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Functional inequalities involving Bessel and modified Bessel functions of the first kind

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Dedicated to Professor Ştefan Cobzaş on the occasion of his sixtieth birthday

Abstract

In this paper, we extend some known elementary trigonometric inequalities, and their hyperbolic analogues to Bessel and modified Bessel functions of the first kind. In order to prove our main results, we present some monotonicity and convexity properties of some functions involving Bessel and modified Bessel functions of the first kind. We also deduce some Turán and Lazarević-type inequalities for the confluent hypergeometric functions.

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1. Extension of Lazarević inequality to modified Bessel functions

The sine and cosine functions are particular cases of Bessel functions, while the hyperbolic sine and hyperbolic cosine functions are particular cases of modified Bessel functions. Thus, it is natural to generalize some formulas and inequalities involving these elementary functions to Bessel functions and modified Bessel functions, respectively.

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I. Lazarević [14, p. 270] proved that for all $x \neq 0$ the inequality

$$\cosh x < \left(\frac{\sinh x}{x}\right)^3 \tag{1.1}$$

holds and exponent 3 is the least possible.

Our main motivation to write this paper is inequality (1.1) which we wish to extend to modified Bessel functions of the first kind. This paper is organized as follows: in this section, we deduce a known Turán-type inequality and using this we extend (1.1) to the function $\mathscr{I}_p$ defined below. Moreover, we present a generalization of the Turán, Lazarević, and Wilker-type inequalities, in order to improve the known results in the literature. For more details about the Turán-type inequalities, the interested reader is referred to the most recent papers [2,4,10,13] on this topic and to the references therein. At the end of this section, we extend some of the main results to confluent hypergeometric functions and we improve a result of Ismail and Laforgia [10]. In Section 2, we extend the analogous of (1.1), Wilker's inequality (1.15) to Bessel functions, we deduce a known Turán-type inequality for Bessel functions and we present some new inequalities involving the Bessel functions of the first kind.

For $p > -1$ let us consider the function $\mathscr{I}_p : \mathbb{R} \to [1, \infty)$, defined by

$$\mathscr{I}_p(x) := 2^p \Gamma(p+1) x^{-p} I_p(x) = \sum_{n \geqslant 0} \frac{(1/4)^n}{(p+1)_n n!} x^{2n}, \tag{1.2}$$

where $(p+1)_n = (p+1)(p+2)\cdots(p+n) = \Gamma(p+n+1)/\Gamma(p+1)$ is the well-known Pochhammer (or Appell) symbol defined in terms of Euler's gamma function, and I_p is the modified Bessel function of the first kind defined by [20, p. 77]

$$I_p(x) := \sum_{n \geqslant 0} \frac{(x/2)^{2n+p}}{n!\Gamma(p+n+1)} \quad \text{for all } x \in \mathbb{R}. \tag{1.3}$$

It is worth mentioning that in particular we have

$$\mathscr{I}_{-1/2}(x) = \sqrt{\pi/2}x^{1/2} I_{-1/2}(x) = \cosh x, \tag{1.4}$$

$$\mathscr{I}_{1/2}(x) = \sqrt{\pi/2}x^{-1/2} I_{1/2}(x) = \frac{\sinh x}{x}, \tag{1.5}$$

$$\mathscr{I}_{3/2}(x) = 3\sqrt{\pi/2}x^{-3/2} I_{3/2}(x) = -3\left(\frac{\sinh x}{x^3} - \frac{\cosh x}{x^2}\right). \tag{1.6}$$

Thus, the function $\mathscr{I}_p$ is of special interest in this paper because inequality (1.1) is actually equivalent to

$$[\mathscr{I}_{-1/2}(x)]^{-1/2+1} \leqslant [\mathscr{I}_{-1/2+1}(x)]^{-1/2+2}. \tag{1.7}$$

So in view of inequality (1.7), it is natural to ask: what is the analogue of this inequality for modified Bessel functions of the first kind? In order to answer this question we prove the following results.

Theorem 1. *Let $p > -1$ and $x \in \mathbb{R}$. Then the following assertions are true*:

(a) *the function $p \mapsto \mathscr{I}_p(x)$ is decreasing and log-convex*;
(b) *the functions $p \mapsto \mathscr{I}_{p+1}(x)/\mathscr{I}_p(x)$, $p \mapsto [\mathscr{I}_p(x)]^{p+1}$ are increasing*;
(c) *the following inequalities*:

$$[\mathscr{I}_{p+1}(x)]^2 \leqslant \mathscr{I}_p(x)\mathscr{I}_{p+2}(x), \tag{1.8}$$

$$[\mathscr{I}_p(x)]^{p+1} \leqslant [\mathscr{I}_{p+1}(x)]^{p+2}, \tag{1.9}$$

$$[\mathscr{I}_p(x)]^{(p+1)/(p+2)} \leqslant \mathscr{I}_{p+1}(x) \leqslant \mathscr{I}_p(x), \tag{1.10}$$

$$[\mathscr{I}_{p+1}(x)]^{1/(p+1)} + \frac{\mathscr{I}_{p+1}(x)}{\mathscr{I}_p(x)} \geqslant 2, \tag{1.11}$$

hold true for all $p > -1$ and $x \in \mathbb{R}$. In (1.9), *the exponent p is the best possible in the sense that $\tau = (p+2)/(p+1)$ is the smallest value of τ for which $\mathscr{I}_p(x) \leqslant [\mathscr{I}_{p+1}(x)]^{\tau}$ holds. Moreover*, *if $x > 0$ is fixed and $p \to \infty$, then $[I_p(x)]^2 \sim I_{p-1}(x)I_{p+1}(x)$*;
(d) *the inequality*

$$\frac{\mathscr{I}_p(x)}{\mathscr{I}_{p+1}(x)} - 1 \leqslant \log[\mathscr{I}_{p+1}(x)] \leqslant \log[\mathscr{I}_p(x)] \tag{1.12}$$

holds true for all $p \geqslant 0$ and $x \in \mathbb{R}$.

Proof. (a) For convenience, let us write

$$\mathscr{I}_p(x) = \sum_{n \geqslant 0} b_n(p) x^{2n} \quad \text{where } b_n(p) := \frac{(1/4)^n}{(p+1)_n n!}, \quad n \geqslant 0.$$

Clearly if $p \geqslant q > -1$, then $(p+1)_n \geqslant (q+1)_n$, and consequently $b_n(p) \leqslant b_n(q)$, for all $n \geqslant 0$. This implies that $\mathscr{I}_p(x) \leqslant \mathscr{I}_q(x)$ for all $x \in \mathbb{R}$, i.e. the function $p \mapsto \mathscr{I}_p(x)$ is decreasing. Now for log-convexity of $p \mapsto \mathscr{I}_p(x)$, we observe that it is enough to show the log-convexity of each individual term and to use the fact that sums of log-convex functions are log-convex too. Thus, we just need to show that for each $n \geqslant 0$ we have

$$\partial^2 \log[b_n(p)]/\partial p^2 = \psi'(p+1) - \psi'(p+n+1) \geqslant 0,$$

where $\psi(x) = \Gamma'(x)/\Gamma(x)$ is the so-called digamma function. But ψ is concave, and consequently the function $p \mapsto b_n(p)$ is log-convex on $(-1, \infty)$.

We note that there is another proof of the log-convexity of $p \mapsto \mathscr{I}_p(x)$. Namely, if we consider the infinite product representation of the modified Bessel function of the first kind I_p, then we have [20]

$$\mathscr{I}_p(x) = \prod_{n \geqslant 1} \left(1 + \frac{x^2}{j_{p,n}^2}\right), \tag{1.13}$$

where $j_{p,n}$ is the nth positive zero of the Bessel function J_p. Using (1.13) we have

$$\log[\mathscr{I}_p(x)] = \sum_{n\geqslant 1} \log\left(1+\frac{x^2}{j_{p,n}^2}\right).$$

Owing to Elbert [6], it is known that $p \mapsto j_{p,n}$ is concave on $(-n,\infty)$ for all $n\geqslant 1$. Consequently, we have that $p\mapsto j_{p,n}$ and $p\mapsto \log j_{p,n}$ are concave on $(-1,\infty)$ for all $n\geqslant 1$. Hence, $p\mapsto -2\log j_{p,n}$ is convex, i.e. $p\mapsto 1/j_{p,n}^2$ is log-convex on $(-1,\infty)$. But this implies that for all $n\geqslant 1$ the function $p\mapsto \log(1+x^2/j_{p,n}^2)$ is convex on $(-1,\infty)$, and consequently the function $p\mapsto \log \mathscr{I}_p(x)$ is convex too on $(-1,\infty)$ as a sum of convex functions.

(b) First we prove that the function $p\mapsto \mathscr{I}_{p+1}(x)/\mathscr{I}_p(x)$ is increasing. From part (a) of the this theorem, the function $p\mapsto \log[\mathscr{I}_p(x)]$ is convex, and hence it follows that $p\mapsto \log[\mathscr{I}_{p+a}(x)]-\log[\mathscr{I}_p(x)]$ is increasing for each $a>0$. Thus, choosing $a=1$, we obtain that indeed the function $p\mapsto \mathscr{I}_{p+1}(x)/\mathscr{I}_p(x)$ is increasing.

Now suppose that $p\geqslant q>-1$ and define the function $\varphi_1:\mathbb{R}\to\mathbb{R}$ with relation

$$\varphi_1(x) := \frac{p+1}{q+1}\log[\mathscr{I}_p(x)] - \log[\mathscr{I}_q(x)].$$

On the other hand,

$$\varphi_1'(x) = \frac{p+1}{q+1}\left[\frac{\mathscr{I}_p'(x)}{\mathscr{I}_p(x)}\right] - \frac{\mathscr{I}_q'(x)}{\mathscr{I}_q(x)} = 2xb_1(q)\left[\frac{\mathscr{I}_{p+1}(x)}{\mathscr{I}_p(x)} - \frac{\mathscr{I}_{q+1}(x)}{\mathscr{I}_q(x)}\right],$$

where we used the differentiation formula $\mathscr{I}_p'(x)=2xb_1(p)\mathscr{I}_{p+1}(x)$, which can be derived easily from (1.2). Since $p\geqslant q$ we have $\mathscr{I}_{p+1}(x)/\mathscr{I}_p(x)\geqslant \mathscr{I}_{q+1}(x)/\mathscr{I}_q(x)$ and from this conclude that the function φ_1 is increasing on $[0,\infty)$ and is decreasing on $(-\infty,0]$. Consequently, $\varphi_1(x)\geqslant\varphi_1(0)=0$, i.e. $[\mathscr{I}_p(x)]^{p+1}\geqslant[\mathscr{I}_q(x)]^{q+1}$ holds for all $x\in\mathbb{R}$.

(c) Since $p\mapsto \mathscr{I}_p(x)$ is log-convex, for all $p_1, p_2>-1$, $x\in\mathbb{R}$ and $\alpha\in[0,1]$ we have

$$\mathscr{I}_{\alpha p_1+(1-\alpha)p_2}(x)\leqslant[\mathscr{I}_{p_1}(x)]^{\alpha}[\mathscr{I}_{p_2}(x)]^{1-\alpha}.$$

Now choosing $\alpha=1/2$, $p_1=p$ and $p_2=p+2$ we conclude that (1.8) holds. Inequality (1.9) follows from the monotonicity of $p\mapsto[\mathscr{I}_p(x)]^{p+1}$, while (1.10) is an immediate consequence of (1.9) and the monotonicity of $p\mapsto\mathscr{I}_p(x)$. Moreover, since $\mathscr{I}_p(x)=1+b_1(p)x^2+\cdots$ and $[\mathscr{I}_{p+1}(x)]^{\tau}=1+\tau b_1(p+1)x^2+\cdots$, we infer that $\tau=b_1(p)/b_1(p+1)=(p+2)/(p+1)$ is the smallest value of τ such that $\mathscr{I}_p(x)\leqslant[\mathscr{I}_{p+1}(x)]^{\tau}$ holds. For inequality (1.11), we use the generalized Lazarević inequality (1.9) and the arithmetic–geometric mean inequality

$$\frac{1}{2}\left[[\mathscr{I}_{p+1}(x)]^{1/(p+1)}+\frac{\mathscr{I}_{p+1}(x)}{\mathscr{I}_p(x)}\right]\geqslant\sqrt{\frac{[\mathscr{I}_{p+1}(x)]^{(p+2)/(p+1)}}{\mathscr{I}_p(x)}}\geqslant 1.$$

It remains just to prove the asymptotic formula $[I_p(x)]^2\sim I_{p-1}(x)I_{p+1}(x)$. In order to prove the asserted result, we show that for $p>0$ and $x>0$ we have

$$1<\frac{[I_p(x)]^2}{I_{p-1}(x)I_{p+1}(x)}<1+\frac{1}{p}. \tag{1.14}$$

The left-hand side of (1.14) is the well-known Amos inequality [3, p. 243]. The right-hand side of (1.14) can be deduced easily from (1.8) using the difference equation $\Gamma(a+1) = a\Gamma(a)$.

(d) Using the the recurrence formula [20, p. 79] $xI_{p-1}(x) - xI_{p+1}(x) = 2pI_p(x)$ and the Mittag–Leffler expansion [7, Eq. 7.9.3]

$$\frac{I_{p+1}(x)}{I_p(x)} = \sum_{n\geqslant 1} \frac{2x}{x^2 + j_{p,n}^2},$$

where $0 < j_{p,1} < j_{p,2} < \cdots < j_{p,n} < \cdots$ are the positive zeros of the Bessel function J_p, we obtain that

$$\begin{aligned}\frac{\mathscr{I}_p(x)}{\mathscr{I}_{p+1}(x)} - 1 &= \frac{x}{2(p+1)}\frac{I_p(x)}{I_{p+1}(x)} - 1 = \frac{x}{2(p+1)}\frac{I_{p+2}(x)}{I_{p+1}(x)} \\ &= \frac{1}{p+1}\sum_{n\geqslant 1}\frac{x^2}{x^2 + j_{p+1,n}^2}.\end{aligned}$$

On the other hand using the infinite product representation of the function $\mathscr{I}_p$, i.e.

$$\mathscr{I}_p(x) = \prod_{n\geqslant 1}\left(1 + \frac{x^2}{j_{p,n}^2}\right),$$

we have

$$\log[\mathscr{I}_{p+1}(x)] = \log\left[\prod_{n\geqslant 1}\left(1 + \frac{x^2}{j_{p+1,n}^2}\right)\right] = \sum_{n\geqslant 1}\log\left(1 + \frac{x^2}{j_{p+1,n}^2}\right).$$

Now using the equivalent form of inequality [14, p. 279] $x^x \geqslant e^{x-1}$, i.e. $\log x \geqslant 1 - 1/x$, which holds for all $x > 0$, we conclude that for all $p \geqslant 0$, $n \geqslant 1$ and $x \in \mathbb{R}$ we have

$$\log\left(1 + \frac{x^2}{j_{p+1,n}^2}\right) \geqslant \frac{x^2}{x^2 + j_{p+1,n}^2} \geqslant \frac{1}{p+1}\frac{x^2}{x^2 + j_{p+1,n}^2}$$

and consequently

$$\log[\mathscr{I}_{p+1}(x)] \geqslant \frac{\mathscr{I}_p(x)}{\mathscr{I}_{p+1}(x)} - 1.$$

Finally, because the function $p \mapsto \mathscr{I}_p(x)$ is decreasing, we conclude that $\log[\mathscr{I}_{p+1}(x)] \leqslant \log[\mathscr{I}_p(x)]$, and with this the proof is complete. □

Concluding remarks:

1. First, we note that the Turán-type inequality (1.8) was proved earlier in 1951 by Thiruvenkatachar and Nanjundiah [19], while in 1991 Joshi and Bissu [12] examined an alternate derivation of (1.8) and slightly extended this inequality. However, our proof is

completely different; moreover, part (a) of the above theorem provides a generalization of (1.8). Recently, Ismail and Laforgia [10, Remark 2.4] proved for all $p > -1/2$ and $x > 0$ the inequality

$$\mathscr{I}_p(x)\mathscr{I}_{p+2}(x) \geqslant \frac{(2p+1)(p+2)}{(2p+3)(p+1)}\mathscr{I}^2_{p+1}(x).$$

We note that, since $(2p+3)(p+1) > (2p+1)(p+2)$, the above Turán-type inequality is weaker than (1.8).

2. On the other hand, observe that using (1.4), (1.5) and (1.6) in particular for $p = -1/2$ the Turán-type inequality (1.8) becomes

$$x \sinh^2(x) \leqslant 3(\cosh x)(x \cosh x - \sinh x),$$

which holds for all $x \geqslant 0$. Moreover, when $x \leqslant 0$, the above inequality is reversed. We note here that using (1.4) and (1.5), from inequality (1.9) we get (1.7), while from $\mathscr{I}_{p+1}(x) < \mathscr{I}_p(x)$ we obtain the well-known inequality $\tanh x < x$, where $x > 0$.

3. Inequality (1.11) is a natural extension of the hyperbolic analogue of Wilker's inequality [21]

$$\left(\frac{\sin x}{x}\right)^2 + \frac{\tan x}{x} > 2, \tag{1.15}$$

where $x \in (0, \pi/2)$. Namely, if we choose $p = -1/2$ in (1.11), then in view of (1.4) and (1.5) we have the hyperbolic analogue of (1.15)

$$\left(\frac{\sinh x}{x}\right)^2 + \frac{\tanh x}{x} > 2,$$

where $x \neq 0$. This inequality was proved recently by Zhu [22].

4. Recently, Stolarsky [17], among other things, proved that the monotonicity of the Hölder mean is actually a consequence of a certain inequality for $x \mapsto \log\cosh x$. In this spirit, he proved the following interesting inequalities:

$$\log\left(\frac{\sinh x}{x}\right) \leqslant \frac{\coth x}{x} - 1 \leqslant \log(\cosh x),$$

where $x > 0$ and the first inequality is in fact equivalent to the inequality between the logarithmic and identric means. Inequality (1.12) was motivated by the above result of Stolarsky and based on numerical experiments we conjecture the following: for each $p \in (-1, 0)$ and $x \in \mathbb{R}$ we have

$$\log[\mathscr{I}_{p+1}(x)] \leqslant \frac{\mathscr{I}_p(x)}{\mathscr{I}_{p+1}(x)} - 1 \leqslant \log[\mathscr{I}_p(x)].$$

By a confluent hypergeometric function, also known as a Kummer function, we mean the function

$$\Phi(a, c, x) = {}_1F_1(a, c; x) = \sum_{n\geqslant 0} \frac{(a)_n}{(c)_n}\frac{x^n}{n!} \quad \text{for all } x \in \mathbb{R}$$

defined for $a, c \in \mathbb{R}$ with $c \neq 0, -1, -2, \ldots$. It is known that [1, p. 509]

$$\mathscr{I}_p(x) = 2^p \Gamma(p+1) x^{-p} I_p(x) = \mathrm{e}^{-x} \Phi(p + 1/2, 2p+1, 2x).$$

Thus, inequalities (1.8), (1.9) and (1.11) are equivalent with inequalities

$$[\Phi(a+1, 2a+2, x)]^2 \leqslant \Phi(a, 2a, x)\Phi(a+2, 2a+4, x),$$

$$[\Phi(a, 2a, x)]^{a+1/2} \leqslant \mathrm{e}^{-x/2}[\Phi(a+1, 2a+2, x)]^{a+3/2},$$

$$[\mathrm{e}^{-x/2}\Phi(a+1, 2a+2, x)]^{2/(2a+1)} + \frac{\Phi(a+1, 2a+2, x)}{\Phi(a, 2a, x)} \geqslant 2,$$

where $a > -1/2$ and $x \in \mathbb{R}$. In fact proceeding exactly as in the proof of Theorem 1, we obtain the followings, which complete the above results.

Theorem 2. *If* $a \geqslant c > 0$ *and* $x \geqslant 0$, *then the function* $\mu \mapsto \Phi(a+\mu, c+\mu, x)$ *is log-convex on* $[0, \infty)$. *Moreover, if* $a, c > 0$ *and* $x \geqslant 0$, *then the function* $\mu \mapsto \Phi(a, c+\mu, x)$ *is log-convex too on* $[0, \infty)$. *In particular, the following inequalities*:

$$[\Phi(a+1, c+1, x)]^2 \leqslant \Phi(a, c, x)\Phi(a+2, c+2, x), \tag{1.16}$$

$$[\Phi(a, c, x)]^{(a+1)/(c+1)} \leqslant [\Phi(a+1, c+1, x)]^{a/c}, \tag{1.17}$$

$$[\Phi(a+1, c+1, x)]^{(a-c)/(c(a+1))} + \frac{\Phi(a+1, c+1, x)}{\Phi(a, c, x)} \geqslant 2 \tag{1.18}$$

hold true for all $a \geqslant c > 0$ *and* $x \geqslant 0$, *where the exponent* $\tau = [a(c+1)]/[c(a+1)]$ *is the smallest value of* τ *such that inequality* $\Phi(a, c, x) \leqslant [\Phi(a+1, c+1, x)]^{\tau}$ *holds. Moreover, for all* $a, c > 0$ *and* $x \geqslant 0$, *the following Turán-type inequality holds true*:

$$\Phi(a, c, x)\Phi(a, c+2, x) \geqslant \Phi^2(a, c+1, x). \tag{1.19}$$

Proof. As in the proof of Theorem 1, let us write

$$\Phi(a, c, x) = \sum_{n \geqslant 0} e_n(a, c) x^n \quad \text{where } e_n(a, c) := \frac{(a)_n}{(c)_n n!}, \ n \geqslant 0.$$

Computations show that for each $n \geqslant 0$ we get

$$\partial^2 \log[e_n(a+\mu, c+\mu)]/\partial\mu^2 = f(a) - f(c),$$

where $f : (0, \infty) \to \mathbb{R}$ is defined by $f(x) = \psi'(x+\mu+n) - \psi'(x+\mu)$ and $\mu \geqslant 0$. It is well-known that the function $x \mapsto \psi''(x)$ is increasing on $(0, \infty)$, thus, for all $x > 0$, $\mu, n \geqslant 0$ we have $f'(x) = \psi''(x+\mu+n) - \psi''(x+\mu) \geqslant 0$. Therefore f is increasing, i.e. $f(a) \geqslant f(c)$, and consequently the function $\mu \mapsto e_n(a+\mu, c+\mu)$ is log-convex on $[0, \infty)$. Thus, the function $\mu \mapsto \Phi(a+\mu, c+\mu, x)$ is also log-convex on $[0, \infty)$, as we required. Similarly, we have

$$\partial^2 \log[e_n(a, c+\mu)]/\partial\mu^2 = \psi'(c+\mu) - \psi'(c+\mu+n) \geqslant 0$$

for all $a, c > 0$ and $n \geqslant 0$, since the digamma function $x \mapsto \psi(x)$ is concave, i.e. the trigamma function $x \mapsto \psi'(x)$ is decreasing. Consequently, the function $\mu \mapsto \Phi(a, c+\mu, x)$ is also log-convex on $[0, \infty)$, as we required.

Inequality (1.16) follows from the log-convexity of $\mu \mapsto \Phi(a + \mu, c + \mu, x)$. To prove the inequality (1.17) consider the function $\varphi_2 : [0, \infty) \to \mathbb{R}$, defined by

$$\varphi_2(x) := \frac{a}{c} \log \Phi(a + 1, c + 1, x) - \frac{a+1}{c+1} \log \Phi(a, c, x).$$

Then from (1.16) we have

$$\varphi_2'(x) = \frac{a(a+1)}{c(c+1)} \left[\frac{\Phi(a+2, c+2, x)}{\Phi(a+1, c+1, x)} - \frac{\Phi(a+1, c+1, x)}{\Phi(a, c, x)} \right] \geqslant 0,$$

where we used the differentiation formula $c\Phi'(a, c, x) = a\Phi(a + 1, c + 1, x)$. Thus φ_2 is increasing, and consequently $\varphi_2(x) \geqslant \varphi_2(0) = 0$. Since $\Phi(a, c, x) = 1 + e_1(a, c)x + \cdots$ and $[\Phi(a+1, c+1, x)]^\tau = 1 + \tau e_1(a+1, c+1)x + \cdots$ for x in the neighborhood of the origin, it follows that the smallest value of τ such that inequality $\Phi(a, c, x) \leqslant [\Phi(a+1, c+1, x)]^\tau$ holds is $e_1(a, c)/e_1(a + 1, c + 1) = [a(c + 1)]/[c(a + 1)]$.

Finally, inequality (1.18) follows from (1.17) and the arithmetic–geometric mean inequality, while inequality (1.19) follows from the log-convexity of the function $\mu \mapsto \Phi(a, c + \mu, x)$. $\square$

Concluding remark: Recently, Ismail and Laforgia [10, Theorem 2.7] proved that if $c > a > 0$ and $x > 0$, then the following Turán-type inequality holds true:

$$\Phi(a, c, x)\Phi(a, c + 2, x) \geqslant \frac{(c-a)(c+1)}{(c+1-a)c} \Phi^2(a, c + 1, x). \tag{1.20}$$

We note that our result from Theorem 2, i.e. inequality (1.19) improves (1.20), because for all $c > a > 0$ we have $(c - a)(c + 1) < (c - a + 1)c$.

2. Extensions of some known trigonometric inequalities to Bessel functions

For $p > -1$ let us consider the function $\mathscr{J}_p : \mathbb{R} \to (-\infty, 1]$, defined by

$$\mathscr{J}_p(x) := 2^p \Gamma(p + 1)x^{-p} J_p(x) = \sum_{n \geqslant 0} \frac{(-1/4)^n}{(p+1)_n n!} x^{2n}, \tag{2.1}$$

where

$$J_p(x) = \sum_{n \geqslant 0} \frac{(-1)^n (x/2)^{2n+p}}{n!\Gamma(p+n+1)} \quad \text{for all } x \in \mathbb{R}$$

is the Bessel function of the first kind [20, p. 40]. It is worth mentioning that

$$\mathscr{J}_{-1/2}(x) = \sqrt{\pi/2}x^{1/2} J_{-1/2}(x) = \cos x, \tag{2.2}$$

$$\mathscr{J}_{1/2}(x) = \sqrt{\pi/2}x^{-1/2} J_{1/2}(x) = \frac{\sin x}{x}. \tag{2.3}$$

On the other hand, it is known that if $\tau\leqslant 3$ and $x\in(0,\pi/2)$, then the Lazarević-type inequality

$$\cos x<\left(\frac{\sin x}{x}\right)^{\tau} \tag{2.4}$$

holds [14, p. 238]. Moreover, here the exponent τ is not the least possible, i.e. if $\tau>3$, then there exists $x_1\in(0,\pi/2)$, depending on τ, such that (2.4) holds for all $x\in(x_1,\pi/2)$. Observe that, using (2.2) and (2.3) inequality (2.4) for $\tau=3$ can be rewritten as

$$[\mathcal{J}_{-1/2}(x)]^{-1/2+1}\leqslant[\mathcal{J}_{-1/2+1}(x)]^{-1/2+2}, \tag{2.5}$$

which is similar to (1.7). Note that because both members of (2.4) and (2.5) are even functions, we can deduce that both of inequalities hold for $|x|<\pi/2$. So in view of inequality (2.5), as in the first section, it is natural to ask: what is the analogue of this inequality for Bessel functions?

Our first main result of this section answer the above question. Moreover, we present some new inequalities for Bessel functions of the first kind.

Theorem 3. *Let $p>-1$ and let $j_{p,n}$ be the nth positive zero of the Bessel function J_p. Further, consider the set $\Delta:=\Delta_1\cup\Delta_2$, where*

$$\Delta_1:=\bigcup_{n\geqslant 1}[-j_{p,2n},-j_{p,2n-1}]\quad\textit{and}\quad \Delta_2:=\bigcup_{n\geqslant 1}[j_{p,2n-1},j_{p,2n}].$$

Then the following assertions are true:

(a) *the function $x\mapsto\mathcal{J}_p(x)$ is negative on Δ and is strictly positive on $\mathbb{R}\backslash\Delta$;*
(b) *the function $x\mapsto\mathcal{J}_p(x)$ is increasing on $(-j_{p,1},0]$ and is decreasing on $[0,j_{p,1})$;*
(c) *the function $x\mapsto\mathcal{J}_p(x)$ is strictly log-concave on $\mathbb{R}\backslash\Delta$;*
(d) *the function $x\mapsto J_p(x)$ is strictly log-concave on $(0,\infty)\backslash\Delta_2$, provided $p\geqslant 0$;*
(e) *the function $p\mapsto\mathcal{J}_p(x)$ is increasing and log-concave for each fixed $x\in(-j_{p,1},j_{p,1})$;*
(f) *the function $p\mapsto J_p(x)$ is log-concave for each fixed $x\in(0,j_{p,1})$;*
(g) *the function $p\mapsto\mathcal{J}_{p+1}(x)/\mathcal{J}_p(x)$ is decreasing for each fixed $x\in(-j_{p,1},j_{p,1})$;*
(h) *the function $p\mapsto[\mathcal{J}_p(x)]^{p+1}$ is increasing for each fixed $x\in(-j_{p,1},j_{p,1})$;*
(i) *the following inequalities hold true for each $\alpha\in(0,1)$ and $x,y\in(0,\infty)\backslash\Delta_2$, $x\neq y$*

$$J_p(\alpha x)>\alpha^p J_p(x)[\mathcal{J}_p(x)]^{\alpha-1}, \tag{2.6}$$

$$[xJ'_p(x)]^2>p[J_p(x)]^2+x^2J_p(x)J''_p(x), \tag{2.7}$$

$$J_p^2\left(\frac{x+y}{2}\right)>\left(\frac{x+y}{2\sqrt{xy}}\right)^{2p}J_p(x)J_p(y), \tag{2.8}$$

(j) *the following inequalities hold true for all $x\in(-j_{p,1},j_{p,1})$*

$$[\mathcal{J}_{p+1}(x)]^2\geqslant\mathcal{J}_p(x)\mathcal{J}_{p+2}(x), \tag{2.9}$$

$$[\mathcal{J}_p(x)]^{p+1}\leqslant[\mathcal{J}_{p+1}(x)]^{p+2}, \tag{2.10}$$

$$[\mathcal{J}_{p+1}(x)]^{1/(p+1)}+\frac{\mathcal{J}_{p+1}(x)}{\mathcal{J}_p(x)}\geqslant 2. \tag{2.11}$$

Proof. (a) It is known that [20, p. 498]

$$\mathscr{J}_p(x) = \prod_{n\geqslant 1}\left(1 - \frac{x^2}{j_{p,n}^2}\right). \tag{2.12}$$

Since $0 < j_{p,1} < j_{p,2} < \cdots < j_{p,n} < \cdots$ we have that if $x \in [j_{p,2n-1}, j_{p,2n}]$ or $x \in [-j_{p,2n}, -j_{p,2n-1}]$ then the first $(2n-1)$ terms of the above product are negative, and the remained terms are strictly positive. Hence $\mathscr{J}_p$ becomes negative on Δ. Now, if $x \in (-j_{p,1}, j_{p,1})$, then clearly each terms of the right-hand side of (2.12) are strictly positive. Moreover, if $x \in (j_{p,2n}, j_{p,2n+1})$ or $x \in (-j_{p,2n+1}, -j_{p,2n})$, then the first $2n$ terms are strictly negative, while the rest is strictly positive. From this it follows that for the function $\mathscr{J}_p$ we have $\mathscr{J}_p(x) > 0$ for all $x \in \mathbb{R}\backslash\Delta$.

(b) From part (a) the function $\mathscr{J}_p$ is strictly positive on $(-j_{p,1}, j_{p,1})$. Using the infinite product representation (2.12) we obtain

$$\frac{\mathrm{d}}{\mathrm{d}x}\log[\mathscr{J}_p(x)] = \frac{\mathscr{J}'_p(x)}{\mathscr{J}_p(x)} = -\sum_{n\geqslant 1}\frac{2x}{j_{p,n}^2 - x^2}. \tag{2.13}$$

From this, we deduce that the function $x \mapsto \mathscr{J}_p(x)$ is increasing on $(-j_{p,1}, 0]$ and is decreasing on $[0, j_{p,1})$, as we required.

(c) Using 2.13 and part (a) we conclude that

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2}\log[\mathscr{J}_p(x)] = -2\sum_{n\geqslant 1}\frac{j_{p,n}^2 + x^2}{(j_{p,n}^2 - x^2)^2} < 0$$

for all $x \in \mathbb{R}\backslash\Delta$, and consequently $\mathscr{J}_p$ is strictly log-concave on $\mathbb{R}\backslash\Delta$.

(d) Rewriting (2.1) as

$$J_p(x) = \frac{x^p\mathscr{J}_p(x)}{2^p\Gamma(p+1)}, \tag{2.14}$$

the strict log-concavity of J_p follows from part (c). Indeed, the function $x \mapsto x^p$ is log-concave on $(0, \infty)$ for all $p \geqslant 0$, which implies that J_p is strictly log-concave on $(0, \infty)\backslash\Delta_2$ as a product of a log-concave and a strictly log-concave function.

(e) Using 2.12 we have

$$\log[\mathscr{J}_p(x)] = \sum_{n\geqslant 1}\log\left(1 - \frac{x^2}{j_{p,n}^2}\right).$$

On the other hand, it is known [15, p. 317] that for each $n \geqslant 1$ the function $p \mapsto 1/j_{p,n}^2$ is decreasing and convex on $(-1, \infty)$. Consequently the functions $p \mapsto 1 - x^2/j_{p,n}^2$ are increasing and concave on $(-1, \infty)$, as well as the functions $p \mapsto \log(1 - x^2/j_{p,n}^2)$. Thus, the function $p \mapsto \log[\mathscr{J}_p(x)]$ is increasing and concave for each fixed $x \in (-j_{p,1}, j_{p,1})$ as a sum of increasing and concave functions.

(f) As in part (d) we use (2.14). Since the function $p \mapsto \Gamma(p+1)$ is log-convex and $p \mapsto \mathscr{J}_p(x)$ is log-concave, the function $p \mapsto J_p(x)$ is log-concave as a product of two log-concave functions.

(g) Since $p \mapsto \log[\mathscr{I}_p(x)]$ is concave it follows that the function $p \mapsto \log[\mathscr{I}_{p+a}(x)] - \log[\mathscr{I}_p(x)]$ is decreasing for each $a > 0$. Choosing $a = 1$ we obtain that $p \mapsto \mathscr{I}_{p+1}(x)/\mathscr{I}_p(x)$ is decreasing, as we asserted.

(h) To prove the required result consider the function $\varphi_3 : (-j_{p,1}, j_{p,1}) \to \mathbb{R}$, defined by

$$\varphi_3(x) := \frac{p+1}{q+1}\log[\mathscr{I}_p(x)] - \log[\mathscr{I}_q(x)],$$

where $q \geqslant p > -1$. On the other hand,

$$\varphi_3'(x) = \frac{p+1}{q+1}\left[\frac{\mathscr{I}'_p(x)}{\mathscr{I}_p(x)}\right] - \frac{\mathscr{I}'_q(x)}{\mathscr{I}_q(x)} = -2xb_1(q)\left[\frac{\mathscr{I}_{p+1}(x)}{\mathscr{I}_p(x)} - \frac{\mathscr{I}_{q+1}(x)}{\mathscr{I}_q(x)}\right],$$

where we used the differentiation formula $\mathscr{I}'_p(x) = -2xb_1(p)\mathscr{I}_{p+1}(x)$, which can be derived easily from 2.1. Since $q \geqslant p$ we have $\mathscr{I}_{p+1}(x)/\mathscr{I}_p(x) \geqslant \mathscr{I}_{q+1}(x)/\mathscr{I}_q(x)$, we conclude that the function φ_3 is decreasing on $[0, j_{p,1})$ and is increasing on $(-j_{p,1}, 0]$. Consequently, $\varphi_3(x) \leqslant \varphi_3(0) = 0$, i.e. $[\mathscr{I}_p(x)]^{p+1} \leqslant [\mathscr{I}_q(x)]^{q+1}$ holds for all $x \in (-j_{p,1}, j_{p,1})$.

(i) Because from part (c) $\mathscr{I}_p$ is strictly log-concave, due to definition one has

$$\mathscr{I}_p(\alpha x + (1-\alpha)y) > [\mathscr{I}_p(x)]^{\alpha}[\mathscr{I}_p(y)]^{1-\alpha}, \tag{2.15}$$

where $p > -1$, $\alpha \in (0, 1)$ and $x, y \in \mathbb{R}\backslash\Delta$, $x \neq y$. Choosing $y = 0$ in (2.15) and taking into account (2.1) we obtain (2.6). Moreover, taking in (2.15) $\alpha = 1/2$ from (2.1) yields (2.8). For (2.7) we use again the fact that $\mathscr{I}_p$ is strictly log-concave, that is $x \mapsto \mathscr{I}'_p(x)/\mathscr{I}_p(x) = J'_p(x)/J_p(x) - p/x$ is strictly decreasing.

(j) Inequality (2.9) follows from the log-concavity of $p \mapsto \mathscr{I}_p(x)$, while inequality (2.10) follows from part (h). Finally, the extension of Wilker's inequality, i.e. inequality (2.11) follows from (2.10) and the arithmetic–geometric mean inequality for the values $\mathscr{I}_{p+1}(x)]^{1/(p+1)}$ and $\mathscr{I}_{p+1}(x)/\mathscr{I}_p(x)$. With this the proof is complete. $\square$

Concluding remarks:

1. Recently Giordano et al. [8] proved that the Bessel function $x \mapsto J_p(x)$ is log-concave on $(0, j_{p,1})$ for each $p > -1$. We note that part (d) of the above theorem states that this property for $p \geqslant 0$ remains true on $(0, \infty)\backslash\Delta_2$ too. Moreover, following the proof of part (d), it is easy to see that the function $x \mapsto J_p(x)/x$ is also log-concave on $(0, \infty)\backslash\Delta_2$ for all $p \geqslant 1$. This was proved in [8] for $x \in (0, j_{p,1})$, please see also [9] for more details.
2. Part (f) was proved earlier by Muldoon [15] using a different argument. Moreover, Ismail and Muldoon [11] showed that the function $p \mapsto J_{p+1}(x)/J_p(x)$ is decreasing when $p > -1$, $x > 0$ and $x \neq j_{p,n}$. We note that using part (g) and (2.1) we obtain that the function

$$p \mapsto \frac{J_{p+1}(x)}{J_p(x)} = \frac{x}{2(p+1)}\frac{\mathscr{I}_{p+1}(x)}{\mathscr{I}_p(x)}$$

is decreasing, but just for each $x \in (-j_{p,1}, j_{p,1})$.

3. It is worth mentioning that the analogous of (2.6), (2.7) and (2.8) for modified Bessel functions can be found in [5,16], while the Turán-type inequality (2.9) was proved earlier for each $x \in \mathbb{R}$ by Szász [18], and later by Joshi and Bissu [12] using recursions. Finally, observe that inequality (2.10) in particular for $p = -1/2$ reduces to the Lazarević-type inequality (2.5), while (2.11) reduces to Wilker's inequality (1.15). Here we used that $j_{-1/2,1} = \pi/2$, which can be verified using the infinite product representation of the cosine function [1, p. 75] and formula (2.12).

Recently, Neuman [16] proved that the function $x \mapsto \mathscr{I}_p(x)$ is strictly log-convex on $\mathbb{R}$ for all $p > -1/2$. We note that $x \mapsto \mathscr{I}_{-1/2}(x) = \cosh(x)$ is also strictly log-convex on $\mathbb{R}$, furthermore, we conjectured in [5] that $x \mapsto \mathscr{I}_p(x)$ is strictly log-convex on $\mathbb{R}$ for each $p > -1$. The following result provides a partial positive answer to the above conjecture and is motivated by the proof of part (c) of Theorem 3.

Theorem 4. *If $p > -1$, then the function $x \mapsto \mathscr{I}_p(x)$ is strictly log-convex on $[-j_{p,1}, j_{p,1}]$, where $j_{p,1}$ is the first positive zero of the Bessel function J_p. Moreover, the function $x \mapsto \mathscr{I}_p(x)/\mathscr{J}_p(x) = I_p(x)/J_p(x)$ is strictly log-convex too on $(-j_{p,1}, j_{p,1})$. In particular, the following inequalities:*

$$\frac{\left[I_p\left(\frac{x+y}{2}\right)\right]^2}{\left[J_p\left(\frac{x+y}{2}\right)\right]^2} \leqslant \frac{I_p(x)I_p(y)}{J_p(x)J_p(y)}, \quad x, y \in (-j_{p,1}, j_{p,1}),$$

$$\frac{\left[\cosh\left(\frac{x+y}{2}\right)\right]^2}{(\cosh x)(\cosh y)} \leqslant \frac{\left[\cos\left(\frac{x+y}{2}\right)\right]^2}{(\cos x)(\cos y)}, \quad x, y \in (-\pi/2, \pi/2),$$

$$\frac{\left[\sinh\left(\frac{x+y}{2}\right)\right]^2}{(\sinh x)(\sinh y)} \leqslant \frac{\left[\sin\left(\frac{x+y}{2}\right)\right]^2}{(\sin x)(\sin y)}, \quad x, y \in (-\pi, \pi)$$

hold true and equality hold in each of the above inequalities if and only if $x = y$.

Proof. It is known that, using (2.12), the function $\mathscr{I}_p$ may be represented as

$$\mathscr{I}_p(x) = \prod_{n \geqslant 1}\left(1 + \frac{x^2}{j_{p,n}^2}\right),$$

which implies that

$$\frac{d^2}{dx^2}\log[\mathscr{I}_p(x)] = 2\sum_{n \geqslant 1}\frac{j_{p,n}^2 - x^2}{(j_{p,n}^2 + x^2)^2} > 0$$

for all $x \in [-j_{p,1}, j_{p,1}]$, and consequently $\mathscr{I}_p$ is strictly log-convex. Now using part (c) of Theorem 3 the function $x \mapsto 1/\mathscr{J}_p(x)$ is strictly log-convex on $(-j_{p,1}, j_{p,1})$. Hence, the function $x \mapsto \mathscr{I}_p(x)/\mathscr{J}_p(x) = I_p(x)/J_p(x)$ is strictly log-convex too on $(-j_{p,1}, j_{p,1})$, as a product of two strictly log-convex functions. Finally, the inequalities follows easily from the definition of log-convexity, taking into account that $j_{-1/2,1} = \pi/2$ and $j_{1/2,1} = \pi$. □

Let us note the following trigonometric inequality which represent a partial answer to the problem E 1277 proposed by Oppenheim and solved by Carver in American Mathematical Monthly 65, 206–209 (1958): if $a \in (0, 1/2]$ and $|x| \leqslant \pi/2$, then [14, p. 238]

$$\frac{(a+1)\sin x}{1+a\cos x} \leqslant x \leqslant \frac{\pi}{2}\frac{\sin x}{1+a\cos x}. \tag{2.16}$$

The following result extends (2.16) to the function $\mathscr{J}_p$.

Theorem 5. *If* $0 < a \leqslant 1/2$, $p \geqslant -1/2$ *and* $x \in [-\pi/2, \pi/2]$, *then*

$$\begin{aligned}&[a(2p+1)+(a+1)]\frac{\mathscr{J}_{p+1}(x)}{1+2a(p+1)\mathscr{J}_p(x)}\\ &\quad \leqslant 1 \leqslant [a(2p+1)+\pi/2]\frac{\mathscr{J}_{p+1}(x)}{1+2a(p+1)\mathscr{J}_p(x)}.\end{aligned} \tag{2.17}$$

Proof. Observe that when $p = -1/2$ from (2.2) and (2.3) it follows that (2.17) reduces to (2.16), which is equivalent to

$$(a+1)\mathscr{J}_{1/2}(x) \leqslant 1 + a\mathscr{J}_{-1/2}(x) \leqslant (\pi/2)\mathscr{J}_{1/2}(x). \tag{2.18}$$

Recall the well-known Sonine integral formula [20, p. 373] for Bessel functions

$$J_{q+p+1}(x) = \frac{x^{p+1}}{2^p\Gamma(p+1)}\int_0^{\pi/2} J_q(x\sin\theta)\sin^{q+1}\theta\cos^{2p+1}\theta\,\mathrm{d}\theta,$$

where $p, q > -1$ and $x \in \mathbb{R}$. From this, we obtain the following formula:

$$\mathscr{J}_{q+p+1}(x) = \frac{2}{B(p+1,q+1)}\int_0^{\pi/2} \mathscr{J}_q(x\sin\theta)\sin^{2q+1}\theta\cos^{2p+1}\theta\,\mathrm{d}\theta, \tag{2.19}$$

which will be useful in the sequel. Here, $B(p,q) = \Gamma(p)\Gamma(q)/\Gamma(p+q)$ is the well-known Euler's beta function. Changing in (2.19) p with $p-1/2$ and taking $q = -1/2$ ($q = 1/2$, respectively) one has for all $p > -1/2$ and $x \in \mathbb{R}$

$$\mathscr{J}_p(x) = \frac{2}{B(p+1/2,1/2)}\int_0^{\pi/2} \mathscr{J}_{-1/2}(x\sin\theta)\cos^{2p}\theta\,\mathrm{d}\theta, \tag{2.20}$$

$$\mathscr{J}_{p+1}(x) = \frac{2}{B(p+1/2,3/2)}\int_0^{\pi/2} \mathscr{J}_{1/2}(x\sin\theta)\sin^2\theta\cos^{2p}\theta\,\mathrm{d}\theta. \tag{2.21}$$

Thus, in view of (2.20) and (2.21), if we change x with $x\sin\theta$ in (2.18), and multiply (2.18) with $\sin^2\theta\cos^{2p}\theta$, then after integration it follows that the expression

$$\begin{aligned} E_p(x) &= \int_0^{\pi/2}\sin^2\theta\cos^{2p}\theta\,\mathrm{d}\theta + a\int_0^{\pi/2}\mathscr{J}_{-1/2}(x\sin\theta)(1-\cos^2\theta)\cos^{2p}\theta\,\mathrm{d}\theta \\ &= \frac{1}{2}B\left(p+\frac{1}{2},\frac{3}{2}\right)+\frac{a}{2}B\left(p+\frac{1}{2},\frac{1}{2}\right)\mathscr{J}_p(x)-\frac{a}{2}B\left(p+\frac{3}{2},\frac{1}{2}\right)\mathscr{J}_{p+1}(x) \end{aligned}$$

satisfies the following:

$$\frac{a+1}{2}B\left(p+\frac{1}{2},\frac{3}{2}\right)\mathscr{J}_{p+1}(x)\leqslant E_p(x)\leqslant\frac{\pi}{4}B\left(p+\frac{1}{2},\frac{3}{2}\right)\mathscr{J}_{p+1}(x).$$

After simplifications we obtain that (2.17) holds. □

Following the proof of the above theorem the next result is quite obvious.

Theorem 6. *For each* $p\geqslant -1/2$ *the function* $\mathscr{J}_p$ *is concave on* $[-\pi/2,\pi/2]$.

Proof. Since the cosine function is concave on $[-\pi/2,\pi/2]$, one has

$$\mathscr{J}_{-1/2}(\alpha x+(1-\alpha)y)\geqslant\alpha\mathscr{J}_{-1/2}(x)+(1-\alpha)\mathscr{J}_{-1/2}(y),$$

where $\alpha\in[0,1]$ and $x,y\in[-\pi/2,\pi/2]$. Changing x with $x\sin\theta$, y with $y\sin\theta$, from (2.20) it follows that $\mathscr{J}_p(\alpha x+(1-\alpha)y)\geqslant\alpha\mathscr{J}_p(x)+(1-\alpha)\mathscr{J}_p(y)$ holds for all $p\geqslant -1/2$, i.e. the function $\mathscr{J}_p$ is concave on $[-\pi/2,\pi/2]$. □

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