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The ergodicity and extinction of stochastically perturbed SIR and SEIR epidemic models with saturated incidence ☆

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ABSTRACT

In this paper, we include stochastic perturbations into SIR and SEIR epidemic models with saturated incidence and investigate their dynamics according to the basic reproduction number R_0. The long time behavior of the two stochastic systems is studied. Mainly, we utilize stochastic Lyapunov functions to show under some conditions, the solution has the ergodic property as $R_0 > 1$, while exponential stability as $R_0 \leqslant 1$. At last, we make simulations to conform our analytical results.

1. Introduction

Mathematical modeling has been an important approach in analyzing the spread and control of infectious diseases. In recent years, attempts have been made to develop realistic mathematical models for the transmission dynamics of infectious diseases, see, e.g., [18,35,36,40,47,49,52] and the references therein. In modeling of communicable diseases, the incidence function has been considered to play a key role in ensuring that the models indeed give reasonable qualitative description of the epidemic dynamics (see [1,2]). In many epidemiological models, the corresponding incidence rate is bilinear with respect to the numbers of susceptible and infective individuals (see [2,22]). More specifically, if $S(t)$ and $I(t)$ are the fractions of susceptible and infective individuals in the population, and if β is the per capita contact rate, then the principle of mass action implies that the infection spreads with the rate βSI. This contact law is more appropriate for communicable diseases such as influenza, but not for sexually transmitted diseases. There is a number of reasons why this standard bilinear incidence rate may require modification. For instance, the underlying assumption of homogeneous mixing and homogeneous environment may be invalid. In this case the necessary population structure and heterogeneous mixing may be incorporated into a model with a specific form of nonlinear transmission. If the population is saturated with the infective, the incidence rate may have a nonlinear dependence on I. This saturation effect was observed, for example, by Capasso and Serio [9] who studied the cholera epidemic spread in Bari in 1973, and by Brown and Hasibuan [8] who studied infection of the two-spotted spider mites, Tetranychus urticae, with the entomopathogenic fungus, Neozygites floridana. They introduced a saturated incidence rate $g(I)S$ into epidemic models, where $g(I)$ tends to a saturation level when I gets large, i.e.,

$$g(I) = \frac{\beta I}{1 + \alpha I},$$

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where βI measures the infection force of the disease and $1/(1+\alpha I)$ measures the inhibition effect from the behavioral change of the susceptible individuals when their number increases or from the crowding effect of the infective individuals. This incidence rate seems more reasonable than the bilinear incidence rate βIS, because it includes the behavioral change and crowding effect of the infective individuals and prevents the unboundedness of the contact rate by choosing suitable parameters. Furthermore, the details of transmission of infectious diseases are generally unknown, and may vary for different conditions, therefore, models of infectious diseases with nonlinear incidence rates have been attracted considerable attention over the last two decades. A variety of nonlinear incidence rates have been used in the literature (see, for example, [13,6,7, 33,30,16,34,50] and the references cited therein).

Following classical assumptions, we divide the host population into the susceptible, the infective and the recovered subpopulations, and denote the fractions of these in the population by S, I and R, respectively. In this case, once infected, each susceptible individual becomes infectious instantaneously and later recovers with a temporary acquired immunity. An epidemic model based on these assumptions is called SIR (susceptible, infectious, recovered) model. If the transmission of the disease is governed by the saturated incidence rate $\beta S/(1+\alpha I)$, the SIR model is described by the following ordinary differential equations:

$$\begin{aligned}
\frac{dS}{dt} &= \lambda - \frac{\beta SI}{1+\alpha I} - d_S S,\\
\frac{dI}{dt} &= \frac{\beta SI}{1+\alpha I} - (d_I+\delta+\gamma)I,\\
\frac{dR}{dt} &= \gamma I - d_R R,
\end{aligned} \tag{1.1}$$

where λ is the birth rate, d_S, d_I and d_R are the natural death rates of S, I and R, respectively. δ is the additional disease-caused rate suffered by the infectious individuals, and γ is the recovery rate of infectious individuals. Throughout this paper, we assume that the parameters are all positive.

In the natural world, for some diseases (for example, tuberculosis, influenza, measles) on adequate contact with an infective, a susceptible individual becomes exposed, that is, infected but not infective. This individual remains in the exposed class for a certain latent period before becoming infective (see, for example, Cooke and van den Driessche [13], Hethcote and van den Driessche [20,21]). Hence, it is realistic to introduce an extra class, the class of exposed hosts to the system. The resulting model is called SEIR (susceptible, exposed, infectious, recovered) model. The SEIR infectious disease model is very important and has been studied by many authors (see, for example, [33,17]). We assume that the average duration of the latent state is $1/\theta$, and that transmission of the infection is governed by a saturated incidence rate $\beta S/(1+\alpha I)$. Then the basic SEIR model is described by the following ordinary differential equations:

$$\begin{aligned}
\frac{dS}{dt} &= \lambda - \frac{\beta SI}{1+\alpha I} - d_S S,\\
\frac{dE}{dt} &= \frac{\beta SI}{1+\alpha I} - (d_E+\theta)E,\\
\frac{dI}{dt} &= \theta E - (d_I+\delta+\gamma)I,\\
\frac{dR}{dt} &= \gamma I - d_R R.
\end{aligned} \tag{1.2}$$

Since the dynamics of R has no effects on the transmission dynamics, the last equations of (1.1) and (1.2) can be omitted in analysis. Obviously, system (1.1) or (1.2) has only two kinds of equilibria: the infection-free equilibrium $Q_0 = (\lambda/d_S, 0, 0, 0)$ and the endemic equilibrium $Q^* = (S^*, I^*, E^*, R^*)$. Global behavior of these equilibriums crucially depends on the basic reproduction number, that is an average number of secondary cases produced by a single infective introduced into an entirely susceptible population. For the SIR and the SEIR models, the basic reproduction number is (see [46])

$$R_0 = \frac{1}{B}\cdot\frac{\lambda\beta}{d_S(d_I+\delta+\gamma)},$$

where $B = 1$ for the SIR model and $B = \frac{\theta+d_E}{\theta}$ for the $SEIR$ model. Korobeinikov [31] showed if $R_0 \leqslant 1$, the infection-free equilibrium Q_0 is globally asymptotically stable, while $R_0 > 1$, the disease-free equilibrium Q_0 is unstable, and the endemic equilibrium Q^* is globally asymptotically stable.

In fact, epidemic models are inevitably affected by environmental white noise which is an important component in realism, because it can provide an additional degree of realism in compared to their deterministic counterparts. Therefore, many stochastic models for the epidemic populations have been developed. In addition, both from a biological and from a mathematical perspective, there are different possible approaches to include random effects in the model. Here, we mainly mention three approaches. The first one is through time Markov chain model to consider environment noise in HIV epidemic (see, e.g., [42–45]). The second is with parameters perturbation. There is an intensive literature on this area, such as [14,15, 24,25,27,26,28,29,51]. Recently, Meng [37] discusses a stochastic epidemic model with double epidemic hypothesis, where he inputs the randomness on the infectious rates. Wanduku and Ladde [48] study a SIR delayed stochastic dynamic epidemic process in a two-scale dynamic structured population and the parameter perturbation is similar to that in [37]. The last important issue to model stochastic epidemic system is to robust the positive equilibria of deterministic models. In this situation, it is mainly to investigate whether the stochastic system preserves the asymptotic stability properties of the positive equilibria of deterministic models, see [6,11,12].

In this paper, we introduce randomness into the model (1.1) by replacing the parameters d_S, d_I and d_R by $d_S \to d_S + \sigma_1\, dB_1(t)$, $d_I \to d_I + \sigma_2\, dB_3(t)$ and $d_R \to d_R + \sigma_3\, dB_4(t)$ with the second approaches as [5] and [25]. This is only a first step in introducing stochasticity into the model. Ideally we would also like to introduce stochastic environmental variation into the other parameters, but to do this would make the analysis much too difficult.

Hence, we show a reasonable stochastic analogue of system (1.1) is given by

$$\begin{aligned}
dS &= \left(\lambda - \frac{\beta SI}{1+\alpha I} - d_S S\right) dt + \sigma_1 S\, dB_1(t),\\
dI &= \left[\frac{\beta SI}{1+\alpha I} - (d_I + \delta + \gamma) I\right] dt + \sigma_2 I\, dB_2(t),\\
dR &= (\gamma I - d_R R)\, dt + \sigma_3 R\, dB_3(t),
\end{aligned} \tag{1.3}$$

where $B_1(t)$, $B_2(t)$, $B_3(t)$ are independent Brownian motions, and σ_1, σ_2, σ_3 are their intensities.

Similarly, the parameters d_S, d_E, d_I and d_R are replaced by $d_S \to d_S + \sigma_1\, dB_1(t)$, $d_E \to d_E + \sigma_2\, dB_2(t)$, $d_I \to d_I + \sigma_3\, dB_3(t)$ and $d_R \to d_R + \sigma_4\, dB_4(t)$, stochastic analogue of system (1.2) is given by

$$\begin{aligned}
dS &= \left(\lambda - \frac{\beta SI}{1+\alpha I} - d_S S\right) dt + \sigma_1 S\, dB_1(t),\\
dE &= \left[\frac{\beta SI}{1+\alpha I} - (d_E + \theta) E\right] dt + \sigma_2 E\, dB_2(t),\\
dI &= \big[\theta E - (d_I + \delta + \gamma) I\big] dt + \sigma_3 I\, dB_3(t),\\
dR &= (\gamma I - d_R R)\, dt + \sigma_4 R\, dB_4(t).
\end{aligned} \tag{1.4}$$

The paper is organized as follows. In Section 2, we mainly study system (1.1). First, we show there is a unique nonnegative solution of system (1.3) for any positive initial value. Next, we investigate its asymptotic behavior according to $R_0 \leqslant 1$ or $R_0 > 1$. We conclude, although the solution of system (1.3) does not converge to Q_0 or Q^*, under some conditions, there is a unique stationary distribution for system (1.3) and it has ergodic property, provided the diffusion coefficients are sufficiently small. At last, we show that the positive solution of system (1.3) converges to the infection-free equilibrium exponentially as the diffusion coefficients are sufficiently large. In Section 3, we discuss corresponding property of system (1.4) by Lyapunov functions. In Section 4, we make simulations to conform our analytical results.

Throughout this paper, unless otherwise specified, let $(\Omega, \{\mathcal{F}_t\}_{t\geqslant 0}, P)$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t\geqslant 0}$ satisfying the usual conditions (i.e., it is right continuous and $\mathcal{F}_0$ contains all P-null sets). Denote

$$R^d_+ = \big\{x \in R^d\colon\ x_i > 0 \text{ for all } 1 \leqslant i \leqslant d\big\}.$$

In general, consider d-dimensional stochastic differential equation [38]

$$dx(t) = f\big(x(t), t\big)\, dt + g\big(x(t), t\big)\, dB(t) \quad \text{on } t \geqslant t_0 \tag{1.5}$$

with initial value $x(t_0) = x_0 \in R^d$. $B(t)$ denotes d-dimensional standard Brownian motions defined on the above probability space. Define the differential operator L associated with Eq. (1.5) by

$$L = \frac{\partial}{\partial t} + \sum_{i=1}^{d} f_i(x,t)\frac{\partial}{\partial x_i} + \frac{1}{2}\sum_{i,j=1}^{d}\big[g^T(x,t)g(x,t)\big]_{ij}\frac{\partial^2}{\partial x_i \partial x_j}.$$

If L acts on a function $V \in C^{2,1}(S_h \times \bar{R}_+; \bar{R}_+)$, then

$$LV(x,t) = V_t(x,t) + V_x(x,t) f(x,t) + \frac{1}{2}\, trace\big[g^T(x,t) V_{xx}(x,t) g(x,t)\big],$$

where $V_t = \frac{\partial V}{\partial t}$, $V_x = (\frac{\partial V}{\partial x_1}, \ldots, \frac{\partial V}{\partial x_d})$ and $V_{xx} = (\frac{\partial^2 V}{\partial x_i \partial x_j})_{d\times d}$. By Itô's formula,

$$dV\big(x(t),t\big) = LV\big(x(t),t\big)\, dt + V_x\big(x(t),t\big) g\big(x(t),t\big)\, dB(t).$$

2. The dynamics of system (1.3)

In this section, we consider system (1.3). First, we show there is a unique nonnegative solution no matter how large the intensities of noises are. In the next two parts, we mainly study the long time behavior of the solution.

2.1. Existence and uniqueness of the positive solution of (1.3)

Before investigating the dynamical behavior, the first concern thing is whether the solution exists globally. Hence in this section we first show the solution of system (1.3) is global and positive. In order for a stochastic differential equation to have a unique global (i.e. no explosion in a finite time) solution for any given initial value, the coefficients of the equation are generally required to satisfy the linear growth condition and local Lipschitz condition (cf. Arnold [3], Mao [38]). However, the coefficients of system (1.3) do not satisfy the linear growth condition (as the incidence is nonlinear), so the solution of system (1.3) may explode at a finite time (cf. Arnold [3], Mao [38]). In this section, using Lyapunov analysis method (mentioned in Mao [39]), we show the solution of system (1.3) is positive and global.

Theorem 2.1. *There is a unique solution $(S(t), I(t), R(t))$ of system (1.3) on $t \geqslant 0$ for any initial value $(S(0), I(0), R(0)) \in R^3_+$, and the solution will remain in R^3_+ with probability 1, namely $(S(t), I(t), R(t)) \in R^3_+$ for all $t \geqslant 0$ almost surely.*

Proof. Consider the diffusion process as follows

$$\begin{cases} dS = \left(\lambda - \dfrac{\beta S e^v}{1+\alpha e^v} - d_S S\right) dt + \sigma_1 S\, dB_1(t), \\ dv = \left[\dfrac{\beta S}{1+\alpha e^v} - \left(d_I + \delta + \gamma + \dfrac{\sigma_2^2}{2}\right)\right] dt + \sigma_2\, dB_2(t), \\ dR = [\gamma e^v - d_R R]\, dt + \sigma_3 R\, dB_3(t). \end{cases} \tag{2.1}$$

Since the coefficients of system (2.1) are locally Lipschitz continuous, there is a unique local solution of system (2.1). Let $I = e^v$, Itô's formula implies that system (1.3) has a unique local solution. Hence it suffices to prove that the unique local solution of system (1.3) is global and positive.

By the above discussion, we show that there is a unique local solution $(S(t), I(t), R(t))$ on $t \in [0, \tau_e)$, where τ_e is the explosion time (see Arnold [3]). To show this solution is global, we need to show that $\tau_e = \infty$ a.s. Let $m_0 \geqslant 0$ be sufficiently large so that $S(0), I(0), R(0)$ all lie within the interval $[1/m_0, m_0]$. For each integer $m \geqslant m_0$, define the stopping time

$$\tau_m = \inf\bigl\{t \in [0, \tau_e)\colon \min\bigl\{S(t), I(t), R(t)\bigr\} \leqslant 1/m \text{ or } \max\bigl\{S(t), I(t), R(t)\bigr\} \geqslant m\bigr\},$$

where throughout this paper, we set $\inf\emptyset = \infty$ (as usual $\emptyset$ denotes the empty set). Clearly, τ_m is increasing as $m \to \infty$. Set $\tau_\infty = \lim_{m\to\infty} \tau_m$, whence $\tau_\infty \leqslant \tau_e$ a.s. If we can show that $\tau_\infty = \infty$ a.s., then $\tau_e = \infty$ and $(S(t), I(t), R(t)) \in R^3_+$ a.s. for all $t \geqslant 0$. In other words, to complete the proof all we need to show is that $\tau_\infty = \infty$ a.s. If this statement is false, then there is a pair of constants $T > 0$ and $\epsilon \in (0, 1)$ such that

$$P\{\tau_\infty \leqslant T\} > \epsilon.$$

Hence there is an integer $m_1 \geqslant m_0$ such that

$$P\{\tau_m \leqslant T\} \geqslant \epsilon \quad \text{for all } m \geqslant m_1. \tag{2.2}$$

Define a C^2-function $V : R^3_+ \to R_+$ by

$$V(S, I, R) = \left(S - C - C\log\frac{S}{C}\right) + (I - 1 - \log I) + (R - 1 - \log R),$$

where $C > 0$ is a positive constant determined later. The non-negativity of this function can been derived from $u - 1 - \log u \geqslant 0$, $\forall u > 0$. Using Itô's formula, we get

$$\begin{aligned} dV &= \left[\lambda - d_S S - (d_I + \delta)I - d_R R - \frac{C\lambda}{S} - \frac{\beta S}{1+aI} - \frac{\gamma I}{R} + \frac{C\beta I}{1+aI} + C d_S + d_I + \delta + \gamma + d_R - \frac{C^2\sigma_1^2 + \sigma_2^2 + \sigma_3^2}{2}\right] \\ &\quad + \sigma_1\left(1 - \frac{C}{S}\right) S\, dB_1(t) + \sigma_2\left(1 - \frac{1}{I}\right) I\, dB_2(t) + \sigma_3\left(1 - \frac{1}{R}\right) R\, dB_3(t) \\ &:= LV\, dt + \sigma_1\left(1 - \frac{C}{S}\right) S\, dB_1(t) + \sigma_2\left(1 - \frac{1}{I}\right) I\, dB_2(t) + \sigma_3\left(1 - \frac{1}{R}\right) R\, dB_3(t), \end{aligned}$$

where

$$LV = \lambda - d_S S - d_R R - \frac{C\lambda}{S} - \frac{\beta S}{1+aI} - \frac{\gamma I}{R} + \left[\frac{C\beta}{1+aI} - (d_I + \delta)\right] I$$
$$+ C d_S + d_I + \delta + \gamma + d_R - \frac{C^2\sigma_1^2 + \sigma_2^2 + \sigma_3^2}{2}.$$

Choosing C such that $C\beta < d_I + \delta$, then

$$LV \leqslant \lambda + C d_S + d_I + \delta + \gamma + d_R - \frac{C^2\sigma_1^2 + \sigma_2^2 + \sigma_3^2}{2} := K.$$

Therefore,

$$\int_0^{\tau_m \wedge T} dV\big(S(r), I(r), R(r)\big) \leqslant \int_0^{\tau_m \wedge T} K\,dr + \int_0^{\tau_m \wedge T} \left[\sigma_1\left(1 - \frac{C}{S(r)}\right) S(r)\,dB_1(r) + \sigma_2\left(1 - \frac{1}{I(r)}\right) I(r)\,dB_2(r) + \sigma_3\left(1 - \frac{1}{R(r)}\right) R(r)\,dB_3(r)\right].$$

Taking expectation yields

$$E\big[V\big(S(\tau_m \wedge T), I(\tau_m \wedge T), R(\tau_m \wedge T)\big)\big] \leqslant V\big(S(0), I(0), R(0)\big) + E\int_0^{\tau_m \wedge T} K\,dr \leqslant V\big(S(0), I(0), R(0)\big) + KT. \tag{2.3}$$

Set $\Omega_m = \{\tau_m \leqslant T\}$ for $m \geqslant m_1$ and by (2.2), $P(\Omega_m) \geqslant \epsilon$. Note that for every $\omega \in \Omega_m$, there is at least one of $S(\tau_m, \omega)$, $I(\tau_m, \omega)$, $R(\tau_m, \omega)$ equals either m or $1/m$. If $S(\tau_m, \omega) = m$ or $1/m$, then

$$V\big(S(\tau_m \wedge T), I(\tau_m \wedge T), R(\tau_m \wedge T)\big) \geqslant \left(m - C - C\log\frac{m}{C}\right) \wedge \left(\frac{1}{m} - C - C\log\frac{1}{Cm}\right)$$
$$= C\left[\left(\frac{m}{C} - 1 - \log\frac{m}{C}\right) \wedge \left(\frac{1}{Cm} - 1 - \log\frac{1}{Cm}\right)\right];$$

while either $I(\tau_m, \omega) = m$ or $1/m$ or $R(\tau_m, \omega) = m$ or $1/m$, then

$$V\big(S(\tau_m \wedge T), I(\tau_m \wedge T), R(\tau_m \wedge T)\big) \geqslant (m - 1 - \log m) \wedge \left(\frac{1}{m} - 1 - \log\frac{1}{m}\right).$$

Consequently,

$$V\big(S(\tau_m \wedge T), I(\tau_m \wedge T), R(\tau_m \wedge T)\big)$$
$$\geqslant \left[C\left[\left(\frac{m}{C} - 1 - \log\frac{m}{C}\right) \wedge \left(\frac{1}{Cm} - 1 - \log\frac{1}{Cm}\right)\right]\right] \wedge \big[(m - 1 - \log m)\big] \wedge \left(\frac{1}{m} - 1 - \log\frac{1}{m}\right).$$

It then follows from (2.2) and (2.3) that

$$V\big(S(0), I(0), R(0)\big) + KT \geqslant E\big[1_{\Omega_m(\omega)} V\big(S(\tau_m \wedge T), I(\tau_m \wedge T), R(\tau_m \wedge T)\big)\big]$$
$$\geqslant \epsilon\left\{\left[C\left[\left(\frac{m}{C} - 1 - \log\frac{m}{C}\right) \wedge \left(\frac{1}{Cm} - 1 - \log\frac{1}{Cm}\right)\right]\right] \wedge \big[(m - 1 - \log m)\big] \wedge \left(\frac{1}{m} - 1 - \log\frac{1}{m}\right)\right\},$$

where $1_{\Omega_m(\omega)}$ is the indicator function of Ω_m. Letting $m \to \infty$ leads to the contradiction $\infty > V(S(0), I(0), R(0)) + KT = \infty$. So we must have $\tau_\infty = \infty$ a.s. □

2.2. Asymptotic behavior around the disease-free equilibrium of the deterministic model (1.1)

Obviously, $Q_0 = (\lambda/d_S, 0, 0)$ is the solution of system (1.1), which is called the disease-free equilibrium. If $R_0 \leqslant 1$, then Q_0 is globally asymptotically stable, which means the disease will vanish after some period of time. Therefore, it is interesting to study the disease-free equilibrium for controlling infectious disease. But, there is no disease-free equilibrium in system (1.3). It is natural to ask how we can consider the disease will extinct. In this subsection we mainly estimate the average of oscillation around Q_0 in time to exhibit whether the disease will die out.

Theorem 2.2. *Let $(S(t), I(t), R(t))$ be the solution of system* (1.3) *with initial value $(S(0), I(0), R(0)) \in R_+^3$. If $R_0 = \frac{\lambda\beta}{d_S(d_I+\delta+\gamma)} \leqslant 1$, $d_S > \sigma_1^2 \vee \sigma_3^2$ and $\frac{d_I+\delta+\gamma}{2} > \sigma_2^2$, then*

$$\limsup_{t\to\infty} \frac{1}{t} E \int_0^t \left[\left(d_S - \sigma_1^2\right)\left(S(r) - \frac{\lambda}{d_S}\right)^2 + \left(\frac{d_I+\delta+\gamma}{4} - \frac{\sigma_2^2}{2}\right) I^2(r) + \frac{(d_S - \sigma_3^2) d_S (d_I+\delta+\gamma)}{4\gamma} R^2(r)\right] dr$$
$$\leqslant \sigma_1^2 \left[\frac{\lambda^2}{d_S^2} + \frac{(d_S + d_I + \delta + \gamma)^2}{4 d_S (d_I+\delta+\gamma)}\right]. \tag{2.4}$$

Proof. Define C^2 functions $V_1, V_2, V_4 : R_+ \to R_+$, and $V_3 : R_+^2 \to R_+$, respectively by

$$V_1(S) = \frac{(S - \frac{\lambda}{d_S})^2}{2}, \qquad V_2(I) = I, \qquad V_3(R) = \frac{R^2}{2}, \qquad V_4(S, I) = \frac{(S - \frac{\lambda}{d_S} + I)^2}{2}.$$

Along the trajectories of system (1.3), we have

$$dV_1 = \left[\left(S - \frac{\lambda}{d_S}\right)\left(\lambda - d_S - \frac{\beta S I}{1 + aI}\right) + \frac{\sigma_1^2 S^2}{2}\right] dt + \left(S - \frac{\lambda}{d_S}\right) dB_1 := LV_1\, dt + \left(S - \frac{\lambda}{d_S}\right) dB_1,$$

where

$$LV_1 = \left(S - \frac{\lambda}{d_S}\right)\left(\lambda - d_S - \frac{\beta S I}{1 + aI}\right) + \frac{\sigma_1^2 S^2}{2}.$$

By computation,

$$LV_1 = -d_S\left(S - \frac{\lambda}{d_S}\right)^2 - \frac{\beta(S - \frac{\lambda}{d_S})^2 I}{1 + \alpha I} - \frac{\beta\lambda(S - \frac{\lambda}{d_S}) I}{d_S(1 + \alpha I)} + \frac{\sigma_1^2 S^2}{2}. \tag{2.5}$$

Similarly, we have

$$LV_2 = -\frac{\beta(S - \frac{\lambda}{d_S}) I}{1 + \alpha I} + I\left[\frac{\beta\lambda}{d_S(1 + \alpha I)} - (d_I + \delta + \gamma)\right], \qquad LV_3 = \gamma I R - d_R R^2 + \frac{\sigma_3^2 R^2}{2},$$

$$\begin{aligned} LV_4 = {} & -d_S\left(S - \frac{\lambda}{d_S}\right)^2 - (d_I + \delta + \gamma) I^2 - \frac{\beta(S - \frac{\lambda}{d_S})^2 I}{1 + \alpha I} \\ & - \frac{\beta\lambda(S - \frac{\lambda}{d_S}) I}{d_S(1 + \alpha I)} - (d_S + d_I + \delta + \gamma)\left(S - \frac{\lambda}{d_S}\right) I + \frac{\sigma_1^2 S^2}{2} + \frac{\sigma_2^2 I^2}{2}. \end{aligned} \tag{2.6}$$

As $R_0 \leqslant 1$,

$$LV_2 = -\frac{\beta(S - \frac{\lambda}{d_S}) I}{1 + \alpha I} + (d_I + \delta + \gamma) I\left[\frac{R_0}{1 + \alpha I} - 1\right] + \frac{\sigma_2^2}{2} \leqslant -\frac{\beta(S - \frac{\lambda}{d_S}) I}{1 + \alpha I}. \tag{2.7}$$

By (2.5)–(2.7), we have

$$LV_1 + \frac{\lambda}{d_S} LV_2 \leqslant -d_S\left(S - \frac{\lambda}{d_S}\right)^2 + \frac{\sigma_1^2 S^2}{2},$$

and

$$\begin{aligned} LV_4 + \frac{\lambda}{d_S} LV_2 & \leqslant -d_S\left(S - \frac{\lambda}{d_S}\right)^2 - (d_I + \delta + \gamma) I^2 - (d_S + d_I + \delta + \gamma)\left(S - \frac{\lambda}{d_S}\right) I + \frac{\sigma_1^2 S^2}{2} + \frac{\sigma_2^2 I^2}{2} \\ & \leqslant \left(S - \frac{\lambda}{d_S}\right)^2\left[-d_S + \frac{(d_S + d_I + \delta + \gamma)^2}{2(d_I + \delta + \gamma)}\right] - \frac{d_I + \delta + \gamma}{2} I^2 + \frac{\sigma_1^2 S^2}{2} + \frac{\sigma_2^2 I^2}{2}. \end{aligned}$$

Considering positive definite C^2 functions $V : R_+^3 \to R_+$ such that

$$V = V_4 + \frac{\lambda}{d_S} V_2 + \frac{(d_S + d_I + \delta + \gamma)^2}{2 d_S (d_I + \delta + \gamma)}\left(V_1 + \frac{\lambda}{d_S} V_2\right) + \frac{d_S(d_I + \delta + \gamma)}{2\gamma} V_3.$$

By computation,

$$
\begin{aligned}
LV &= LV_4 + \frac{\lambda}{d_S} LV_2 + \frac{(d_S+d_I+\delta+\gamma)^2}{2d_S(d_I+\delta+\gamma)}\left(LV_1 + \frac{\lambda}{d_S}LV_2\right) + \frac{d_S(d_I+\delta+\gamma)}{2\gamma} LV_3 \\
&\leqslant -d_S\left(S-\frac{\lambda}{d_S}\right)^2 - \frac{d_I+\delta+\gamma}{4} I^2 - \frac{d_S^2(d_I+\delta+\gamma)R^2}{4\gamma} + \frac{\sigma_1^2 S^2}{2} \\
&\quad + \frac{\sigma_2^2 I^2}{2} + \frac{\sigma_1^2(d_S+d_I+\delta+\gamma)^2}{4d_S(d_I+\delta+\gamma)} + \frac{\sigma_3^2 d_S(d_I+\delta+\gamma)R^2}{4\gamma}. \qquad (2.8)
\end{aligned}
$$

Since $a^2 \leqslant 2(a-b)^2 + 2b^2$, (2.8) implies that

$$
\begin{aligned}
LV &\leqslant -\big(d_S-\sigma_1^2\big)\left(S-\frac{\lambda}{d_S}\right)^2 - \left(\frac{d_I+\delta+\gamma}{4} - \frac{\sigma_2^2}{2}\right) I^2 - \frac{(d_S-\sigma_3^2)d_S(d_I+\delta+\gamma)R^2}{4\gamma} \\
&\quad + \frac{\sigma_1^2\lambda^2}{d_S^2} + \frac{\sigma_1^2(d_S+d_I+\delta+\gamma)^2}{4d_S(d_I+\delta+\gamma)}. \qquad (2.9)
\end{aligned}
$$

Taking expectation above, yields

$$
\begin{aligned}
EV(t) - V(0) = E\int_0^t LV(r)\,dr &\leqslant -\big(d_S-\sigma_1^2\big)E\int_0^t \left(S(r)-\frac{\lambda}{d_S}\right)^2 dr - \left(\frac{d_I+\delta+\gamma}{4} - \frac{\sigma_2^2}{2}\right)E\int_0^t I^2(r)\,dr \\
&\quad - \frac{(d_S-\sigma_3^2)d_S(d_I+\delta+\gamma)}{4\gamma}E\int_0^t R^2(r)\,dt + \left[\frac{\sigma_1^2\lambda^2}{d_S^2} + \frac{\sigma_1^2(d_S+d_I+\delta+\gamma)^2}{4d_S(d_I+\delta+\gamma)}\right]t.
\end{aligned}
$$

Hence

$$
\begin{aligned}
&\limsup_{t\to\infty} \frac{1}{t} E\int_0^t \left[\big(d_S-\sigma_1^2\big)\left(S(r)-\frac{\lambda}{d_S}\right)^2 + \left(\frac{d_I+\delta+\gamma}{4} - \frac{\sigma_2^2}{2}\right) I^2(r) + \frac{(d_S-\sigma_3^2)\,d_S(d_I+\delta+\gamma)}{4\gamma}R^2(r)\right] dr \\
&\quad \leqslant \sigma_1^2\left[\frac{\lambda^2}{d_S^2} + \frac{(d_S+d_I+\delta+\gamma)^2}{4d_S(d_I+\delta+\gamma)}\right]. \qquad \square
\end{aligned}
$$

Remark 2.1. From Theorem 2.3, we display, under some conditions, the solution of system (1.3) will oscillate around the disease-free equilibrium in time, and the disturbance intensity is proportional to the intensity of the white noise. In a biological view, as the intensity of stochastic perturbations is small, the solution of system (1.3) will fluctuate around the disease-free equilibrium of system (1.3) most of the time.

Besides, Q_0 becomes the disease-free equilibrium of system (1.1) as $\sigma_1 = 0$. In the proof of Theorem 2.2, we see

$$
LV \leqslant -d_S\left(S-\frac{\lambda}{d_S}\right)^2 - \left(\frac{d_I+\delta+\gamma}{4} - \frac{\sigma_2^2}{2}\right) I^2 - \frac{(d_S-\sigma_3^2)d_S(d_I+\delta+\gamma)R^2}{4\gamma}. \qquad (2.10)
$$

Thus, the solution of system (1.1) is stochastically asymptotically stable in the large (Mao [38]) as $d_S > \sigma_1^2 \vee \sigma_3^2$, $\frac{d_I+\delta+\gamma}{2} > \sigma_2^2$.

2.3. Asymptotic behavior around the endemic equilibrium of the deterministic model (1.1) in system (1.3)

In studying epidemic dynamical system, we are interested in two problems. One is the occurring of extinction, which has been shown in the above part, another is the persistent presence in a population. In the deterministic models, the second problem is solved by showing that the endemic equilibrium of corresponding model is a global attractor or is globally asymptotic stable. But, there is none of endemic equilibrium in system (1.3). We obtain a unique stationary distribution of system (1.3) instead of the endemic equilibrium (see [10]). Furthermore, since system (1.3) is the perturbed system of system (1.1) which has an endemic equilibrium Q^*, it seems reasonable to consider the disease will prevail if the solution of system (1.3) has the ergodic property. Before giving the main theorem, we first give a lemma (see [19]).

Let $X(t)$ be a regular temporally homogeneous Markov process in $E_l \subset R^l$ described by the stochastic differential equation

$$
dX(t) = b(X)\,dt + \sum_{r=1}^{k} \sigma_r(X)\,dB_r(t),
$$

and the diffusion matrix is defined as follows

$$A(x) = \big(a_{i,j}(x)\big), \quad a_{i,j}(x) = \sum_{r=1}^{k} \sigma_r^i(x)\sigma_r^j(x).$$

Lemma 2.1. *(See [19].) We assume that there exists a bounded domain $U \subset E_l$ with regular boundary, having the following properties:*

(B.1) *In the domain U and some neighborhood thereof, the smallest eigenvalue of the diffusion matrix $A(x)$ is bounded away from zero.*
(B.2) *If $x \in E_l \backslash U$, the mean time τ at which a path issuing from x reaches the set U is finite, and $\sup_{x \in K} E_x\tau < \infty$ for every compact subset $K \subset E_l$.*

Then, the Markov process $X(t)$ has a stationary distribution $\mu(\cdot)$ with density in E_l such that for any Borel set $B \subset E_l$

$$\lim_{t\to\infty} P(t, x, B) = \mu(B),$$

and

$$P_x\left\{\lim_{T\to\infty} \frac{1}{T}\int_0^T f\big(x(t)\big)\,dt = \int_{E_l} f(x)\mu(dx)\right\} = 1,$$

for all $x \in E_l$ and $f(x)$ being a function integrable with respect to the measure μ.

Remark 2.2. The proof is given by [19] in detail. Exactly, the existence of stationary distribution with density is referred to Theorem 4.1, p. 119, and Lemma 9.4, p. 138. The ergodicity and the weak convergence is obtained in Theorem 5.1, p. 121, and Theorem 7.1, p. 130.

To validate assumptions (B.1) and (B.2), it suffices to prove that there exists some neighborhood U and a nonnegative C^2-function such that $A(x)$ is uniformly elliptical in U and for any $x \in E_l \backslash U$, $LV(x) \leqslant -C$ for some $C > 0$ (details refer to [53], p. 1163).

Applying Lemma 2.1, we get the following result.

Theorem 2.3. *Let $(S(t), I(t), R(t))$ be the solution of system (1.3) with any initial value $(S(0), I(0), R(0)) \in R_+^3$. If $R_0 = \frac{\lambda\beta}{d_S(d_I+\delta+\gamma)} > 1$, $\sigma_i > 0$, $1 \leqslant i \leqslant 3$ and $\min\{m_1 S^{*2}, m_2 I^{*2}, m_3 R^{*2}\} > \delta > 0$, then there is a unique stationary distribution μ for system (1.3) and the ergodicity holds. Here (S^*, I^*, R^*) is the unique endemic equilibrium of (1.1), $m_1 = d_S - \sigma_1^2$, $m_2 = \frac{d_I+\delta+\gamma}{2} - \sigma_2^2$, $m_3 = \frac{d_R(d_I+\delta+\gamma)}{\gamma^2}(\frac{d_R^2}{2} - \sigma_3^2)$ and*

$$\delta = \sigma_1^2 S^{*2} + \sigma_2^2 I^{*2} + \frac{(1+aI^*)(d_S + d_I + \delta + \gamma)}{2\beta}\sigma_2^2 + \frac{d_R(d_I+\delta+\gamma)}{\gamma^2}\sigma_3^2. \tag{2.11}$$

Especially, we have

$$\lim_{t\to\infty} \frac{1}{t} E\int_0^t \big[m_1\big(S(r) - S^*\big)^2 + m_2\big(I(r) - I^*\big)^2 + m_3\big(R(r) - R^*\big)^2\big]\,dr \leqslant \delta.$$

Proof. When $R_0 > 1$, there is a unique endemic equilibrium $Q^* = (S^*, I^*, R^*)$. Setting the right sides of system (1.1) to be zero, we see

$$\lambda = \frac{\beta S^* I^*}{1+\alpha I^*} + d_S S^*, \qquad \frac{\beta S^*}{1+\alpha I^*} = d_I + \delta + \gamma, \qquad \gamma I^* = d_R R^*. \tag{2.12}$$

Define C^2 functions as follows

$$V_1(S, I) = \frac{(S - S^* + I - I^*)^2}{2}, \qquad V_2(I) = I - I^* - I^* \log\frac{I}{I^*}, \qquad V_3(R) = \frac{(R - R^*)^2}{2}.$$

(2.12) implies

$$\begin{aligned}dV_1 &= \left[\left(S-S^*+I-I^*\right)\left(d_S\left(S^*-S\right)+(d_I+\delta+\gamma)\left(I^*-I\right)\right)+\frac{\sigma_1^2S^2+\sigma_2^2I^2}{2}\right]ds\\ &\quad+\left(S-S^*+I-I^*\right)(\sigma_1\,dB_1+\sigma_2\,dB_2)\\ &:= LV_1\,dt+\left(S-S^*+I-I^*\right)(\sigma_1\,dB_1+\sigma_2\,dB_2).\end{aligned}$$

By computation,

$$LV_1=-d_S\left(S-S^*\right)^2-(d_I+\delta+\gamma)\left(I-I^*\right)^2-(d_S+d_I+\delta+\gamma)\left(S-S^*\right)\left(I^*-I\right)+\frac{\sigma_1^2S^2+\sigma_2^2I^2}{2}. \tag{2.13}$$

We also note that

$$\begin{aligned}LV_2 &= \left(I-I^*\right)\left(\frac{\beta S}{1+aI}-\frac{\beta S^*}{1+aI^*}\right)+\frac{\sigma_2^2}{2}=\left(I-I^*\right)\left[\beta S\left(\frac{1}{1+aI}-\frac{1}{1+aI^*}\right)+\frac{\beta(S-S^*)}{1+aI^*}\right]+\frac{\sigma_2^2}{2}\\ &= -\frac{a\beta S(I-I^*)^2}{(1+aI)(1+aI^*)}+\frac{\beta(S-S^*)(I-I^*)}{1+aI^*}+\frac{\sigma_2^2}{2}\leqslant\frac{\beta(S-S^*)(I-I^*)}{1+aI^*}+\frac{\sigma_2^2}{2}\end{aligned} \tag{2.14}$$

and

$$\begin{aligned}LV_3 &= \left(R-R^*\right)\left[\gamma\left(I-I^*\right)-d_R\left(R-R^*\right)\right]+\frac{\sigma_3^2R^2}{2}=\gamma\left(I-I^*\right)\left(R-R^*\right)-d_R\left(R-R^*\right)^2+\frac{\sigma_3^2R^2}{2}\\ &\leqslant\frac{\gamma^2(I-I^*)^2}{2d_R}-\left(\frac{d_R}{2}-\sigma_3^2\right)\left(R-R^*\right)^2+\sigma_3^2R^{*2},\end{aligned} \tag{2.15}$$

where the last inequality is derived from $ab\leqslant\frac{a^2+b^2}{2}$.

Now, we define C^2 function $V: R_+^3\to R_+$ as follows

$$V(S,I,R)=V_1(S,I)+\frac{(1+aI^*)(d_S+d_I+\delta+\gamma)}{\beta}V_2(I)+\frac{d_R(d_I+\delta+\gamma)}{\gamma^2}V_3.$$

Together with (2.13)–(2.15), this implies

$$\begin{aligned}LV &= -d_S\left(S-S^*\right)^2-\frac{d_I+\delta+\gamma}{2}\left(I-I^*\right)^2-\frac{d_R(d_I+\delta+\gamma)}{\gamma^2}\left(\frac{d_R^2}{2}-\sigma_3^2\right)\left(R-R^*\right)^2\\ &\quad+\frac{\sigma_1^2S^2+\sigma_2^2I^2}{2}+\frac{(1+aI^*)(d_S+d_I+\delta+\gamma)}{2\beta}\sigma_2^2+\frac{d_R(d_I+\delta+\gamma)}{\gamma^2}\sigma_3^2\\ &= -\left(d_S-\sigma_1^2\right)\left(S-S^*\right)^2-\left(\frac{d_I+\delta+\gamma}{2}-\sigma_2^2\right)\left(I-I^*\right)^2-\frac{d_R(d_I+\delta+\gamma)}{\gamma^2}\left(\frac{d_R^2}{2}-\sigma_3^2\right)\left(R-R^*\right)^2\\ &\quad+\sigma_1^2S^{*2}+\sigma_2^2I^{*2}+\frac{(1+aI^*)(d_S+d_I+\delta+\gamma)}{2\beta}\sigma_2^2+\frac{d_R(d_I+\delta+\gamma)}{\gamma^2}\sigma_3^2,\end{aligned}$$

which can be simplified into

$$LV\leqslant -m_1\left(S-S^*\right)^2-m_2\left(I-I^*\right)^2-m_3\left(R-R^*\right)^2+\delta,$$

according to (2.11). Note that if $\delta<\min\{m_1S^{*2},m_2I^{*2},m_3R^{*2}\}$, then the ellipsoid

$$-m_1\left(S-S^*\right)^2-m_2\left(I-I^*\right)^2-m_3\left(R-R^*\right)^2+\delta=0$$

lies entirely in R_+^3. We can take U to be any neighborhood of the ellipsoid with $\bar{U}\subseteq E_l=R_+^3$, so for $x\in U\setminus E_l$, $LV\leqslant -C$, which implies condition (B.2) in Lemma 2.1 is satisfied. Besides, there is $M>0$ such that

$$\sum_{i,j=1}^{n}\left(\sum_{k=1}^{n}a_{ik}(x)a_{jk}(x)\right)\xi_i\xi_j=\sigma_1^2x_1^2\xi_1^2+\sigma_2^2x_2^2\xi_2^2+\sigma_3^2x_3^2\xi_3^3\geqslant M|\xi|^2\quad\text{for all } x\in\bar{U},\ \xi\in R^3.$$

Applying Rayleigh's principle (see [41], p. 349), condition (B.1) is satisfied. Therefore, the stochastic system (1.3) has a unique stationary distribution $\mu(\cdot)$ and it is ergodic. $\square$

2.4. Exponential stability of system (1.3)

In this subsection, we investigate the exponential decay of the global solution of system (1.3) as the intensity of white noise is great. It can be shown below, even if the endemic equilibrium exists in the system (1.1), the stochastic effect may make washout more likely in system (1.3).

Theorem 2.4. *Let $(S(t), I(t), R(t))$ be the solution of system* (1.3) *with any initial value $(S(0), I(0), R(0)) \in R^3_+$. If $(d_I + \delta + \gamma)(R_0 - 1) < \frac{\sigma_2^2}{2}$, then*

$$\limsup_{t\to\infty} \frac{1}{t} \log I(t) \leqslant (d_I + \delta + \gamma)(R_0 - 1) - \frac{\sigma_2^2}{2},$$

$$\limsup_{t\to\infty} \frac{1}{t} \log R(t) \leqslant \left[-\left(d_R + \frac{\sigma_3^2}{2}\right)\right] \vee \left[(d_I + \delta + \gamma)(R_0 - 1) - \frac{\sigma_2^2}{2}\right],$$

$$\lim_{t\to\infty} \frac{1}{t} \int_0^t S(u)\,du = \frac{\lambda}{d_S}, \quad a.e., \qquad S(t) \to^w \nu, \quad as\ t \to \infty,$$

where $\to^w$ means the convergence in distribution and ν is a probability measure in R^1_+ such that $\int_0^\infty x\nu(dx) = \frac{\lambda}{d_S}$. In particularly, ν has density $(A\sigma_1^2 x^2 p(x))^{-1}$, where A is a normal constant,

$$p(x) = \exp\left(-\frac{2\lambda}{\sigma_1^2}\right) x^{\frac{2d_S}{\sigma_1^2}} \exp\left(\frac{2\lambda}{\sigma_1^2 x}\right), \quad x > 0. \tag{2.16}$$

Proof. By comparison theorem, we see that $S(t) \leqslant X(t)$, where $X(t)$ is the global solution of the following stochastic system with initial value $X(0) = S(0)$:

$$dX = (\lambda - d_S X)\,dt + \sigma_1 X\,dB_1(t). \tag{2.17}$$

Obviously, (2.17) is a diffusion process lying in R^1_+.

Firstly, we show (2.17) is stable in distribution and ergodic. Let $Y(t) = X(t) - \frac{\lambda}{d_S}$, then $Y(t)$ satisfies

$$dY = -d_S Y\,dt + \sigma_1\left(Y + \frac{\lambda}{d_S}\right) dB_1(t). \tag{2.18}$$

Theorem 2.1 (a) in [4] with $C = 1$ implies that the diffusion process $Y(t)$ is stable in distribution as $t \to \infty$, so does $X(t)$.

To prove the ergodicity of $X(t)$, we define

$$p(x) = \exp\left(-2\int_1^x \frac{\lambda - d_S y}{\sigma_1^2 y^2}\,dy\right).$$

By computation,

$$p(x) = \exp\left(-\frac{2\lambda}{\sigma_1^2}\right) x^{\frac{2d_S}{\sigma_1^2}} \exp\left(\frac{2\lambda}{\sigma_1^2 x}\right),$$

and it is noted that for each integer $n \geqslant 1$, there exist positive constants $C_1(n)$, $C_2(n)$ and $M(n)$ such that

$$p(x) \geqslant C_1(n) x^{\frac{2d_S}{\sigma_1^2} - n}, \quad \text{as } 0 < x < \frac{1}{M(n)},$$

$$p(x) \geqslant C_2(n) x^{\frac{2d_S}{\sigma_1^2}}, \quad \text{as } x > M(n). \tag{2.19}$$

Therefore, together with (2.19) we see

$$\int_1^\infty p(x)\,dx = \infty, \qquad \int_0^1 p(x)\,dx = \infty, \qquad \int_0^\infty \frac{dx}{\sigma_1^2 p(x) x^2} < \infty.$$

So $X(t)$ is ergodic (Theorem 1.16 in [32]), and with respect to the Lebesgue measure its invariant measure ν has density $(A\sigma_1^2 x^2 p(x))^{-1}$, where A is a normal constant.

Now, we show that $f(t) := EX^p(t)$ is uniformly bounded for some $p > 1$ determined later. Applying Itô's formula to X^p, we have

$$dX^p = \left(p\lambda X^{p-1} - pd_S X^p + \frac{\sigma_1^2 p(p-1)X^p}{2}\right)dt + p\sigma_1 X^p\,dB_1(t).$$

Taking expectation of equation above, and using the fact $a^{\frac{1}{p}} b^{\frac{p-1}{p}} \leqslant \frac{a}{p} + \frac{b(p-1)}{p}$, $a, b > 0$,

$$f'(t) \leqslant \frac{\lambda^p}{\varepsilon^{p-1}} + \frac{p\varepsilon}{p-1} f(t) - p\left[d_S - \frac{\sigma_1^2(p-1)}{2}\right] f(t) \leqslant \frac{\lambda^p}{\varepsilon^{p-1}} + p\left[\frac{\varepsilon}{p-1} - \left(d_S - \frac{\sigma_1^2(p-1)}{2}\right)\right] f(t).$$

Choosing $\varepsilon > 0$ sufficiently small and $p > 1$ closely enough to 1 such that

$$d_S - \frac{\sigma_1^2(p-1)}{2} < 0, \qquad \frac{\varepsilon}{p-1} - \left(d_S - \frac{\sigma_1^2(p-1)}{2}\right) < 0.$$

Hence, $\sup_{t\geqslant 0} EX^p(t) = \sup_{t\geqslant 0} f(t) < \infty$, implying that $\int_0^\infty x^p \nu(dx) < \infty$. Together with its ergodicity we have

$$P_x\left\{\lim_{T\to\infty} \frac{1}{T}\int_0^T X(t)\,dt = \int_0^\infty x\nu(dx)\right\} = 1, \tag{2.20}$$

for all $x \in R_+^1$. On the other hand, Jensen's inequality yields

$$E\left[\frac{1}{T}\int_0^T X(t)\,dt\right]^p \leqslant E\frac{1}{T}\int_0^T X^p(t)\,dt \leqslant \sup_{t\geqslant 0} EX^p(t) < \infty,$$

therefore, $\{\frac{1}{T}\int_0^T X(t)\,dt,\ t \geqslant 0\}$ is uniformly integrable. Together with (2.20), we have

$$E\frac{1}{T}\int_0^T X(t)\,dt \to \int_0^\infty x\nu(dx). \tag{2.21}$$

Taking expectation of (2.17), we have

$$\frac{EX(t)}{t} = \lambda - \frac{d_S}{t} E\int_0^t X(s)\,ds.$$

Let $t \to \infty$, taking (2.21) into account, then we see

$$\int_0^\infty x\nu(dx) = \frac{\lambda}{d_S}.$$

Secondly, using Itô's formula to $\log I$ and the fact that $S(t) \leqslant X(t)$ show

$$\begin{aligned} d\log I(t) &= \left(\frac{\beta S(t)}{1+\alpha I(t)} - \left(d_I + \delta + \gamma + \frac{\sigma_2^2}{2}\right)\right)dt + \sigma_2\,dB_2(t) \\ &\leqslant \left[\beta X(t) - \left(d_I + \delta + \gamma + \frac{\sigma_2^2}{2}\right)\right]dt + \sigma_2\,dB_2(t). \end{aligned}$$

Integrating the above inequality, together with (2.20) and the fact that $\lim_{t\to\infty} \frac{B_2(t)}{t} = 0$, yields for almost sure $\omega \in \Omega$,

$$\limsup_{t\to\infty} \frac{1}{t}\log I(t,\omega) \leqslant \frac{\beta\lambda}{d_S} - \left(d_I + \delta + \gamma + \frac{\sigma_2^2}{2}\right) \leqslant (d_I + \delta + \gamma)(R_0 - 1) - \frac{\sigma_2^2}{2}. \tag{2.22}$$

Alike the proof of Theorem 5.2 in [15], pp. 1093–1096, we introduce another diffusion process $\tilde{R}(t)$ which is defined by the initial condition $\tilde{R}(0) = R(0)$ and the stochastic differential equation

$$d\tilde{R} = -d_R\tilde{R}\,dt + \sigma_3\tilde{R}\,dB_3(t).$$

Then

$$d(R-\tilde{R}) = \big(\gamma I - d_R(R-\tilde{R})\big)\,dt + \sigma_3(R-\tilde{R})\,dB_3(t).$$

The solution is given by

$$R(t)-\tilde{R}(t) = \gamma \exp\left\{-\left(d_R+\frac{\sigma_3^2}{2}\right)t + \sigma_3 B_3(t)\right\}\int_0^t \exp\left\{\left(d_R+\frac{\sigma_3^2}{2}\right)s - \sigma_3 B_3(s)\right\} I(s)\,ds.$$

By (2.22) and the fact that the trajectory of B_3 is continuous, $\lim_{t\to\infty}\frac{B_3(t)}{t}=0$, a.e. it has been shown there exists some null set $\mathcal{N}$ such that $P(\mathcal{N})=0$ and for any $\omega\notin\mathcal{N}$, $B_3(\cdot,\omega)$ is continuous,

$$\lim_{t\to\infty}\frac{1}{t}\log I(t,\omega) \leqslant (d_I+\delta+\gamma)(R_0-1)-\frac{\sigma_2^2}{2}, \qquad \lim_{t\to\infty}\frac{\max_{s\leqslant t}|B_3(s,\omega)|}{t}=0,$$

where the last equation is derived the continuity of $B_3(\cdot,\omega)$ and $\lim_{t\to\infty}\frac{B_3(t,\omega)}{t}=0$ for any $\omega\notin\mathcal{N}$. Thus for any $\tilde{\varepsilon}>0$, there exists $T=T(\omega)$ such that

$$I(t,\omega)\leqslant \exp\big((m+\tilde{\varepsilon})t\big), \quad \forall t\geqslant T, \tag{2.23}$$

where $m=(d_I+\delta+\gamma)(R_0-1)-\frac{\sigma_2^2}{2}$. Hence for all $\omega\in\Omega$, if $t>T(\omega)$, then

$$\begin{aligned}\big|R(t,\omega)-\tilde{R}(t,\omega)\big| \leqslant{}& \gamma\exp\left\{-\left(d_R+\frac{\sigma_3^2}{2}\right)t+\sigma_3 B_3(t,\omega)\right\}\int_0^T \exp\left\{\left(d_R+\frac{\sigma_3^2}{2}\right)s-\sigma_3 B_3(s,\omega)\right\} I(s,\omega)\,ds\\ &+\gamma\exp\left\{-\left(d_R+\frac{\sigma_3^2}{2}\right)t+\sigma_3 B_3(t,\omega)+\sigma_3\max_{s\leqslant t}\big|B_3(s,\omega)\big|\right\}\\ &\cdot\int_T^t \exp\left\{\left(d_R+\frac{\sigma_3^2}{2}+m+\tilde{\varepsilon}\right)s\right\}ds.\end{aligned}$$

Therefore, we get for any $\omega\notin\mathcal{N}$,

$$\limsup_{t\to\infty}\frac{1}{t}\log\big|R(t,\omega)-\tilde{R}(t,\omega)\big| \leqslant \left[-\left(d_R+\frac{\sigma_3^2}{2}\right)\right]\vee[m+\tilde{\varepsilon}].$$

Let $\tilde{\varepsilon}\to 0$, we get $\limsup_{t\to\infty}\frac{1}{t}\log|R(t)-\tilde{R}(t)|\leqslant[-(d_R+\frac{\sigma_3^2}{2})]\vee m$, a.e. On the other hand,

$$\tilde{R}(t) = R(0)\exp\left\{-\left(d_R+\frac{\sigma_3^2}{2}\right)t+\sigma_3 B_3(t)\right\},$$

and hence, $\limsup_{t\to\infty}\frac{1}{t}\log\tilde{R}(t) = -(d_R+\frac{\sigma_3^2}{2})$. Therefore,

$$\limsup_{t\to\infty}\frac{1}{t}\log R(t)\leqslant\left[-\left(d_R+\frac{\sigma_3^2}{2}\right)\right]\vee\left[(d_I+\delta+\gamma)(R_0-1)-\frac{\sigma_2^2}{2}\right], \quad \text{a.e.}$$

At last, we concentrate on $S(t)$. We shall eventually show that $S(t)$ is stable in distribution. To do this, as in [15], we introduce a new stochastic process $S_\varepsilon(t)$ which is defined by its initial condition $S_\varepsilon(0)=S(0)$ and the stochastic differential equation

$$dS_\varepsilon = \big[\lambda-(d_S+\varepsilon)S_\varepsilon\big]\,dt + \sigma_1 S_\varepsilon\,dB_1(t).$$

First we prove that

$$\lim_{t\to\infty}\big(S(t)-S_\varepsilon(t)\big)\geqslant 0, \quad \text{a.e.} \tag{2.24}$$

Therefore consider

$$d(S-S_\varepsilon) = \left[\left(\varepsilon-\frac{\beta I}{1+aI}\right)S-(d_S+\varepsilon)(S-S_\varepsilon)\right]dt+\sigma_1(S-S_\varepsilon)\,dB_1(t).$$

The solution is given by

$$S(t)-S_\varepsilon(t)=\exp\left\{-\left(d_S+\varepsilon+\frac{\sigma_1^2}{2}\right)t+\sigma_1 B_1(t)\right\}\int_0^t \exp\left\{\left(d_S+\varepsilon+\frac{\sigma_1^2}{2}\right)s-\sigma_1 B_1(s)\right\}\left(\varepsilon-\frac{\beta I(s)}{1+\alpha I(s)}\right)S(s)\,ds.$$

By (3.21), for almost $\omega\in\Omega$, $\exists T=T(\omega)$ such that

$$\varepsilon>\frac{\beta I(t)}{1+\alpha I(t)},\quad \forall t>T.$$

Hence as the proof of Theorem 5.2 in [15], pp. 1092–1093, for almost $\omega\in\Omega$, for any $t>T$,

$$S(t)-\tilde S(t)\geqslant \exp\left\{-\left(d_S+\varepsilon+\frac{\sigma_1^2}{2}\right)t+\sigma_1 B_1(t)\right\}\cdot\int_0^T \exp\left\{\left(d_S+\varepsilon+\frac{\sigma_1^2}{2}\right)s-\sigma_1 B_1(s)\right\}\left(\varepsilon-\frac{\beta I(s)}{1+\alpha I(s)}\right)S(s)\,ds.$$

Therefore,

$$\liminf_{t\to\infty}\big(S(t)-S_\varepsilon(t)\big)\geqslant 0,\quad \text{a.e.}$$

Next, it is noted that

$$d(X-S_\varepsilon)=\big[\varepsilon S_\varepsilon-d_S(X-S_\varepsilon)\big]dt+\sigma_1(X-S_\varepsilon)\,dB_1(t).$$

Taking the expectation of above equation, we see

$$E\big|X(t)-S_\varepsilon(t)\big|=\int_0^t\big[\varepsilon S_\varepsilon(u)-d_S\big(X(u)-S_\varepsilon(u)\big)\big]du\leqslant\int_0^t\big[\varepsilon X(u)-d_S\big|X(u)-S_\varepsilon(u)\big|\big]du,$$

where the last inequality is using the fact that $S_\varepsilon(t)\leqslant X(t)$. Hence, we have

$$E\big|X(t)-S_\varepsilon(t)\big|\leqslant\frac{\varepsilon\sup_{u\geqslant 0}EX_u}{d_S}\big(1-\exp(-d_S t)\big).$$

This implies that

$$\lim_{\varepsilon\to 0}\inf\lim_{t\to\infty}E\big|X(t)-S_\varepsilon(t)\big|=0. \tag{2.25}$$

Combining (2.24), (2.25) and the fact that $S(t)\leqslant X(t)$, we get

$$\lim_{t\to\infty}\big(X(t)-S(t)\big)=0,\quad \text{in probability.}$$

It has been shown that $X(t)$ converges weakly to distribution ν, so does $S(t)$ as $t\to\infty$. □

Remark 2.3. Note that Theorem 2.4 does not assume $R_0<1$. Note also that the conditions of Theorem 2.4 cannot possibly be satisfied in the deterministic model when $\sigma_1=\sigma_2=0$ as $R_0>1$. However, if the variance σ_1^2 is large enough, these conditions will always be satisfied. This is an interesting result as it says that if the noise variance is large enough then the population of infected and recovered will always die out, whatever the other parameter values, even if $R_0>1$. Thus the behavior of the system with added environmental noise can be very different from the behavior of the deterministic system (1.1).

3. The dynamics of system (1.4)

In this section we study the dynamics of system (1.4). Compared with system (1.3), system (1.4) concludes the incubation of the communicable disease. As in Section 2, we show there is a unique nonnegative solution and mainly investigate its ergodicity and extinction under different conditions, respectively.

3.1. Existence and uniqueness of the positive solution of (1.4)

Theorem 3.1. *There is a unique solution $(S(t), E(t), I(t), R(t))$ of system (1.4) on $t \geqslant 0$ for any initial value $(S(0), E(0), I(0), R(0)) \in R_+^4$, and the solution will remain in R_+^4 with probability 1, namely, for almost sure, $(S(t), E(t), I(t), R(t)) \in R_+^4$ for all $t \geqslant 0$.*

Proof. Alike the proof of Theorem 2.1, it suffices to define a C^2 function $V : R_+^4 \to R_+$ such that V approach infinity at the boundary of R_+^4 and $LV(x) \leqslant C$, $\forall x \in R_+^4$ for some positive constant C. Consider C^2 function $V : R_+^4 \to R_+$ as follows

$$V(S,E,I,R) = \left(I - \frac{d_I+\delta}{\beta} - \frac{d_I+\delta}{\beta}\log\frac{\beta I}{d_I+\delta}\right) + (E-1-\log E) + (I-1-\log I) + (R-1-\log R).$$

By computation,

$$\begin{aligned} LV(S,E,I,R) &= \left(\lambda + \frac{(d_I+\delta)d_S}{\beta} + d_E + \theta + d_I + \delta + \gamma + d_R\right) - d_S S - d_E E - d_R R \\ &\quad - \frac{\lambda(d_I+\delta)}{\beta S} - \frac{\beta SI}{E(1+\alpha I)} - \frac{\theta E}{I} - \frac{\gamma I}{R} - (d_I+\delta)I + \frac{(d_I+\delta)I}{1+\alpha I} \\ &\leqslant \lambda + \frac{(d_I+\delta)d_S}{\beta} + d_E + \theta + d_I + \delta + \gamma + d_R. \end{aligned}$$

This completes the proof of Theorem 3.1. □

3.2. Asymptotic behavior around the disease-free equilibrium of the deterministic model (1.2)

It is noted the disease-free equilibrium $Q_0 = (\lambda/d_S, 0, 0, 0)$ is the solution of system (1.2). If $R_0 = \frac{\lambda\beta\theta}{d_S(d_I+\delta+\gamma)(\theta+d_E)} \leqslant 1$, then Q_0 is globally asymptotically stable, and the disease will vanish after some period of time. But, there is no disease-free equilibrium in system (1.4). In this subsection we mainly estimate the average oscillation around Q_0 in time.

Theorem 3.2. *Let $(S(t), E(t), I(t), R(t))$ be the solution of system (1.4) with initial value $(S(0), E(0), I(0), R(0)) \in R_+^4$. If $R_0 = \frac{\lambda\beta\theta}{d_S(d_I+\delta+\gamma)(\theta+d_E)} \leqslant 1$, $d_S > \sigma_1^2 + \frac{(d_S+d_E+\theta)^2\sigma_1^2}{2d_S(d_E+\theta)}$, $d_E > \sigma_1^2$, $d_I > 2\sigma_3^2$ and $d_R > \sigma_4^2$, then*

$$\begin{aligned} &\limsup_{t\to\infty} \frac{1}{t} E\int_0^t \left[\left(d_S - \sigma_1^2 - \frac{(d_S+d_E+\theta)^2\sigma_1^2}{2d_S(d_E+\theta)}\right)\left(S - \frac{\lambda}{d_S}\right)^2 + \frac{d_E-\sigma_1^2}{2}E^2 + \frac{d_I^2 - 2d_I\sigma_3^2}{4\theta}I^2 + \frac{d_R^2 d_I^2 - d_R d_I^2\sigma_4^2}{4\theta\delta^2}R^2\right] \\ &\quad \leqslant \frac{\lambda^2\sigma_1^2}{d_S^2}\left[\frac{(d_S+d_E+\theta)^2}{2d_S(d_E+\theta)} + 1\right]. \end{aligned} \tag{3.1}$$

Proof. Define C^2 functions $V_1, V_2, V_3 : R_+ \to R_+$, and $V_4, V_5 : R_+^2 \to R_+$, respectively by

$$\begin{aligned} &V_1(S) = \frac{(S-\frac{\lambda}{d_S})^2}{2}, \qquad V_2(I) = \frac{I^2}{2}, \qquad V_3(R) = \frac{R^2}{2}, \\ &V_4(E,I) = E + \frac{d_E+\theta}{\theta}I, \qquad V_5(S,E) = \frac{(S-\frac{\lambda}{d_S}+E)^2}{2}. \end{aligned}$$

Along the trajectories of system (1.4), we have

$$\begin{aligned} LV_1 &= -d_S\left(S-\frac{\lambda}{d_S}\right)^2 - \frac{\beta(S-\frac{\lambda}{d_S})^2 I}{1+\alpha I} - \frac{\beta\lambda(S-\frac{\lambda}{d_S})I}{d_S(1+\alpha I)} + \frac{\sigma_1^2 S^2}{2}, \\ LV_2 &= \theta EI - d_I I^2 + \frac{\sigma_3^2 I^2}{2}, \\ LV_3 &= \delta IR - d_R R^2 + \frac{\sigma_4^2 R^2}{2}, \\ LV_4 &= \frac{\beta(S-\frac{\lambda}{d})I}{1+\alpha I} + \frac{\theta I}{(d_E+\theta)(d_I+\delta+\gamma)}\left(\frac{R_0}{1+\alpha I} - 1\right), \\ LV_5 &= -d_S\left(S-\frac{\lambda}{d}\right)^2 - (d_E+\theta)E^2 - (d_S+d_E+\theta)\left(S-\frac{\lambda}{d}\right)E + \frac{\sigma_1^2 S^2}{2} + \frac{\sigma_2^2 E^2}{2}. \end{aligned}$$

As $R_0 \leqslant 1$, we see

$$LV_1 + \frac{\lambda}{d_S} LV_4 \leqslant -d_S\left(S - \frac{\lambda}{d_S}\right)^2 + \frac{\sigma_1^2 S^2}{2}. \tag{3.2}$$

Since $2ab \leqslant a^2 + b^2$,

$$\begin{aligned}
LV_1 + \frac{\lambda}{d_S} LV_4 &\leqslant -\left(d_S - \sigma_1^2\right)\left(S - \frac{\lambda}{d_S}\right)^2 + \frac{\sigma_1^2\lambda^2}{d_S^2},\\
LV_2 &\leqslant \frac{\theta^2 E^2}{2d_I} - \frac{d_I I^2}{2} + \frac{\sigma_3^2 I^2}{2},\\
LV_3 &\leqslant \frac{\delta^2 I^2}{2d_R} - \frac{d_R R^2}{2} + \frac{\sigma_4^2 R^2}{2},\\
LV_5 &\leqslant \left[\frac{(d_S + d_E + \theta)^2}{2(d_E + \theta)} - d_S + \sigma_1^2\right]\left(S - \frac{\lambda}{d_S}\right)^2 - \frac{d_E + \theta}{2}E^2 + \frac{\sigma_1^2\lambda^2}{d_S^2} + \frac{\sigma_2^2 E^2}{2}.
\end{aligned} \tag{3.3}$$

Hence,

$$LV_2 + \frac{d_R d_I}{2\delta^2} LV_3 \leqslant \frac{\theta^2 E^2}{2d_I} - \frac{d_I}{4} I^2 - \frac{d_R^2 d_I}{4\delta^2} R^2 + \frac{\sigma_3^2}{2} I^2 + \frac{d_R d_I \sigma_4^2}{4\delta^2} R^2. \tag{3.4}$$

Combining (3.2), (3.3) with (3.4), define positive definite C^2 function $V : R_+^4 \to R_+$ such that

$$V := \frac{(d_S + d_E + \theta)^2}{2d_S(d_E + \theta)}\left(V_1 + \frac{\lambda}{d_S} V_4\right) + \frac{d_I}{\theta}\left(V_2 + \frac{d_R d_I}{2\delta^2} V_3\right) + V_5.$$

By computation,

$$\begin{aligned}
LV &= \frac{(d_S + d_E + \theta)^2}{2d_S(d_E + \theta)}\left(LV_1 + \frac{\lambda}{d_S} LV_4\right) + \frac{d_I}{\theta}\left(LV_2 + \frac{d_R d_I}{2\delta^2} LV_3\right) + LV_5\\
&\leqslant -\left(d_S - \sigma_1^2 - \frac{(d_S + d_E + \theta)^2\sigma_1^2}{2d_S(d_E + \theta)}\right)\left(S - \frac{\lambda}{d_S}\right)^2 - \frac{d_E - \sigma_1^2}{2}E^2 - \frac{d_I^2 - 2d_I\sigma_3^2}{4\theta} I^2\\
&\quad - \frac{d_R^2 d_I^2 - d_R d_I^2 \sigma_4^2}{4\theta\delta^2} R^2 + \left[\frac{(d_S + d_E + \theta)^2}{2d_S(d_E + \theta)} + 1\right]\frac{\lambda^2\sigma_1^2}{d_S^2}.
\end{aligned} \tag{3.5}$$

Taking expectation above, (3.5) yields

$$\begin{aligned}
EV(t) - EV(0) = \int_0^t ELV(r)\,dr &\leqslant -\left(d_S - \sigma_1^2 - \frac{(d_S + d_E + \theta)^2\sigma_1^2}{2d_S(d_E + \theta)}\right)\int_0^t \left(S(r) - \frac{\lambda}{d_S}\right)^2 dr - \frac{d_E - \sigma_1^2}{2}\int_0^t E^2(r)\,dr\\
&\quad - \frac{d_I^2 - 2d_I\sigma_3^2}{4\theta}\int_0^t I^2(r)\,dr - \frac{d_R^2 d_I^2 - d_R d_I^2\sigma_4^2}{4\theta\delta^2}\int_0^t R^2(r)\,dr + \left[\frac{(d_S + d_E + \theta)^2}{2d_S(d_E + \theta)} + 1\right]\frac{\lambda^2\sigma_1^2}{d_S^2} t.
\end{aligned}$$

Hence,

$$\begin{aligned}
&\limsup_{t\to\infty} \frac{1}{t} E\int_0^t \left[\left(d_S - \sigma_1^2 - \frac{(d_S + d_E + \theta)^2\sigma_1^2}{2d_S(d_E + \theta)}\right)\left(S - \frac{\lambda}{d_S}\right)^2 + \frac{d_E - \sigma_1^2}{2}E^2 + \frac{d_I^2 - 2d_I\sigma_3^2}{4\theta} I^2 + \frac{d_R^2 d_I^2 - d_R d_I^2\sigma_4^2}{4\theta\delta^2} R^2\right]\\
&\quad \leqslant \frac{\lambda^2\sigma_1^2}{d_S^2}\left[\frac{(d_S + d_E + \theta)^2}{2d_S(d_E + \theta)} + 1\right]. \qquad \square
\end{aligned} \tag{3.6}$$

Remark 3.1. It is seen, under some conditions, the solution of system (1.4) oscillates around the disease-free equilibrium, and the intensity of fluctuation is propositional to the intensity of the white noise.

Besides, if $\sigma_1 = 0$, then Q_0 is also the disease-free equilibrium of system (1.4). In the proof of Theorem 3.2, we see

$$LV \leqslant -d_S\left(S - \frac{\lambda}{d_S}\right)^2 - \frac{d_E}{2}E^2 - \frac{d_I^2 - 2d_I\sigma_3^2}{4\theta} I^2 - \frac{d_R^2 d_I^2 - d_R d_I^2\sigma_4^2}{4\theta\delta^2} R^2 \tag{3.7}$$

which is strictly negative-definite, if $d_I > 2\sigma_3^2$ and $d_R > \sigma_4^2$. Therefore the solution of system (1.4) is stochastically asymptotically stable in the large (see [38]).

3.3. Asymptotic behavior around the endemic equilibrium of the deterministic model (1.2) in system (1.4)

In this subsection, we will show there is a unique stationary distribution for system (1.4) instead of asymptotically stable equilibrium (see [10]).

Theorem 3.3. *Let $(S(t), I(t), R(t))$ be the solution of system* (1.3) *with any initial value* $(S(0), I(0), R(0)) \in R_+^4$. *If* $R_0 = \frac{\lambda\beta\theta}{d_S(d_I+\delta+\gamma)(\theta+d_E)} > 1$, $\sigma_i > 0$, $1 \leqslant i \leqslant 4$ *and* $\min\{\kappa_1 S^{*2}, \kappa_2 E^{*2}, \kappa_3 I^{*2}, \kappa_4 R^{*2}\} > \rho > 0$, *then there is a unique stationary distribution* ν *for system* (1.3) *and the ergodicity holds. Here* (S^*, E^*, I^*, R^*) *is the unique endemic equilibrium of* (1.2), $\kappa_1 = d_S - \sigma_1^2 - \frac{\sigma_1^2(d_E+d_S+\theta)^2 S^*}{d_S(d_E+\theta)}$, $\kappa_2 = \frac{d_E+\theta}{4} - \sigma_2^2$, $\kappa_3 = \frac{(d_I+\delta+\gamma)(d_E+\theta)}{8\theta^2}(d_I+\delta+\gamma-4\sigma_3^2)$, $\kappa_4 = \frac{d_R(d_I+\delta+\gamma)^2(d_E+\theta)}{8\theta^2\gamma^2}(d_R - 2\sigma_4^2)$, *and*

$$\begin{aligned}\varrho = {}& \sigma_1^2\left[\frac{(d_E+d_S+\theta)^2 S^{*3}}{d_S(d_E+\theta)} + S^{*2} + \frac{\beta S^{*2} I^*}{d_S(1+\alpha I^*)}\right] + \sigma_2^2\left[E^{*2} + \frac{S^* E^*}{2} + \frac{\beta S^* E^* I^*}{2d_S(1+\alpha I^*)}\right] \\ & + \sigma_3^2\left[\frac{(d_I+\delta+\gamma)(d_E+\theta) I^{*2}}{2\theta^2} + \frac{S^* I^{*2}(d_E+d_S+\theta)^2}{4\theta d_S}\left(1 + \frac{\beta}{d_S(1+\alpha I^*)}\right)\right] \\ & + \frac{\sigma_4^2 d_R (d_I+\delta+\gamma)^2 (d_E+\theta) R^{*2}}{4\theta^2\gamma^2}.\end{aligned} \tag{3.8}$$

Especially, we have

$$\lim_{t\to\infty} \frac{1}{t} E\int_0^t \left[\kappa_1\big(S(r)-S^*\big)^2 + \kappa_2\big(E(r)-E^*\big)^2 + \kappa_3\big(I(r)-I^*\big)^2 + \kappa_4\big(R(r)-R^*\big)^2\right] dr \leqslant \rho.$$

Proof. As $R_0 > 1$, there is a unique endemic equilibrium $Q^* = (S^*, E^*, I^*, R^*)$ such that

$$\lambda = \frac{\beta S^* I^*}{1+\alpha I^*} + d_S S^*, \qquad \frac{\beta S^* I^*}{1+\alpha I^*} = (d_E+\theta)E^*, \qquad \theta E^* = (d_I+\delta+\gamma)I^*, \qquad \gamma I^* = d_R R^*. \tag{3.9}$$

Firstly, define C^2 function $V_1 : R_+^3 \to R_+$ as follows

$$V_1(S, E, I) = \left(S - S^* - S^*\log\frac{S}{S^*}\right) + \left(E - E^* - E^*\log\frac{E}{E^*}\right) + \frac{d_E+\theta}{\theta}\left(I - I^* - I^*\log\frac{I}{I^*}\right).$$

By computation,

$$\begin{aligned} dV_1 = {}& \left[\lambda - d_S S - \frac{(d_E+\theta)(d_I+\delta+\gamma)I}{\theta} - \frac{\lambda S^*}{S} + \frac{\beta S^* I}{1+\alpha I} + d_S S^* - \frac{\beta S I E^*}{(1+\alpha I)E} + (d_E+\theta)E^* \right.\\ & \left. - \frac{(d_E+\theta)E I^*}{I} + \frac{(d_E+\theta)(d_I+\delta+\gamma)I^*}{\theta} + \frac{\sigma_1^2 S^*}{2} + \frac{\sigma_2^2 E^*}{2} + \frac{\sigma_3^2 I^*(d_E+\theta)}{2\theta}\right] dt \\ & + \big(S-S^*\big)\sigma_1\, dB_1 + \big(E-E^*\big)\sigma_2\, dB_2 + \frac{d_E+\theta}{\theta}\big(I-I^*\big)\sigma_3\, dB_3 \\ := {}& LV_1\, dt + \big(S-S^*\big)\sigma_1\, dB_1 + \big(E-E^*\big)\sigma_2\, dB_2 + \frac{d_E+\theta}{\theta}\big(I-I^*\big)\sigma_3\, dB_3. \end{aligned}$$

Taking (3.9), yields

$$\begin{aligned} LV_1 = {}& 3\frac{\beta S^* I^*}{1+\alpha I^*} - \frac{\beta S^* I^*}{1+\alpha I^*}\left[\frac{S^*}{S} + \frac{S}{S^*}\frac{I(1+\alpha I^*)}{I^*(1+\alpha I)}\frac{E^*}{E} + \frac{E}{E^*}\frac{I^*}{I}\right] + \frac{\beta S^* I^*}{1+\alpha I^*}\left[\frac{I(1+\alpha I^*)}{I^*(1+\alpha I)} - \frac{I}{I^*}\right] \\ & + d_S S^*\left(2 - \frac{S}{S^*} - \frac{S^*}{S}\right) + \frac{\sigma_1^2 S^*}{2} + \frac{\sigma_2^2 E^*}{2} + \frac{\sigma_3^2 I^*(d_E+\theta)}{2\theta}. \end{aligned} \tag{3.10}$$

Since $x - 1 \geqslant \log x$, $\forall x \geqslant 0$, we note

$$\frac{S^*}{S} + \frac{S}{S^*}\frac{I(1+\alpha I^*)}{I^*(1+\alpha I)}\frac{E^*}{E} + \frac{E}{E^*}\frac{I^*}{I} \geqslant 3 - \log\frac{1+\alpha I}{1+\alpha I^*} \geqslant 4 - \frac{1+\alpha I}{1+\alpha I^*}. \tag{3.11}$$

Instituting (3.11) into (3.10),

$$LV_1 \leqslant \frac{\beta S^* I^*}{1+\alpha I^*}\left[\frac{I(1+\alpha I^*)}{I^*(1+\alpha I)} - \frac{I}{I^*} + \frac{1+\alpha I}{1+\alpha I^*} - 1\right] + d_S S^*\left(2 - \frac{S}{S^*} - \frac{S^*}{S}\right) + \frac{\sigma_1^2 S^*}{2} + \frac{\sigma_2^2 E^*}{2} + \frac{\sigma_3^2 I^*(d_E+\theta)}{2\theta}.$$

By computation,

$$\frac{I(1+\alpha I^*)}{I^*(1+\alpha I)} - \frac{I}{I^*} + \frac{1+\alpha I}{1+\alpha I^*} - 1 = -\frac{\alpha (I-I^*)^2}{I^*(1+\alpha I^*)(1+\alpha I)} \leqslant 0. \tag{3.12}$$

Therefore,

$$LV_1 \leqslant d_S S^*\left(2 - \frac{S}{S^*} - \frac{S^*}{S}\right) + \frac{\sigma_1^2 S^*}{2} + \frac{\sigma_2^2 E^*}{2} + \frac{\sigma_3^2 I^*(d_E+\theta)}{2\theta}. \tag{3.13}$$

Secondly, define C^2 function $V_2 : R_+^2 \to R_+$ such that

$$V_2(E,I) = \left(E - E^* - E^* \log\frac{E}{E^*}\right) + \frac{d_E+\theta}{\theta}\left(I - I^* - I^*\log\frac{I}{I^*}\right).$$

It is seen (3.9) implies

$$\begin{aligned}
LV_2 &= \left(S-S^*\right)\left(\frac{\beta I}{1+\alpha I} - \frac{\beta I^*}{1+\alpha I^*}\right) + \frac{\beta S I^*}{1+\alpha I^*} + \frac{\beta S^* I}{1+\alpha I} - \frac{\beta S^* I^*}{1+\alpha I^*}\cdot\frac{I}{I^*} \\
&\quad + \frac{\beta S^* I^*}{1+\alpha I^*}\left[1 - \frac{S}{S^*}\cdot\frac{I(1+\alpha I^*)}{I^*(1+\alpha I)}\cdot\frac{E^*}{E} - \frac{E}{E^*}\cdot\frac{I^*}{I}\right] + \frac{\sigma_2^2 E^*}{2} + \frac{(d_E+\theta)\sigma_3^2 I^{*2}}{2\theta} \\
&\leqslant \left(S-S^*\right)\left(\frac{\beta I}{1+\alpha I} - \frac{\beta I^*}{1+\alpha I^*}\right) + \frac{\beta S^* I^*}{1+\alpha I^*}\left(\frac{S}{S^*} + \log\frac{S^*}{S} - 1\right) \\
&\quad + \frac{\beta S^* I^*}{1+\alpha I^*}\left[\frac{I(1+\alpha I^*)}{I^*(1+\alpha I)} - \frac{I}{I^*} + \log\frac{1+\alpha I}{1+\alpha I^*}\right] + \frac{\sigma_2^2 E^*}{2} + \frac{(d_E+\theta)\sigma_3^2 I^{*2}}{2\theta} \\
&\leqslant \left(S-S^*\right)\left(\frac{\beta I}{1+\alpha I} - \frac{\beta I^*}{1+\alpha I^*}\right) + \frac{\beta S^* I^*}{1+\alpha I^*}\left(\frac{S}{S^*} + \frac{S^*}{S} - 2\right) \\
&\quad + \frac{\beta S^* I^*}{1+\alpha I^*}\left[\frac{I(1+\alpha I^*)}{I^*(1+\alpha I)} - \frac{I}{I^*} + \frac{1+\alpha I}{1+\alpha I^*} - 1\right] + \frac{\sigma_2^2 E^*}{2} + \frac{(d_E+\theta)\sigma_3^2 I^{*2}}{2\theta} \\
&\leqslant \left(S-S^*\right)\left(\frac{I}{1+\alpha I} - \frac{I^*}{1+\alpha I^*}\right) + \frac{\beta S^* I^*}{1+\alpha I^*}\left(\frac{S}{S^*} + \frac{S^*}{S} - 2\right) + \frac{\sigma_2^2 E^*}{2} + \frac{(d_E+\theta)\sigma_3^2 I^{*2}}{2\theta},
\end{aligned} \tag{3.14}$$

where the second and the third inequality is derived from the fact $x - 1 \geqslant \log x$, $\forall x \geqslant 0$ and the last inequality is implied by (3.12).

Thirdly, define C^2 function $V_3 : R_+ \to R_+$ as follows

$$V_3(S) = \frac{(S-S^*)^2}{2}.$$

By computation,

$$\begin{aligned}
LV_3 &= -d_S\left(S-S^*\right)^2 - \left(S-S^*\right)^2\frac{\beta I}{1+\alpha I} - S^*\left(S-S^*\right)\left(\frac{I}{1+\alpha I} - \frac{I^*}{1+\alpha I^*}\right) + \frac{\sigma_1^2 S^2}{2} \\
&\leqslant -d_S\left(S-S^*\right)^2 - S^*\left(S-S^*\right)\left(\frac{I}{1+\alpha I} - \frac{I^*}{1+\alpha I^*}\right) + \frac{\sigma_1^2 S^2}{2}.
\end{aligned} \tag{3.15}$$

Combining (2.13)–(2.15), we have

$$\begin{aligned}
L\left[V_3 + S^* V_2 + \frac{\beta S^* I^*}{d_S(1+\alpha I^*)}V_1\right] &\leqslant -d_S\left(S-S^*\right)^2 + \frac{\sigma_1^2 S^* S^2}{2} + \frac{\sigma_2^2 S^* E^*}{2} + \frac{(d_E+\theta)\sigma_3^2 S^* I^{*2}}{2\theta} \\
&\quad + \frac{\beta\sigma_1^2 S^{*2} I^*}{d_S(1+\alpha I^*)} + \frac{\beta\sigma_2^2 S^* E^* I^*}{2d_S(1+\alpha I^*)} + \frac{\beta\sigma_3^2 S^* I^{*2}(d_E+\theta)}{2\theta d_S(1+\alpha I^*)}.
\end{aligned} \tag{3.16}$$

Let $V_4 : R_+^2 \to R_+$ such that

$$V_4(S,E) = \frac{(S-S^*+E-E^*)^2}{2}.$$

Then,

$$LV_4 = -d_S(S-S^*)^2 - (d_E+\theta)(E-E^*)^2 - (d_E+d_S+\theta)(S-S^*)(E-E^*) + \frac{\sigma_1^2 S^2}{2} + \frac{\sigma_2^2 E^2}{2}$$
$$\leqslant -d_S(S-S^*)^2 - \frac{d_E+\theta}{2}(E-E^*)^2 + \frac{(d_E+d_S+\theta)^2}{2(d_E+\theta)}(S-S^*)^2 + \frac{\sigma_1^2 S^2}{2} + \frac{\sigma_2^2 E^2}{2}. \tag{3.17}$$

For C^2 function $V_5(I,R) = \frac{d_R(d_I+\delta+\gamma)}{4\gamma^2}(R-R^*)^2 + \frac{(I-I^*)^2}{2}$, Itô's formula and inequality $2ab \leqslant a^2+b^2$ yield

$$LV_5 \leqslant -\frac{d_R^2(d_I+\delta+\gamma)}{4\gamma^2}(R-R^*)^2 - \frac{(d_I+\delta+\gamma)}{4}(I-I^*)^2 + \frac{\theta^2(E-E^2)}{2(d_I+\delta+\gamma)} + \frac{\sigma_4^2 d_R(d_I+\delta+\gamma)R^2}{4\gamma^2} + \frac{\sigma_3^2 I^2}{2}. \tag{3.18}$$

Combining (3.17) and (3.18),

$$L\left[V_4 + \frac{(d_I+\delta+\gamma)(d_E+\theta)}{2\theta^2}V_5\right] \leqslant \frac{(d_E+d_S+\theta)^2}{2(d_E+\theta)}(S-S^*)^2 - d_S(S-S^*)^2 - \frac{d_E+\theta}{4}(E-E^*)^2$$
$$- \frac{d_R^2(d_I+\delta+\gamma)^2(d_E+\theta)}{8\gamma^2\theta^2}(R-R^*)^2 - \frac{(d_I+\delta+\gamma)^2(d_E+\theta)}{8\theta^2}(I-I^*)^2$$
$$+ \frac{\sigma_1^2 S^2}{2} + \frac{\sigma_2^2 E^2}{2} + \frac{\sigma_4^2 d_R(d_I+\delta+\gamma)^2(d_E+\theta)R^2}{8\theta^2\gamma^2}$$
$$+ \frac{\sigma_3^2(d_I+\delta+\gamma)(d_E+\theta)I^2}{4\theta^2}. \tag{3.19}$$

Define positive definite function $V: R_+^4 \to R_+$ such that

$$V := \frac{(d_E+d_S+\theta)^2}{2d_S(d_E+\theta)}\left[V_3 + S^*V_2 + \frac{\beta S^* I^*}{d_S(1+\alpha I^*)}V_1\right] + V_4 + \frac{(d_I+\delta+\gamma)(d_E+\theta)}{2\theta^2}V_5.$$

At last, we have

$$LV = \frac{(d_E+d_S+\theta)^2}{2d_S(d_E+\theta)}\left[LV_3 + S^*LV_2 + \frac{\beta S^* I^*}{d_S(1+\alpha I^*)}LV_1\right] + LV_4 + \frac{(d_I+\delta+\gamma)(d_E+\theta)}{2\theta^2}LV_5$$
$$\leqslant -d_S(S-S^*)^2 - \frac{d_E+\theta}{4}(E-E^*)^2 + \frac{\sigma_1^2(d_E+d_S+\theta)^2 S^* S^2}{2d_S(d_E+\theta)}$$
$$- \frac{d_R^2(d_I+\delta+\gamma)^2(d_E+\theta)}{8\gamma^2\theta^2}(R-R^*)^2 - \frac{(d_I+\delta+\gamma)^2(d_E+\theta)}{8\theta^2}(I-I^*)^2$$
$$+ \frac{\sigma_1^2 S^2}{2} + \frac{\sigma_2^2 E^2}{2} + \frac{\sigma_4^2 d_R(d_I+\delta+\gamma)^2(d_E+\theta)R^2}{8\theta^2\gamma^2} + \frac{\sigma_3^2(d_I+\delta+\gamma)(d_E+\theta)I^2}{4\theta^2}$$
$$+ \frac{(d_E+d_S+\theta)^2}{2d_S(d_E+\theta)}\left[\frac{\sigma_2^2 S^* E^*}{2} + \frac{(d_E+\theta)\sigma_3^2 S^* I^{*2}}{2\theta} + \frac{\beta\sigma_1^2 S^{*2} I^*}{d_S(1+\alpha I^*)} + \frac{\beta\sigma_2^2 S^* E^* I^*}{2d_S(1+\alpha I^*)} + \frac{\beta\sigma_3^2 S^* I^{*2}(d_E+\theta)}{2\theta d_S(1+\alpha I^*)}\right]$$
$$\leqslant -\kappa_1(S-S^*)^2 - \kappa_2(E-E^*)^2 - \kappa_3(I-I^*)^2 - \kappa_4(R-R^*)^2 + \rho,$$

where the last inequality is derived by the inequality $2ab \leqslant a^2+b^2$ and (3.8).

Note that if $\rho < \min\{\kappa_1 S^{*2}, \kappa_2 E^{*2}, \kappa_3 I^{*2}, \kappa_4 R^{*2}\}$, then the ellipsoid

$$-\kappa_1(S-S^*)^2 - \kappa_2(E-E^*)^2 - \kappa_3(I-I^*)^2 - \kappa_4(R-R^*)^2 + \rho = 0$$

lies entirely in R_+^4. We can take U to be any neighborhood of the ellipsoid with $\bar{U} \subseteq E_l = R_+^4$, so for $x \in U \setminus E_l$, $LV \leqslant -C$, which implies condition (B.2) in Lemma 2.1 is satisfied. Besides, there is $M > 0$ such that

$$\sum_{i,j=1}^{n}\left(\sum_{k=1}^{n} a_{ik}(x)a_{jk}(x)\right)\xi_i\xi_j = \sigma_1^2 x_1^2\xi_1^2 + \sigma_2^2 x_2^2\xi_2^2 + \sigma_3^2 x_3^2\xi_3^3 + \sigma_4^2 x_4^2\xi_4^3 \geqslant M|\xi|^2 \quad \text{for all } x \in \bar{U}, \xi \in R^4,$$

where Rayleigh's principle (see [41], p. 349) implies condition (B.1) is also satisfied. Therefore, the stochastic system (1.3) has a unique stationary distribution ν and it is ergodic. $\square$

3.4. Exponential stability of system (1.4)

In this subsection, we investigate the exponential decay of the global solution of system (1.4) as the intensity of white noise is great.

Theorem 3.4. *Let* $(S(t), E(t), I(t), R(t))$ *be the solution of system* (1.4) *with any initial value* $(S(0), E(0), I(0), R(0)) \in R^4_+$. *If* $\frac{\lambda\beta(d_E+\theta)}{\theta d_S} < (\frac{\sigma_3^2}{2} + d_I + \delta + \gamma) \wedge (\frac{\sigma_2^2}{2})$, *then*

$$\limsup_{t\to\infty} \frac{1}{t}\log\left[E(t) + \frac{d_E+\theta}{\theta}I(t)\right] \leqslant \frac{\lambda\beta\theta}{d_S(d_E+\theta)} - \left(\frac{\theta}{d_E+\theta}\right)^2\left[\left(\frac{\sigma_3^2}{2} + d_I + \delta + \gamma\right) \wedge \left(\frac{\sigma_2^2}{2}\right)\right],$$

$$\limsup_{t\to\infty} \frac{1}{t}\log R(t) \leqslant \left[-\left(d_R + \frac{\sigma_4^2}{2}\right)\right] \vee \left[\frac{\lambda\beta\theta}{d_S(d_E+\theta)} - \left(\frac{\theta}{d_E+\theta}\right)^2\left[\left(\frac{\sigma_3^2}{2} + d_I + \delta + \gamma\right) \wedge \left(\frac{\sigma_3^2}{2}\right)\right]\right],$$

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t S(u)\,du = \frac{\lambda}{d_S}, \quad \text{a.e.}, \qquad S(t) \to^w \nu, \quad \text{as } t \to \infty,$$

where ν *is defined in* (2.16).

Proof. By comparison theorem, we see that $S(t) \leqslant X(t)$, where $X(t)$ is a diffusion process defined in (2.17). Applying Itô's formula to $\log[E + \frac{d_E+\theta}{\theta}I]$, we see

$$\begin{aligned}
&d\log\left[E + \frac{d_E+\theta}{\theta}I\right] \\
&= \left[\frac{\beta SI}{(1+\alpha I(t))(E + \frac{d_E+\theta}{\theta}I)} - \frac{(d_E+\theta)(d_I+\delta+\gamma)I}{\theta(E + \frac{d_E+\theta}{\theta}I)} - \frac{\sigma_2^2E^2 + \sigma_3^2(\frac{d_E+\theta}{\theta})^2I^2}{2(E + \frac{d_E+\theta}{\theta}I)^2}\right]dt \\
&\quad + \frac{\sigma_2 E}{E + \frac{d_E+\theta}{\theta}I}\,dB_2(t) + \frac{\sigma_3(d_E+\theta)I}{\theta(E + \frac{d_E+\theta}{\theta}I)}\,dB_3(t) \\
&\leqslant \frac{\beta\theta X\,dt}{d_E+\theta} - \frac{1}{(E + \frac{d_E+\theta}{\theta}I)^2}\left[\left(\frac{d_E+\theta}{\theta}\right)(d_I+\delta+\gamma)IE + \left(\frac{\sigma_3^2}{2} + d_I + \delta + \gamma\right)\left(\frac{d_E+\theta}{\theta}\right)^2 I^2 + \frac{\sigma_2^2}{2}E^2\right]dt \\
&\quad + \frac{\sigma_2 E}{E + \frac{d_E+\theta}{\theta}I}\,dB_2(t) + \frac{\sigma_3(d_E+\theta)I}{\theta(E + \frac{d_E+\theta}{\theta}I)}\,dB_3(t) \\
&\leqslant \frac{\beta\theta X\,dt}{d_E+\theta} - \frac{1}{(E + \frac{d_E+\theta}{\theta}I)^2}\left[\left(\frac{\sigma_3^2}{2} + d_I + \delta + \gamma\right)\left(\frac{d_E+\theta}{\theta}\right)^2 I^2 + \frac{\sigma_2^2}{2}E^2\right]dt \\
&\quad + \frac{\sigma_2 E}{E + \frac{d_E+\theta}{\theta}I}\,dB_2(t) + \frac{\sigma_3(d_E+\theta)I}{\theta(E + \frac{d_E+\theta}{\theta}I)}\,dB_3(t) \\
&\leqslant \frac{\beta\theta X\,dt}{d_E+\theta} - \left(\frac{\theta}{d_E+\theta}\right)^2\left[\left(\frac{\sigma_3^2}{2} + d_I + \delta + \gamma\right) \wedge \left(\frac{\sigma_2^2}{2}\right)\right]dt + \frac{\sigma_2 E}{E + \frac{d_E+\theta}{\theta}I}\,dB_2(t) + \frac{\sigma_3(d_E+\theta)I}{\theta(E + \frac{d_E+\theta}{\theta}I)}\,dB_3(t),
\end{aligned}$$

where the last inequality is obtained by $S(t) \leqslant X(t)$. Integrating the above inequality from 0 to t, together with (2.20) and the fact that $\lim_{t\to\infty} \frac{|B_i(t)|}{t} = 0$, $i = 2, 3$ (Mao [38]), yields

$$\limsup_{t\to\infty} \frac{1}{t}\log\left[E + \frac{d_E+\theta}{\theta}I\right] \leqslant \frac{\lambda\beta\theta}{d_S(d_E+\theta)} - \left(\frac{\theta}{d_E+\theta}\right)^2\left[\left(\frac{\sigma_3^2}{2} + d_I + \delta + \gamma\right) \wedge \left(\frac{\sigma_2^2}{2}\right)\right] =: \varsigma. \tag{3.20}$$

To help with the proof we introduce another diffusion process $\tilde{R}(t)$ which is defined by the initial condition $\tilde{R}(0) = R(0)$ and the stochastic differential equation

$$d\tilde{R} = -d_R\tilde{R}\,dt + \sigma_4\tilde{R}\,dB_4(t).$$

Then consider

$$d(R - \tilde{R}) = \big(\gamma I - d_R(R - \tilde{R})\big)\,dt + \sigma_4(R - \tilde{R})\,dB_4(t).$$

The solution is given by

$$R(t)-\tilde{R}(t)=\gamma\exp\left\{-\left(d_R+\frac{\sigma_4^2}{2}\right)t+\sigma_4 B_4(t)\right\}\int_0^t \exp\left\{\left(d_R+\frac{\sigma_4^2}{2}\right)s-\sigma_4 B_4(s)\right\}I(s)\,ds.$$

By (2.22) and the fact that $\lim_{t\to\infty}\frac{B_4(t)}{t}=0$, it has been shown that, for any $\tilde{\varepsilon}>0$ and almost $\omega\in\Omega$, $\exists T=T(\omega)$ such that

$$I(t)\leqslant\frac{\theta}{d_E+\theta}\exp\big((\varsigma+\tilde{\varepsilon})t\big),\quad \forall t\geqslant T, \tag{3.21}$$

where ς is defined in (3.20). Hence for all $\omega\in\Omega$, if $t>T(\omega)$, then

$$\begin{aligned}\big|R(t)-\tilde{R}(t)\big|\leqslant{}&\gamma\exp\left\{-\left(d_R+\frac{\sigma_4^2}{2}\right)t+\sigma_4 B_4(t)\right\}\int_0^T \exp\left\{\left(d_R+\frac{\sigma_4^2}{2}\right)s-\sigma_4 B_4(s)\right\}I(s)\,ds\\&+\frac{\gamma\theta}{d_E+\theta}\exp\left\{-\left(d_R+\frac{\sigma_4^2}{2}\right)t+\sigma_4 B_4(t)+\sigma_4\max_{s\leqslant t}\big|B_4(s)\big|\right\}\int_T^t \exp\left\{\left(d_R+\frac{\sigma_4^2}{2}+\varsigma+\tilde{\varepsilon}\right)s\right\}ds.\end{aligned}$$

Therefore,

$$\limsup_{t\to\infty}\frac{1}{t}\log\big|R(t)-\tilde{R}(t)\big|\leqslant\left[-\left(d_R+\frac{\sigma_4^2}{2}\right)\right]\vee[\varsigma+\tilde{\varepsilon}],\quad \text{a.e.}$$

Let $\tilde{\varepsilon}\to 0$, we get $\limsup_{t\to\infty}\frac{1}{t}\log|R(t)-\tilde{R}(t)|\leqslant[-(d_R+\frac{\sigma_4^2}{2})]\vee\varsigma$, a.e. On the other hand,

$$\tilde{R}(t)=R(0)\exp\left\{-\left(d_R+\frac{\sigma_4^2}{2}\right)t+\sigma_4 B_4(t)\right\},$$

and hence, $\limsup_{t\to\infty}\frac{1}{t}\log\tilde{R}(t)=-(d_R+\frac{\sigma_4^2}{2})$. Therefore,

$$\limsup_{t\to\infty}\frac{1}{t}\log R(t)\leqslant\left[-\left(d_R+\frac{\sigma_4^2}{2}\right)\right]\vee\varsigma.$$

Similar to the proof of Theorem 2.4, we conclude $S(t)\to^{w}\nu$, where ν is defined in (2.16). □

Remark 3.2. Note that the conditions of Theorem 2.4 cannot possibly be satisfied in the deterministic model when $\sigma_1=\sigma_2=0$ as $R_0>1$. If the variance σ_1^2 is large enough such that these conditions are satisfied, then the stochastic effect may make the system (1.4) wash out likely.

4. Simulation

To conform the analytical results above, we use Milstein's higher order method (see [23]) to find the strong solutions of system (1.3) and system (1.4) with given initial value and the parameters. The corresponding discretization equations are

$$\begin{cases} S_{k+1}=S_k+\left(\lambda-\dfrac{\beta S_k I_k}{1+\alpha I_k}-d_S S_k\right)\Delta t+\sigma_1 S_k\xi_{1,k}\sqrt{\Delta t}+\dfrac{\sigma_1^2}{2}S_k\big(\xi_{1,k}^2\Delta t-\Delta t\big),\\ I_{k+1}=I_k+\left(\dfrac{\beta S_k I_k}{1+\alpha I_k}-(d_I+\delta+\gamma)I_k\right)+\sigma_2 I_k\xi_{1,k}\sqrt{\Delta t}+\dfrac{\sigma_2^2}{2}I_k\big(\xi_{2,k}^2\Delta t-\Delta t\big),\\ R_{k+1}=R_k+(\gamma I_k-d_R R_k)\Delta t+\sigma_3 R_k\xi_{3,k}\sqrt{\Delta t}+\dfrac{\sigma_3^2}{2}R_k\big(\xi_{3,k}^2\Delta t-\Delta t\big),\end{cases}$$

and

$$\begin{cases} S_{k+1}=S_k+\left(\lambda-\dfrac{\beta S_k I_k}{1+\alpha I_k}-d_S S_k\right)\Delta t+\sigma_1 S_k\xi_{1,k}\sqrt{\Delta t}+\dfrac{\sigma_1^2}{2}S_k\big(\xi_{1,k}^2\Delta t-\Delta t\big),\\ E_{k+1}=E_k+\left(\dfrac{\beta S_k I_k}{1+\alpha I_k}-(d_E+\theta)E_k\right)+\sigma_2 E_k\xi_{1,k}\sqrt{\Delta t}+\dfrac{\sigma_2^2}{2}E_k\big(\xi_{2,k}^2\Delta t-\Delta t\big),\\ I_{k+1}=I_k+\big(\theta E_k-(d_I+\delta+\gamma)I_k\big)+\sigma_3 I_k\xi_{1,k}\sqrt{\Delta t}+\dfrac{\sigma_3^2}{2}I_k\big(\xi_{3,k}^2\Delta t-\Delta t\big),\\ R_{k+1}=R_k+(\gamma I_k-d_R R_k)\Delta t+\sigma_4 R_k\xi_{3,k}\sqrt{\Delta t}+\dfrac{\sigma_4^2}{2}R_k\big(\xi_{4,k}^2\Delta t-\Delta t\big),\end{cases}$$

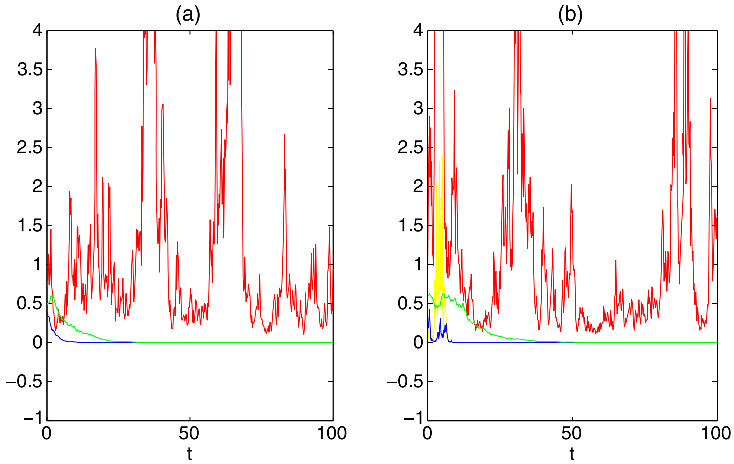

Fig. 1. (Color online.) The solutions of system (1.3) and system (1.4) with $R_0 < 1$. In the simulation, $n = 500$, $\Delta t = 0.2$. The susceptible, the exposed, the infective and the recovered fraction of system (1.3) and system (1.4) are represented by red lines, yellow lines, blue lines and green lines, respectively.

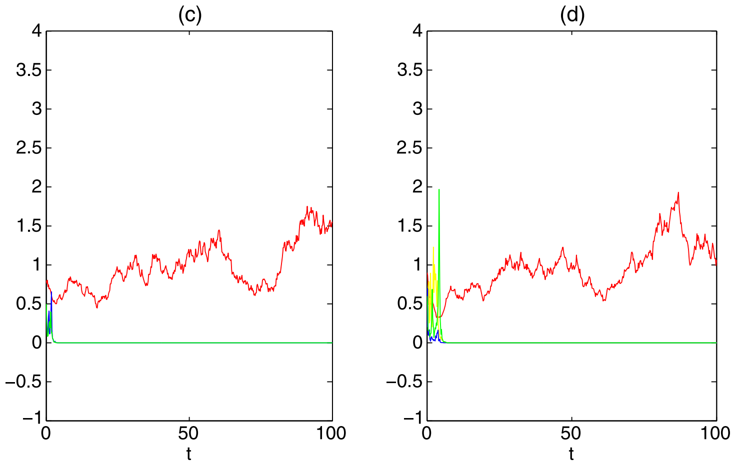

Fig. 2. (Color online.) The solution of system (1.3) and system (1.4) with $R_0 > 1$. In the simulation, $n = 500$, $\Delta t = 0.2$. The susceptible, the exposed, the infective and the recovered fraction of system (1.3) and system (1.4) are represented by red lines, yellow lines, blue lines and green lines, respectively.

where $\xi_{1,k}, \xi_{2,k}, \xi_{3,k}$ and $\xi_{4,k}$, $k = 1, 2, \ldots, n$, are independent Gaussian random variables $N(0, 1)$, and σ_i, $1 \leqslant i \leqslant 4$, are intensities of white noises.

We choose the initial value $(S(0), E(0), I(0), R(0)) = (1.8, 2.7, 2.4, 1.2)$ and the parameters $\lambda = 0.1$, $d_S = d_E = d_I = d_R = 0.1$, $\theta = 0.1$, $\delta = 0.2$, $\gamma = 0.3$. By Matlab software, we simulate the solution of system (1.3) and (1.4) with different values of β and σ_k, $k = 1, 2, 3, 4$.

In (a), $\sigma_1 = 0.8$, $\sigma_2 = 0.2$, $\sigma_3 = 0.1$, $\beta = 0.4$ such that $R_0 < 1$, where the conditions of Theorem 2.4 are satisfied; in (b), $\sigma_1 = 0.8$, $\sigma_2 = 2.4$, $\sigma_3 = 1.2$, $\sigma_4 = 0.1$, $\beta = 0.4$ such that $R_0 < 1$ and the conditions of Theorem 3.4 are satisfied; in (c) and (d), $\beta = 0.9$, $\sigma_1 = 0.1$, $\sigma_2 = 1.5$, $\sigma_3 = 1.6$, $\sigma_4 = 1.5$ such that $R_0 > 1$ and the conditions of Theorems 2.4 and 3.4 hold, respectively.

Figs. 1 and 2 give the solutions of system (1.3) and system (1.4). In both figures, we choose parameters such that the conditions said in theorems are satisfied. Hence, as theorems said, there is some stability. From the figures, we can see, the exposed, the infected, and the recovered parts of system (1.3) and (1.4) are exponentially, while the susceptible converges weakly to the stationary distribution ν. Besides, the parameters chosen in Fig. 2 are the same as Fig. 1's, except the increasing intensities of white noises. It is clear that with the increasing intensities of white noises, the strength of the exponential extinction is getting large.

In Fig. 3, the parameter values of (e) and (f) are the same as in Fig. 2. As can be clearly seen from Fig. 3, $\frac{1}{t}\int_0^t S(u)\,du$ tends to $\frac{\lambda}{d_S}$ in the stochastic models.

In Fig. 4 and Fig. 5, we represent the histograms of the values of $S(t)$ and ν. The parameter are the same as in Fig. 2. For convenience, we only discuss the histograms and the kernel density of the susceptible in system (1.3). We use statistical software R to record the values of $S(t)$ at large time $t = 50\,000$, and $\Delta t = 0.01$. Comparing these figures we see that at

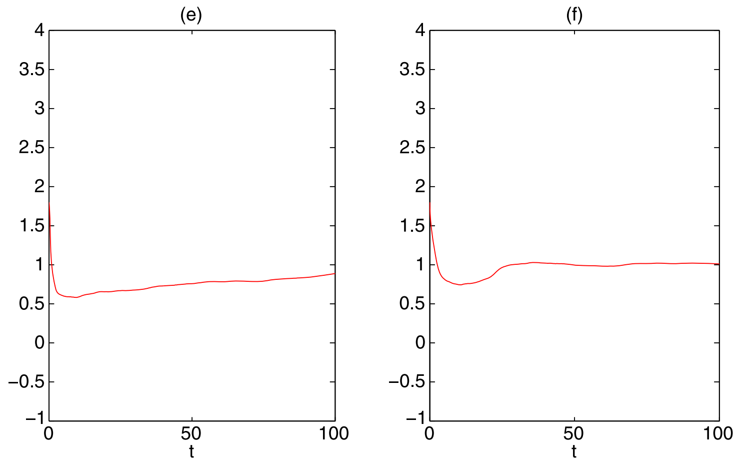

Fig. 3. (Color online.) $\frac{1}{t}\int_0^t S(u)\,du$ in system (1.3) and system (1.4) with $R_0 > 1$. In the simulation, $n = 500$, $\Delta t = 0.2$. Red lines represent $\frac{1}{t}\int_0^t S(u)\,du$.

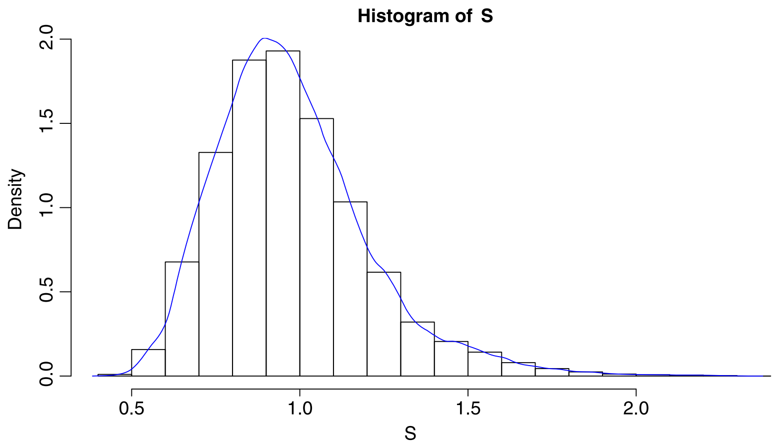

Fig. 4. (Color online.) The kernel density of $S(t)$ in system (1.3) with $R_0 > 1$. In the simulation, $n = 500\,000$, $\Delta t = 0.01$. The blue lines represent the kernel density of $S(t)$.

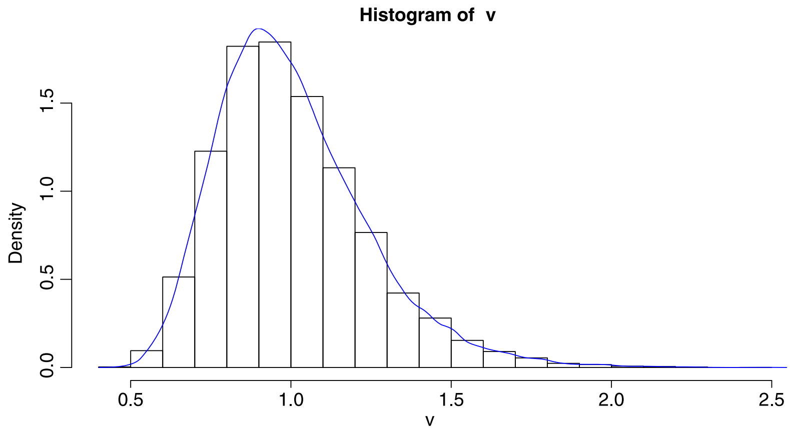

Fig. 5. (Color online.) The density of ν in system (1.3). In the simulation, $n = 10\,000$, $\Delta t = 0.01$. The blue lines represent the density of ν.

large time the kernel density of $S(t)$ looks very like that of ν. Thus $S(t)$ is a good approximation to ν where $E(t)$, $I(t)$ and $R(t)$ tend to zero.

In (g) and (h), we choose $\beta = 0.9$, and $\sigma_1 = 0.2$, $\sigma_2 = 0.1$, $\sigma_3 = 0.1$, $\sigma_4 = 0.1$ such that the conditions of Theorems 2.4 and 3.4 are satisfied. In Fig. 6, the simulating solutions fluctuate around the endemic equilibrium, which conforms the ergodicity of system (1.3) and system (1.4).

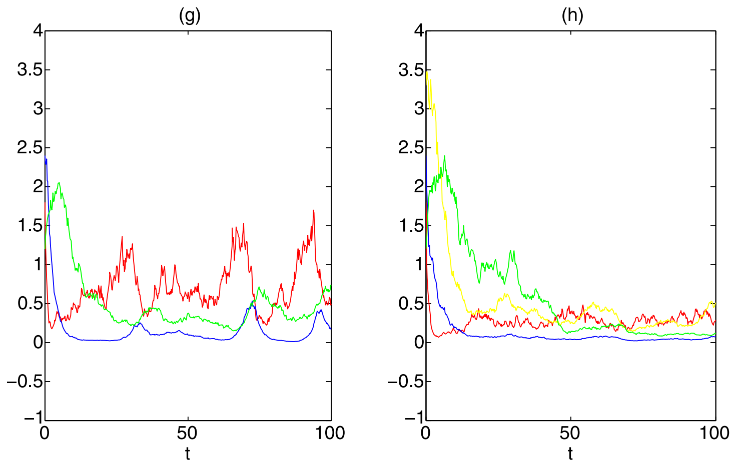

Fig. 6. (Color online.) The solutions of system (1.3) and system (1.4) with $R_0 > 1$. The susceptible, the exposed, the infective and the recovered fraction of system (1.3) and system (1.4) are represented by red lines, yellow lines, blue lines and green lines, respectively.

5. Conclusion

As most real world problems are not deterministic, incorporating stochastic effects into the model gives us a more realistic way of modeling epidemic models. In this paper, we have considered a stochastic SIR and SEIR epidemic models with saturated incidences. We first proved the positivity of the solutions. Then, we investigate the stability of the model. We illustrated the dynamical behavior of SIR and SEIR models according to $R_0 \leqslant 1$ or $R_0 > 1$. We proved that the infective tends asymptotically to zero exponentially almost surely as $R_0 \leqslant 1$ in SIR model. We also proved that the SIR model has the ergodic property as the fluctuation is small, where the positive solution converges weakly to the unique stationary distribution. The SEIR model was also discussed in the latent section. Simulations are also carried out to verify our analytical results.

Our work shows the stochastic differential equations give another insight into modeling epidemic dynamics. It displays a different perspective to this particular problem. Especially, we obtain the ergodicity of stochastic systems which is usually used in statistical inference of unknown parameters in stochastic differential equation. Thus, it gives us the motivation to investigate the stability of stochastic systems.

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