J\}$ for all $J \in \mathcal{J}$.*

Before starting to prove Lemma 5.2, we state a couple of auxiliary results. The first one is a direct consequence of the doubling property.

Lemma 5.3. *Let μ be a doubling measure on a metric space X and let $1 < \Lambda < \infty$. Suppose that $x, y \in X$, $d(x, y) \leq \Lambda r$, and $1/\Lambda \leq r/s \leq \Lambda$. Then $\mu(B(x, r)) \approx \mu(B(y, s))$ where the constants of comparability only depend on Λ and the doubling constant of μ.*

Lemma 5.4. *Under the assumptions of Lemma 5.2, there is $c > 0$ so that the following holds: If $J, J' \in \mathcal{J}$ and K is the interval between J and J', then $|K| \geq c \min\{|J|, |J'|\}$.*

Proof. Let I (resp. I') be the smallest interval from $\mathcal{I}$ containing J (resp. J'). Then $I \subset I'$, $I' \subset I$ or $I \cap I' = \emptyset$. In any case, $J \cap I' = \emptyset$ or $J' \cap I = \emptyset$. We may assume that $J' \cap I = \emptyset$. Using (5.1), we get $|K| = \operatorname{dist}(J, J') \geq \operatorname{dist}(J, \partial I) \geq (1-c)|I|/2 - |J| \geq |I|(1-c)/6 > |J|(1-c)/6$. □

Proof of Lemma 5.2. We assume without loss of generality that $\inf C = 0$, $\sup C = 1$. By $B(x,r)$ we denote the Euclidean interval $B(x,r) =]x-r, x+r[$ whereas $B_M(x,r) = B(x,r) \cap M$, for $x \in M$.

To prove the first assertion, suppose that μ is a doubling measure on $[0,1]$ and let ν be defined as in the lemma. We have to verify that ν is a doubling measure on M. Fix $x \in M$ and $r > 0$. If $B(x,2r) \cap C = \emptyset$, we have $B_M(x,r) = B_M(x,2r) = \{x\}$ and there is nothing to prove. Proving that ν is doubling thus reduces to showing the following. If $x \in M$ and $B(x,2r) \cap C \neq \emptyset$, then

$$\nu(B_M(x,2r)) \lesssim \mu(B(x,r)) \quad \text{and} \tag{5.2}$$
$$\nu(B_M(x,r)) \gtrsim \mu(B(x,r)). \tag{5.3}$$

We may write $\nu(B_M(x,2r)) = \mu[a,b] + \nu(E)$, where $a = \inf(B(x,2r) \cap C)$, $b = \sup(B(x,2r) \cap C)$ and E is either empty or contains one or two isolated points of M. By the construction of ν, we have $\nu(E) \le \mu(B(x,4r))$ and thus

$$\begin{aligned}\nu(B_M(x,2r)) &\le \mu[a,b] + \mu(B(x,4r)) \le \mu(B(x,2r)) + \mu(B(x,4r))\\ &\lesssim \mu(B(x,r))\end{aligned}$$

since μ is doubling. Thus (5.2) follows.

To show (5.3), assume first that $B(x,r/2) \cap C \neq \emptyset$. If we let $a = \inf(B(x,r) \cap C)$, $b = \sup(B(x,r) \cap C)$, then Lemma 5.4 implies $|b-a| \gtrsim r$ and thus

$$\nu(B_M(x,r)) \ge \nu([a,b] \cap M) = \mu[a,b] \gtrsim \mu(B(x,r))$$

by Lemma 5.3. If $B(x,r/2) \cap C = \emptyset$, then $B(x,r/2) \subset J$ for some $J \in \mathcal{J}$ with $|J| \ge r$, and we have

$$\nu(B_M(x,r)) \ge \nu\{x\} = \mu(J) \gtrsim \mu(B(x,r)).$$

Thus we have (5.3) and it follows that ν is a doubling measure on M.

To give the details for the latter claim of the lemma requires a bit more work. Consider a doubling measure ν on M. We define μ by the following procedure: Let $c > 0$ be the constant of Lemma 5.4 and choose $1/(1+c) < t < 1$. For $J =]x-r, x+r[\in \mathcal{J}$, consider its division to Whitney type sub-intervals

$$\begin{aligned}J_k^+ &=]x+r-(1-t)^k r, x+r-(1-t)^{k+1} r[,\\ J_k^- &=]x-r+(1-t)^{k+1} r, x-r+(1-t)^k r[\end{aligned}$$

for $k \in \{0,1,2,\ldots\}$. Next define

$$\begin{aligned}m_{J_k^+} &= \nu([x+r+(1-t)^{k+1} r, x+r+(1-t)^k r[\cap M),\\ m_{J_k^-} &= \nu(]x-r-(1-t)^k r, x-r-(1-t)^{k+1} r] \cap M).\end{aligned}$$

If K is one of the intervals J_k^+ or J_k^-, let $\mu|_K$ be uniformly distributed on K with total measure

$$\mu(K) = \frac{m_K \nu\{x_J\}}{\nu\left((2J \cap M) \setminus \{x_J\}\right)}.$$

Observe that the scaling factor $\nu\{x_J\}/\nu((2J \cap M) \setminus \{x_J\})$ is bounded away form 0 and ∞ as ν is doubling. Thus we have

$$\mu(K) \approx m_K \tag{5.4}$$

for all $K \in \{J_k^+, J_k^-\}_{k=0}^{\infty}$. To complete the definition of μ, we set $\mu|_C = \nu|_C$.

It is now evident that $\mu(J) = \nu\{x_J\}$ for each $J \in \mathcal{J}$ and it remains to show that μ is doubling. For this purpose, we prove the following chain of claims. We formulate some of the claims for u_2 and J_k^+ but due to symmetry, similar claims are valid for u_1 and J_k^- as well.

Let $J =]u_1, u_2[\in \mathcal{J}$. Then

(i) For each J_k^+, there is $y \in M$ with $y - u_2 \approx |J_k^+|$ such that $\nu(B_M(y, |J_k^+|)) \approx \mu(J_k^+)$.
(ii) If K_0 and K_1 are two consecutive intervals among J_k^+, J_k^-, then $m_{K_0} \approx m_{K_1}$.
(iii) $\mu(J_k^+) \approx \mu(\cup_{n>k} J_k^+)$.
(iv) If $0 < s < |J|/2$, then $\nu([u_2, u_2+s] \cap M) \approx \mu[u_2 - s, u_2]$.
(v) If I is an interval with $I \cap C \neq \emptyset$ and $\kappa > 1$, then $\mu(I) \le c(\kappa, t)\nu(\kappa I \cap M)$.
(vi) If $0 < s < |J|$, then $\mu[u_2, u_2+s] \approx \nu([u_2, u_2+s] \cap M)$.

We now start to prove the claims (i)–(vi). Let $c > 0$ be the constant of Lemma 5.4. Since $t > 1/(1+c)$, we may choose $\varrho = \varrho(t) > 0$ such that $1 - t + \varrho t < c(1-2\varrho)t$. Let $K = J_k^+$. By scaling, we may assume that $(1-t)^k r = 1$ so that $|K| = t$ and $\mathrm{dist}(K, u_2) = 1 - t$. Denote

$$\begin{aligned}K' &= [u_2 + (1-t), u_2 + 1[,\\ (1-2\varrho)K' &= [u_2 + (1-t) + \varrho t, u_2 + 1 - \varrho t[.\end{aligned}$$

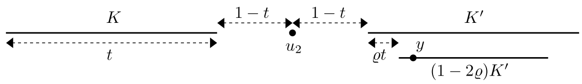

Fig. 3. Illustration for the proof of (i).

It follows from Lemma 5.4 and the choice of t and ϱ that $(1-2\varrho)K'\cap M\neq\emptyset$. Thus, we may choose $y\in M$ so that $B_M(y,\varrho t)\subset K'$. See Fig. 3. Using the doubling property of ν, and the way μ is defined, we get

$$\mu(K)\approx\nu(K'\cap M)\geq\nu(B_M(y,\varrho t))\gtrsim\nu(B_M(y,t))\geq\nu(K'\cap M)\approx\mu(K).$$

As $(1-t)\leq|y-u_2|\leq 1$, we have $|y-u_2|\approx t=|J_k^+|$ and (i) follows.

Let K_0 and K_1 be two consecutive intervals among $\{J_k^+,J_k^-\}$, and $y_0,y_1\in M$ be points given by (i). Then $|y_0-y_1|\lesssim|K_0|\approx|K_1|$ and combined with (5.4) and Lemma 5.3, we get $m_{K_0}\approx\mu(K_0)\approx\nu(B_M(y_0,|K_0|))\approx\nu(B_M(y_1,|K_1|))\approx\mu(K_1)\approx m_{K_1}$ implying (ii).

For $k=0,1,2,\ldots$, let $y_k\in M$ be a point satisfying (i). Since ν is doubling, we get (using (5.4) and Lemma 5.3)

$$\begin{aligned}\mu\left(\bigcup_{n=k+1}^{\infty}J_n^+\right)&=\mu[u_2-\frac{1-t}{t}|J_k^+|,u_2]\approx\sum_{n>k}m_{J_n^+}\\&=\nu\left([u_2,u_2+\frac{1-t}{t}|J_k^+|[\cap M\right)\lesssim\nu B_M(y_k,|J_k^+|)\approx\mu(J_k^+).\end{aligned}$$

On the other hand, using (ii) we see that $\mu(J_k^+)\approx m_{J_k^+}\approx m_{J_{k+1}^+}\approx\mu(J_{k+1}^+)\leq\mu(\bigcup_{n=k+1}^{\infty}J_n^+)$ and (iii) follows.

By construction, we have

$$\mu\left[u_2-\frac{1}{t}|J_k^+|,u_2\right]\approx\sum_{n\geq k}m_{J_n^+}=\nu\Big[u_2,u_2+\frac{1}{t}|J_k^+|\Big[.$$

Combining this with (iii) yields (iv).

Let $s<|J|$ and let K be the largest interval among $\{J_k^+\}$ contained in $[u_2-s,u_2]$. With the help of (i)–(iii), we see that

$$\mu[u_2-s,u_2]\approx\mu(K)\approx m_K\leq\nu([u_2,u_2+s]\cap M)$$

(and similarly $\mu[u_1,u_1+s]\lesssim\nu[u_1-s,u_1]\cap M$). To prove (v), we apply this observation for the components of $I\setminus C$ to obtain $\mu(I\setminus C)\lesssim\nu(3I\cap M)$. Choosing $y\in I\cap C$, we have

$$\begin{aligned}\mu(I)&=\nu(I\cap C)+\mu(I\setminus C)\lesssim 2\nu(3I\cap M)\lesssim c(\kappa,t)\,\nu\left(B_M(y,\frac{\kappa-1}{2}I)\right)\\&\leq c(\kappa,t)\nu(\kappa I\cap M)\end{aligned}$$

for each $\kappa>1$.

To prove (vi), let $v=\sup C\cap[u_2,u_2+s]$. Using Lemma 5.4, we may find $y\in M$ and $r\gtrsim s$ so that $B_M(y,r)\subset[u_2,v]$. Now

$$\nu([u_2,u_2+s]\cap M)\lesssim\nu(B_M(y,r))\leq\nu([u_2,v]\cap M)=\mu[u_2,v]\leq\mu[u_2,u_2+s].$$

On the other hand, we have $\mu[u_2,v]=\nu([u_2,v]\cap M)$ and

$$\mu[v,u_2+s]\lesssim\nu([u_2,v+s]\cap M)\lesssim\nu(B_M(y,r))\leq\nu([u_2,u_2+s]\cap M)$$

using (v). Thus (vi) follows and we have verified all the claims (i)–(vi).

Let $I_1,I_2\subset[0,1]$ be two adjacent closed intervals of the same length. To finish the proof we have to show that

$$\mu(I_1)\approx\mu(I_2).\tag{5.5}$$

To achieve this goal, we consider several different cases and subcases.

Case a: Both intervals I_1 and I_2 are contained in a gap $J=]u_1,u_2[\in\mathcal{J}$. Let $\mathcal{K}=\{J_k^+,J_k^-:k=0,1,2,\ldots\}$.

Subcase a1: If both intervals I_1 and I_2 intersect at most 2 intervals of $\mathcal{K}$, the estimate (5.5) follows directly from (ii).

Subcase a2: If both intervals I_i intersect at least 3 elements of $\mathcal{K}$, let K_i be the largest element $K\in\mathcal{K}$ contained in I_i. Then, it follows from (iii) and (ii) that $\mu(I_i)\approx\mu(K_i)$. On the other hand, there is at most one interval $K\in\mathcal{K}$ in between K_1 and K_2 and thus, using (ii) once again, we get $\mu(K_1)\approx\mu(K_2)$.

Subcase a3: Suppose that I_1 intersects at least three sub-intervals $K\in\mathcal{K}$ whereas I_2 intersects at most two of them. Again, letting K_1 be the largest element of $\mathcal{K}$ contained in I_1, we have $\mu(I_1)\approx\mu(K_1)$. Now, if $K_2\in\mathcal{K}$ and $K_2\cap I_2\neq\emptyset$, there are at most two intervals of $\mathcal{K}$ in between K_1 and K_2. Thus, from (ii) we get $m_{K_2}\approx m_{K_1}$ giving $\mu(I_1)\approx\mu(I_2)$.

Case b: I_1 is contained in a gap but $I_2 \cap C \neq \emptyset$. We may assume by symmetry that $I_1 = [a, b]$, $I_2 = [b, c]$ (where $c - b = b - a$). Let $d = \inf(I_2 \cap C)$.

Subcase b1: If $d - b \geq c - d$, the claims (vi) and (iv) imply $\mu[b, c] \approx \mu[b, d]$ and from the case *a* and Lemma 5.3, we obtain $\mu[b, d] \approx \mu[a, b]$.

Subcase b2: If $d - b \leq c - d$, we first use the case *a* to get $\mu[a, b] \approx \mu[d - (c - d), d]$ and then (vi) and (iv) to conclude $\mu[d - (c - d), d] \approx \mu[d, c] \approx \mu[b, c]$.

Case c: $I_1 \cap C \neq \emptyset \neq I_2 \cap C$. By symmetry, we assume again that $I_1 = [a, b]$, $I_2 = [b, c]$, and denote $r = b - a = c - b$. Let $v_1 = \inf(I_1 \cap C)$, $v_2 = \sup(I_1 \cap C)$, $v_3 = \inf(I_2 \cap C)$, and $v_4 = \sup(I_2 \cap C)$.

Subcase c1: If $v_2 - v_1 \geq r/2$ and $v_4 - v_3 \geq r/2$, we can find $y_1 \in M$ so that $B_M(y_1, r/8) \subset [v_1, v_2] \cap M$ and

$$\mu(I_1) \geq \mu[v_1, v_2] = \nu([v_1, v_2] \cap M) \approx \nu(B_M(y_1, r/8)).$$

As also $\mu(I_1) \lesssim \nu(2I_1 \cap M)$ by (v), $2I_1 \cap M \subset B_M(y_1, 2r)$, and ν is doubling, we thus get $\mu(I_1) \approx \nu(B_M(y_1, r))$. Repeating the argument for I_2 yields $B_M(y_2, r/8) \subset [v_3, v_4] \cap M$ with $\mu(I_2) \approx \nu(B_M(y_2, r))$. Using Lemma 5.3, we get $\nu(B_M(y_1, r)) \approx \nu(B_M(y_2, r))$ yielding (5.5).

Subcase c2: Suppose $v_2 - v_1 \geq r/2$ and $v_4 - v_3 < r/2$. Now, as in subcase *c1*, we find $B_M(y_1, r/8) \subset [v_1, v_2] \cap M$ with $\mu(I_1) \approx \nu(B_M(y_1, r))$. On the other hand, letting I_3 be the longer of the intervals $[b, v_3]$ and $[v_4, c]$, with the help of (i)–(iii), we find $y_2 \in M$ with $\operatorname{dist}(I_3, y_2) \lesssim r$ and $s \approx r$ such that $\mu(I_3) \approx \nu(B_M(y_2, s))$. Again, as ν is doubling we can use Lemma 5.3 to conclude that $\mu(I_1) \approx \nu(B_M(y_1, r)) \approx \nu(B_M(y_2, s)) \approx \mu(I_2)$ as desired.

Subcase c3: Finally, if both $v_2 - v_1 < r/2$ and $v_4 - v_3 < r/2$, we let I_3 be the longer of the intervals $[a, v_1]$, $[v_2, b]$ and I_4 the longer of the sub-intervals $[b, v_3]$, $[v_4, c]$. As above, we find $B_M(y_1, s_1)$ and $B_M(y_2, s_2)$ so that $\mu(I_1) \approx \nu(B_M(y_1, s_1)) \approx \nu(B_M(y_2, s_2)) \approx \mu(I_2)$. □

Proof of Theorem 5.1. Suppose first that C is (α_n)-porous as a subset of $\mathbb{R}$. The Claim (1) follows from the Lemma 4.1 since C is $(\alpha_n/2)$-porous as a subset of M. Indeed, for each $J_{n,i}$, let $x_{n,i} = x_{J_{n,i}}$ and consider $B_{n,i} = B_M(x_{n,i}, |I_{n,i}|)$ and $B'_{n,i} = B_M(x_{n,i}, |J_{n,i}|/2)$. Then $B'_{n,i} \cap B'_{l,j} = \emptyset$ if $(n, i) \neq (l, j)$ and moreover, $|J_{n,i}|/2 \geq (\alpha_n/2)|I_{n,i}|$ for all n and i.

To prove the claims (2)–(4), we use Lemma 5.2. Then (2) follows from Theorem 2.2, (3) from the result of Staples and Ward [5, Theorem 1.4], and (4) from Theorem 2.1. Finally, (5) follows putting (1)–(4) together. □

Remarks 5.5. (a) Our choice to put one isolated point in the middle of each gap is somewhat arbitrary. The Theorem 5.1 (and Lemma 5.2) holds true for many other choices of (collections) of isolated points as well. For instance, instead of choosing the middle point of each $J \in \mathcal{J}$, one could consider a Whitney decomposition $\mathcal{W}_J$ of J and choose all the midpoints of the elements of $\mathcal{W}_J$ to be the collection of isolated points inside J. Doubling measures on this kind of Whitney modification sets have been considered in [10], and [11].

(b) In many situations, the technical assumption (5.1) (used only to prove Lemma 5.4) may be omitted. For the middle interval midpoint sets $M(\alpha_n)$, for instance, the claims (2)–(4) in Theorem 5.1 hold also without this assumption.

Kaufman and Wu [10] have posed the following problem: Does there exist a compact set $X \subset \mathbb{R}$ with $X = \overline{F_X}$ and a doubling measure ν on X so that $\nu|_{E_X}$ is a doubling measure on E_X? Recall that F_X is the set of isolated points of X and $E_X = X \setminus F_X$. The following example yields a positive answer to their question.

Example 5.6. Let $(\alpha_n) \in \ell^1$, $X = M(\alpha_n)$, $C = C(\alpha_n)$, $\mu = \mathcal{L}|_{[0,1]}$, and let ν be a doubling measure on X given by Lemma 5.2. Then $F_X = \cup_{J \in \mathcal{J}} \{x_J\}$, $E_X = C$, and $X = \overline{F_X}$. Moreover, it is easy to see that $\nu|_C = \mathcal{L}|_C$ is a doubling measure on C since there exists $c = c(\alpha_n)$ so that $\mathcal{L}(C \cap (x - r, x + r)) > cr$ for all $x \in C$ and $0 < r < 1$.

5.2. On sets with positive Lebesgue measure

To complete the discussion on purely atomic doubling measures, we answer a question posed by Lou et al. in [14]. As observed by Wu [3, Example 1], see also Lou et al. [14, Theorem 1], it is possible to construct compact sets $X \subset [0, 1]$ with Hausdorff dimension one so that all doubling measures on X are purely atomic. The examples of Wu [3] and Lou et al. [14] are countable unions of self-similar Cantor sets whose dimensions gets closer and closer to one. Another, more direct, way to obtain such a set is given by Theorem 5.1: Choosing $X = M(\alpha_n)$ for any sequence $(\alpha_n) \notin \bigcup_{0<p<\infty} \ell^p$ such that

$$\lim_{n\to\infty} \frac{\log\left(\prod_{k=1}^{n}(1-\alpha_k)\right)}{n} = 0 \tag{5.6}$$

will do. Note that (5.6) always holds if $\lim_{n\to\infty} \alpha_n = 0$. It was asked by Lou et al. [14] whether there are compact sets $X \subset \mathbb{R}$ with positive Lebesgue measure so that all doubling measures μ on X are purely atomic. The answer is negative.

Proposition 5.7. *If $X \subset \mathbb{R}$ is compact and $\mathcal{L}(X) > 0$, there are doubling measures on X with nontrivial continuous part.*

Proof. The claim is a direct consequence of the results of Vol'berg and Konyagin [17], see also [1, Section 13]. For subsets of $\mathbb{R}^n$, they proved the existence of n-homogeneous measures. In our case this gives a constant $c < \infty$ and a measure μ on X so that

$$\mu(B(x, \lambda r)) \leq c\lambda\mu(B(x, r)) \quad \text{for all } x \in X, \lambda \geq 1, \quad \text{and} \quad r > 0.$$

Putting $\lambda = 1/r$, it follows that $c\mu(B(x, r)) \geq r$ for all $x \in X$ and $0 < r < 1$. Now we may define $\nu = \mu + \mathcal{L}|_X$. If c' is the doubling constant of μ, it follows that for all $x \in X$ and $0 < r < 1$,

$$\begin{aligned}\nu(B(x, 2r)) &= \mu(B(x, 2r)) + \mathcal{L}(X \cap B(x, 2r)) \leq \mu(B(x, 2r)) + 2r \\ &\leq (c' + 2c)\mu(B(x, r)) \leq (c' + 2c)\nu(B(x, r))\end{aligned}$$

so ν is a doubling measure on X. As $\mathcal{L}(X) > 0$, it follows that ν has a nontrivial (absolutely) continuous part. □

Remark 5.8. While this paper was in preparation, there has been some independent research on the topics of the last section. Wang and Wen [18] have constructed a set X with the same properties as in Example 5.6 and Lou and Wu [19] have also observed that Proposition 5.7 follows from the above mentioned result of Vol'berg and Konyagin.

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