NOTE

On a Reconstruction Problem for Sequences*

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It is shown that any word of length \( n \) is uniquely determined by all its \( \binom{n}{k} \) subwords of length \( k \), provided \( k \geq \lceil \frac{2}{\sqrt{\pi}} \sqrt{n} \rceil + 5 \). This improves the bound \( k \geq \lceil n/2 \rceil \) given in B. Manvel et al. (Discrete Math. 94 (1991), 209–219).

1. INTRODUCTION

Given a word \( X \) of length \( n \) with terms from an alphabet \( \Sigma \), define the \( k \)-deck of \( X \), \( D_k(X) \), to be the multiset of all \( \binom{n}{k} \) \( k \)-subwords of \( X \). The following reconstruction problem is due to Kalashnik [4].

When is \( X \) uniquely determined by \( D_k(X) \)?

We call \( X \) \( k \)-reconstructible when this is the case. We call words \( X \) and \( Y \) \( k \)-equivalent if \( D_k(X) = D_k(Y) \) (We will only use this term for distinct words).

This question resembles the well-known vertex reconstruction problem (see Bondy [1]) but seems more tractable. The problem has been treated in [5] by Manvel et al. They proved that any word of length \( n \) is \( k \)-reconstructible whenever \( k \geq \lceil n/2 \rceil \). On the other hand, they gave a construction of nonreconstructible words for \( k \leq \log_2 n \). It also has been shown in [5] that without loss of generality we can restrict the problem to words over the alphabet \( \{0, 1\} \), in view of the following:

All words of length \( n \) with terms from an alphabet \( \Sigma \) are \( k \)-reconstructible if and only if all words of the same length with terms from \( \{0, 1\} \) are \( k \)-reconstructible.

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We establish \( k \)-reconstructibility of words of length \( n \) for \( k \geq \lceil \frac{4}{n} \rceil + 5 \). For, we will compare “average” words

\[
\left( \begin{array}{c} n \\ k \end{array} \right)^{-1} \sum_{x \in D_k(X)} X, \quad \left( \begin{array}{c} n \\ k \end{array} \right)^{-1} \sum_{y \in D_k(Y)} Y.
\]

It turned out that this leads to the Prouhet–Tarry–Escott problem of Diophantine analysis and our result immediately follows from the bound obtained in [3].

Many variations of the above problem are possible. For example, what happens if all words are given up to their complements? What can be said about reconstruction of cyclic words?

## 2. RESULTS

In what follows we will deal only with words over the alphabet \( \{0, 1\} \). For convenience we use indices \( 0, 1, \ldots, n-1 \) for letters of words.

We start with the following counting lemma.

**Lemma 2.1.** Let \( X = x_0x_1 \cdots x_{n-1} \), and let \( S_j(X) \) be the total number of 1’s appearing in the \( j \)'s place in all words of \( D_k(X) \). Then

\[
S_j(X) = \sum_{i=0}^{n-1} \left( \begin{array}{c} i \\ j \end{array} \right) \left( \begin{array}{c} n-i-1 \\ k-j-1 \end{array} \right) x_i, \quad j = 0, 1, \ldots, k-1.
\]

**Proof.** If \( x_i = 1 \) then it contributes \( \left( \begin{array}{c} i \\ j \end{array} \right) \left( \begin{array}{c} n-i-1 \\ k-j-1 \end{array} \right) \) ones to the \( j \)'s places of the deck. That is since the first \( j \) letters from 0 to \( j-1 \) of a word in the deck are chosen from the first \( i \) letters of \( X \), while the last \( k-j-1 \) letters starting from the \( j+1 \), are chosen from the last \( n-i-1 \) letters of \( X \). Summing upon all the \( x_i \) we obtain the required result.

Let \( X \) and \( Y \) be two \( k \)-equivalent words. Define the following vector with components from \( \{ -1, 0, 1\} \):

\[
\delta = \delta_0 \delta_1 \cdots \delta_{n-1}, \quad \delta_j = x_j - y_j.
\]

As an immediate corollary of the previous lemma we get:

**Lemma 2.2.**

\[
\sum_{i=0}^{n-1} \left( \begin{array}{c} i \\ j \end{array} \right) \left( \begin{array}{c} n-i-1 \\ k-j-1 \end{array} \right) \delta_i = 0, \quad j = 0, 1, \ldots, k-1.
\]

Consider now the following set of polynomials

\[
f_j^n(t) = f_j(t) = \left( \begin{array}{c} t \\ j \end{array} \right) \left( \begin{array}{c} n-t-1 \\ k-j-1 \end{array} \right), \quad j = 0, 1, \ldots, k-1.
\]
Note that \( f_j(t) \) is a polynomial of degree \( k - 1 \) and has distinct integer roots since \( f_j(i) = 0 \) for \( 0 \leq i < j \) and \( n - k + j < i \leq n - 1 \).

**Lemma 2.3.** For fixed integers \( n \) and \( k \), the set \( \{ f_0(t), \ldots, f_{k-1}(t) \} \) is a basis for the space of polynomials of degree \( k - 1 \).

**Proof.** Consider
\[
\phi(i) = \sum_{j=0}^{k-1} \lambda_j f_j(i) = \sum_{j=0}^{k-1} \lambda_j \binom{i}{n-i-1} / k - j - 1
\]
It is enough to show that \( \phi(i) \) is not identically zero whenever the \( \lambda_j \) are not all zero. Assume the contrary and let \( \lambda_0 \) be the first nonzero coefficient. Then \( \phi(t) = \lambda_0 \binom{t}{n-k-1} \), since \( \binom{j}{i} = 0 \) for \( j > t \). Hence \( \phi(t) = 0 \) implies that \( \lambda_0 = 0 \), a contradiction.

Combining the two previous lemmas we obtain the following necessary condition for nonreconstructibility of words of length \( n \).

**Corollary 2.4.** If \( X \) and \( Y \) are \( k \)-equivalent then for any polynomial \( \phi(j) \) of degree at most \( k - 1 \),
\[
\sum_{j=0}^{n-1} \delta_j \phi(j) = 0.
\]

**Proof.**
\[
\sum_{j=0}^{n-1} \delta_j \phi(j) = \sum_{j=0}^{n-1} \lambda_j \sum_{i=0}^{k-1} \delta_i f_j(i) = \sum_{j=0}^{k-1} \lambda_j \sum_{i=0}^{n-1} \delta_i f_j(i) = 0.
\]
Observe that if \( X \) and \( Y \) have the same \( k \)-decks they have also the same number of ones. To see this just choose \( \phi(i) = 1 \) in (1).

If we denote by \( u_i \) and \( w_i \), \( i = 1, 2, \ldots, s \), the indices of ones of the words \( X \) and \( Y \), respectively, then (1) is equivalent to the following system:
\[
\sum_{h=1}^{k-1} u_h^i + u_h^s + \cdots + u_h^s = \sum_{h=1}^{k-1} w_h^i + w_h^s + \cdots + w_h^s, \quad h = 1, \ldots, k - 1,
\]
\[
u_1 < u_2 < \cdots < u_s, \quad w_1 < w_2 < \cdots < w_s,
\]
and \( u_i \) and \( w_i \) are integers from the interval \([0, n - 1]\).

Of course, this system ever has a trivial solution \( u_i = w_i, \ i = 0, \ldots, s \). Thus, we obtain

**Theorem 1.** If \( X \) and \( Y \) are \( k \)-equivalent then (2) has a nontrivial solution with \( u_i, w_i \in [0, n - 1] \).
A problem of finding two distinct sets of integers \( \{u_i\} \) and \( \{w_i\} \) satisfying (2) is a classic (more than 200 years old) problem of Diophantine analysis, usually referred to as the Prouhet–Tarry–Escott problem (see [2] for extensive discussion). Recently Borwein, Erdelyi, and Kos proved that (2) has only trivial solutions whenever \( k \geq \frac{16}{7} \sqrt{n} + 5 \). They construct a polynomial \( \phi(j) \) of degree \( k - 1 = \frac{16}{7} \sqrt{n} + 4 \), such that \( |\phi(0)| > \sum_{j=1}^{n-1} |\phi(j)| \), clearly contradicting (1). This immediately yields our main result.

**Main Theorem.** A word \( X \) of length \( n \) is reconstructible from \( D_k(X) \) provided \( k \geq \frac{16}{7} \sqrt{n} + 5 \).

Notice that simple counting arguments show that for

\[
k < c \sqrt{n} \log \frac{1}{2} n, \quad c < 2,
\]

(2) already has a nontrivial solution [2] (it is enough to consider all words with about \( n/2 \) ones). Thus, our method cannot yield an essentially better bound. Yet, to obtain (2) we have actually compared the average number of ones appearing at the \( j \)th place in \( D_k(X) \) and \( D_k(Y) \). It seems plausible that taking into consideration higher moments may provide a sharper result.

In conclusion, let us notice that (2) yields also some conditional results, if some information about the number of ones in \( X \) and \( Y \) is known. For instance, considering the words as binary vectors and introducing the usual Hamming distance, we obtain:

**Corollary 2.5.** If \( 0 < \text{dist}(X, Y) < 2k \) then \( D_k(X) \neq D_k(Y) \).

**Proof.** We may assume that \( X \) and \( Y \) have the same number \( s \) of ones. So \( \text{dist}(X, Y) \) is even. Cancelling if necessary, one can take in (2) just \( s = \text{dist}(X, Y)/2 \). It is well known and easily follows from the theory of symmetric functions that for \( k - 1 \geq s \) any solution of (2) satisfies \( u_i = w_i \), that is, \( X = Y \). 

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REFERENCES

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