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KARL MARX AND THE FOUNDATIONS OF DIFFERENTIAL CALCULUS

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SUMMARIES

The publication of the mathematical manuscripts of Karl Marx, suggested by Engels in 1885, announced in 1932, and completed in 1968, brought new awareness of his many-sided talent. A sketch of the history is followed by discussion of Marx's concept of the derivative and the differential, and assessment of the originality and value of his achievements in this field.

Die von Engels im Jahre 1885 vorgeschlagene, in 1932 angekündigte und in 1968 vollendete Veröffentlichung der mathematischen Manuskripte von Karl Marx brachte ein tieferes Verständnis für seine vielseitigen Talente. Einer Skizze deren Geschichte folgt eine Erklärung seiner Ideen über den Begriff der Ableitung und des Differentials, sowie eine Würdigung der Originalität und des Wertes seiner Leistungen auf diesem Gebiet.

Публикация математических рукописей К. Маркса, предложенная Энгельсом еще в 1885 г., объявлянная к печати в 1932 г. и законченная в 1968 г., вызвала новое осознание многосторонностн таланта Маркса. В докладе, кроме наброска истории этих рукописей, предлагается изложение понятий производной функции и дифференциала разработанные Марксом и оценка оригинальности его мысли и достижений в этой области.

I

In his preface to the 2nd edition (1885) of his *Anti-Dühring* Friedrich Engels expressed the desire to publish "the extremely important mathematical manuscripts left by Marx" [MEW 20, 13; Engels 1939, 17] [1] together with the results of his own research in science. This was not done, however, and so the "independent discoveries" of Marx, mentioned by Engels in the graveside ceremony at Highgate Cemetery [MEW 19, 336], remained unpublished for fifty years after Marx' death.

The existence of some 1000 pages of mathematical manuscripts of Marx in the Marx-Engels Institute in Moscow was announced in 1931 by E. Kolman at the International Congress of the History of Science and Technology, London [Kolman 1931]. An extensive excerpt from Marx' mathematical manuscripts was published in 1933 in Russian translation [Marx 1933] along with an analysis of it by S. A. Yanovskaya [1933]. This publication was announced at the International Congress of Mathematicians, Zürich 1932, by E. Kolman, one of the editors of the journal in which it appeared, although his sanguine prediction that "the complete mathematical writings of Marx, under the editorial direction of Professor Yanovskaya, will shortly [demnächst] appear in the works of the Marx-Engels-Lenin Institute (Moscow)" [Kolman 1932] did not come true until 1968 [Marx 1968]. That edition was, in fact, prepared under the direction of S. A. Yanovskaya, although she died two years before its final appearance.

During this period, interest in the mathematical writings of Marx was mainly confined to the Soviet Union, where, for example, an extensive monograph on the subject was published by L. P. Gokieli [Gokieli 1947]. Perhaps the first outside the Soviet Union to give an analysis of Marx' mathematical writings was D. J. Struik [1948]. He had access to the original German text of the Russian publication of 1933 and gave English translations of several pertinent passages [2].

In the 1950's work on the manuscripts continued under the direction of S. A. Yanovskaya, especially by K. A. Rybnikov, who investigated the mathematical sources at Marx' disposal. In addition to writing his doctoral dissertation on Marx' mathematics, Rybnikov also contributed an article on this subject to the 2nd edition of the Great Soviet Encyclopedia [Rybnikov 1954]. (This article has been omitted from the 3rd edition.)

But the manuscripts were not published in their original language--mainly German--until 1968, when the long awaited (nearly) complete text appeared along with a complete translation into Russian [Marx 1968]. This edition contained a preface that was immediately translated into German [Yanovskaya 1969] as well as numerous notes and commentaries. For the few articles that were in [Marx 1933] a new translation into Russian was made and the translation on the whole is, as far as I can tell, excellent, although one egregious error should be pointed out. At one point Marx remarks that Boucharlat "wants some hocus pocus", which has been translated as "nuzhdaetsya v kakom-nibud fokuse [wants some kind of focus]" [Marx 1968, 263]. To this the editor can only plaintively note: "The question of precisely what focus in Boucharlat Marx has in view here presents a certain difficulty" [Marx 1968, 617] [3].

With the publication of [Marx 1968] interest in Marx' mathematical writings spread more rapidly outside the Soviet Union. Already in 1969 an article on Marx' foundation of differential calculus appeared in the German Democratic Republic [Miller 1969]. (This article probably covers some of the same ground as mine. I have not seen it.) An Italian translation of the first article in [Marx 1968] appeared along with a commentary [Marx 1972 and Lombardo Radice 1972], and in 1974 the original German of the first part of [Marx 1968] was published in the Federal Republic of Germany [Marx 1974]. This part, headed "Differential calculus, its nature and history", contains the original and self-contained articles of Marx on the subject. The second, and longer, part is headed "Description of the mathematical manuscripts", a rather misleading title, since it consists mainly of actual writings of Marx and not a mere description of them. An Italian translation of 'Part One' appeared the following year [Marx 1975], prompting further discussion of Marx' mathematical writings (for example, [Bottazzini 1975].) An English translation of the mathematical writings of Marx is expected to be included in the *Collected Works of Marx and Engels*, the publication of which began in 1975 and will include some fifty volumes.

II

Although Marx' Gymnasium certificate said that he had "a good knowledge of mathematics," there is no evidence of further occupation with mathematics for 23 years. Then Marx wrote Engels on 11 January 1858: "During the elaboration of the economic principles I have been so damed delayed by computational errors that out of despair I undertook again a quick scanning of the algebra. Arithmetic was always alien to me. Via the algebraic detour, however, I catch up quickly" [MEW 29, 256]. Marx' new interest in mathematics continued and he wrote Engels on 23 November 1860: "Writing articles is almost out of the question for me. The only activity by which I can keep the necessary quietness of mind is mathematics" [MEW 30, 113]. By 1863 he was well into his study of calculus, writing Engels on 6 July: "In my spare time I do differential and integral calculus. Apropos! I have plenty of books on it and I will send you one if you like to tackle that field. I consider it almost necessary for your military studies. It is also a much easier part of mathematics (as far as the purely technical side is concerned) than for instance the higher parts of algebra. Aside from knowledge of the common algebraic and trigonometric stuff no preparatory study is needed except general acquaintance with the conic sections" [MEW 30, 362]. Engels, however, seems to have felt the need to return to arithmetic, writing Marx on 30 May 1864: "In your Francoeur [Louis Benjamin Francoeur (1773-1849), a French textbook writer] I have immersed myself in arithmetic, from which you seem to have kept yourself rather distant" [MEW 30, 401]. But he, too, prefers algebra "precisely because the general expression in the algebraic form is simpler and more evident [anschaulicher]" [MEW 30, 401]. Indeed, Engels was quite impressed by Marx' advances in mathematics; he wrote F. A. Lange on 29 March 1865: "I cannot leave unmentioned a remark about old Hegel, to whom you deny a deeper mathematical scientific education. Hegel knew so much mathematics that none of his students were capable of editing the numerous mathematical manuscripts that he left. To my knowledge, the only man who understands enough mathematics and philosophy to do this is Marx" [MEW 31, 467].

We see from these letters that Marx was prompted to renew his study of mathematics in order to apply it in his study of political economy. By 1873, however, after searching in vain for a way to mathematically determine the principal laws governing financial crises (something his friend Samuel Moore thought impossible, but which Marx still thought possible "with sufficiently examined material"), he decided to give up the attempt "for the time being" [letter to Engels, 31 May 1873, MEW 33, 82]. But Marx continued to be interested in mathematics for its own sake, and most of his mathematical writings date from the last decade of his life, from 1873 to 1883. He was especially interested in the differential calculus, which he felt lacked an adequate foundation. He agreed with Engels, who had written in his *Anti-Dühring*: "With the introduction of variable magnitudes and the extension of their variablity to the infinitely small and infinitely large, mathematics, in other respects so strictly moral, fell from grace; it ate of the tree of knowledge, which opened up to it a career of most colossal achievements, but at the same time a path of error. The virgin state of absolute validity and irrefutable certainty of everything mathematical was gone forever; mathematics entered the realm of controversy, and we have reached the point where most people differentiate and integrate not because they understand what they are doing but from pure faith, because up to now it has always come out right" [MEW 20, 81; Engels 1939, 98].

Marx saw that with the introduction of variable magnitudes to describe dialectical processes in nature, the mathematical explanation must also be dialectical. He was not satisfied with the explanations of derivatives and differentials in the books at his disposal, his criticisms of them became more frequent in his notebooks, and in the end he produced his own original and independent analyses of these concepts. These were never written up for publication, but his drafts of at least two of them were revised to a 'final' form for Engels to read. The first of these treats the concept of the derivative and the second the differential. They were both written in 1881. Engels was prompted to write to him on 18 August 1881: "Yesterday I found at last the courage to study your mathematical manuscripts even without reference to textbooks, and I was glad to see that I did not need them. I compliment you on your work. The matter is so perfectly clear [sonnenklar] that we cannot be amazed enough how the mathematicians so stubbornly insist upon mystifying it" [MEW 35, 23]. Indeed, Engels waxed enthusiastic: "The matter has taken such a hold of me that it not only turns around in my head the whole day, but that also last week in a dream I gave a fellow my shirt buttons to differentiate and this fellow ran away with them [und dieser mir damit durchbrannte]" [MEW 35, 25]. (These two articles of Marx are described in sections IV and VI below.)

Other writings of Marx that are also included in 'Part One' of [Marx 1968] deal with the history of differential calculus, Taylor's and Maclaurin's Theorems, Lagrange's theory of analytic functions, and the method of D'Alembert. Marx distinguished three periods in the development of differential calculus in the 17th and 18th centuries: (1) the mystical differential calculus (of Leibniz and Newton), (2) the rational differential calculus (of D'Alembert), and (3) the purely algebraic differential calculus (of Lagrange). His commentary on the "mystical" period has often been quoted, but is worth repeating: "Thus: they themselves believed in the mysterious character of the newly discovered calculus, that yielded true (and moreover, particularly in the geometrical application, astonishing) results by a positively false mathematical procedure. They were thus self-mystified, valued the new discovery all the higher, enraged the crowd of old orthodox mathematicians all the more, and thus called forth the cry of opposition, that even in the lay world has an echo and is necessary in order to pave the way for something new" [Marx 1968, 168].

Of the mathematicians not satisfied with this state of affairs, Marx especially appreciated D'Alembert: "D'Alembert had, by stripping the differential calculus from its mystical garb, made an enormous step ahead" [Marx 1968, 174]. D. J. Struik commented on this: "Marx' evaluation of D'Alembert's work as 'an enormous step ahead' still stands. This is the more remarkable, since even modern historians of mathematics have a way of glossing over it" [Struik 1948, 189].

Lagrange's contribution was in making the differential calculus algebraic. For this "Lagrange took as his immediate starting point the theorem of ... Taylor, which in fact is the most general, most comprehensive theorem and at the same time an operational formula of differential calculus, namely that which expresses y_1 or $f(x + h)$ by a development in a series with symbolic differential coefficients" [Marx 1968, 178]. Marx sees Lagrange as "furnishing the truly rational basis of differential calculus" [Marx 1968, 285]. He sums up his judgement of Lagrange's merit in two points:

"(1) The great merit of Lagrange is not only the founding of Taylor's Theorem and differential calculus in general by a purely algebraic analysis, but in particular to have introduced the concept of derived function that all those who have come after him have more or less used without mentioning it. But he was not content with this. He gives the purely algebraic development of all possible functions of $x + h$, in ascending whole positive powers of h and christens them with the names of differential calculus. All the ease and short cuts that differential calculus itself allows (Taylor's Theorem etc.) are thereby forfeited and very often replaced by operations of a much more lengthy and complicated nature.

"(2) So far as it is a question of pure analysis, Lagrange is in fact free of everything that appears to him as metaphysical transcendence in Newton's fluxions, Leibniz' infinitesimals of various orders, the limiting value theory of vanished quantities, the use of $0/0$ $(=dy/dx)$ as symbol for the differential coefficient, etc. This still does not hinder him, in the application of his theory and curves, etc., from constantly using one or the other of these 'metaphysical' notions" [Marx 1968, 202].

III

Marx was not in the mainstream of mathematics and to the end he seems to have been unaware of the advances being made by continental mathematicians in the foundations of differential calculus, including the work of Cauchy. The most mathematical of his acquaintances was Samuel Moore, who, as it turned out, was unable to appreciate the originality of Marx' work, although he was co-translator, with Edward Aveling, of the English translation of the first volume of Marx' *Capital*. Marx was self-taught, and for this he used textbooks based on the work of mathematicians of the 17th and 18th centuries.

He began his study of differential calculus with the *Cours complet de mathématiques* (Paris 1778) of the Abbé Sauri and later worked his way through the 1828 English translation *(An elementary treatise on the differential and integral calculus)* of the widely read work of Jean Louis Boucharlat (1775-1848). The book of Sauri presented the infinitesimal method of Leibniz. (Marx immediately compared this with Newton's method.) Boucharlat's work was a mixture of the ideas of D'Alembert and

Lagrange. Marx also read Euler and MacLaurin, as well as textbooks by Lacroix, John Hind (1796-1866), George Hemming (1821-1905), and others.

IV

Marx' article "On the concept of the derived function" begins with the very simple example $y = ax$, for which: "if x increases to x_1, $y_1 = ax_1$ and $y_1 - y = a(x_1 - x)$. Let the *differential operation* now take place, i.e. let x_1 decrease to x, so that $x_1 = x$; $x_1 - x = 0$, then $a(x_1 - x) = a \cdot 0 = 0$. Further, since y simply went to y_1, because x went to x_1, now likewise $y_1 = y$; $y_1 - y = 0$. Therefore $y_1 - y = a(x_1 - x)$ becomes $0 = 0$.

"First making the differentiation and then removing it leads literally to *nothing*. The entire difficulty in understanding the differential operation (as in that of any *negation of the negation* whatever) lies precisely in seeing *how* it differs from such a simple procedure and so leads to true results" [Marx 1968, 28].

He then proceeds to divide $y_1 - y = a(x_1 - x)$ by $x_1 - x$ to obtain $(y_1 - y)/(x_1 - x) = a$. He comments:

"Since a is a constant, no change in it can occur, and so neither can it occur on the reduced *right side* of the equation. Under such circumstances the *differential process* takes place on the left side

$$(y_1 - y)/(x_1 - x) \text{ or } \Delta y / \Delta x$$

and this is a characteristic of such simple functions as ax.

"If in the denominator of this ratio x_1 decreases, then it approaches x; the limit of its decrease is reached as soon as it becomes x. With this the difference is such that $x_1 - x = x - x = 0$, and hence also $y_1 - y = y - y = 0$. We thus obtain $0/0 = a$.

"Since in the expression $0/0$ every trace of its origin and its meaning has been wiped out, we replace it by dy/dx, in which the finite differences $x_1 - x$ or Δx and $y_1 - y$ or Δy appear symbolized as *removed* or *vanished* differences, or $\Delta y/\Delta x$ is changed into dy/dx. Therefore $dy/dx = a$.

"The closely held consolation of some rationalizing mathematicians, that the quantities dy and dx are in fact only infinitely small and [their ratio] only approaches $0/0$, is a chimera, ..." [Marx 1968, 30-32].

Two things stand out in this presentation of Marx. One is his total rejection of the concept of the derivative as a ratio of infinitesimals. The other is his view that he is analysing a dialectical process, seen especially as a "negation of the negation".

I remark in passing that the explanation of infinitesimals as a dialectical existence of contraries--at the same time both zero and nonzero--has been described as false by S. A. Yanovskaya: "Some so-called Marxists are prepared therefore to see in infinitesimals a truly dialectical concept, forgetting that dialectical materialism does not recognize a static, not connected with movement (by a struggle of opposites and a bringing about of its transition to a new stage) contradiction" [Yanovskaya 1933, 101-102].

But what is Marx' objection to the infinitesimal method of Leibniz? Let us see how Marx interprets it with another of his examples: $y = x^2$. Here $x_1 = x + \Delta x$ is changed to $x_1 = x + dx$, so that $y + dy = (x + dx)^2 = x^2 + 2xdx + dx^2$. The original function $(y = x^2)$ is removed from both sides, giving: $dy = 2xdx + dx^2$. The last term on the right is then suppressed, giving $dy = 2xdx$, and finally $dy/dx = 2x$. "The one question that could still be asked: why the forceful suppression of the term standing in the way? This presupposes that one knows it stands in the way and does not truly belong to the derivative" [Marx 1968, 166]. Here was the mystical aspect that Marx objected to. The suppression of this term was mere "sleight of hand [Eskamotage]."

We may now see how D'Alembert overcame this difficulty. Marx writes: "D'Alembert starts directly from the point of departure of Newton and Leibniz: $x_1 = x + dx$. But he immediately makes the fundamental correction: $x_1 = x + \Delta x$, x and an indeterminant but prima facie finite increment, that he calls h. The change of this h or Δx into dx (like all the French, he uses the Leibniz notation) takes place only as a last result of the development or at least just at the eleventh hour [knapp vor Toresschluss], while with the mystics and the initiators of the calculus it appears at the starting point" [Marx 1968, 168-170]. Marx illustrates this with the example: $f(x) = x^3$. He obtains $[f(x + h) - f(x)]/h = 3x^2 + 3xh + h^2$. Now, setting $h = 0$, the left side is changed into the expression dy/dx, "while through the setting of $h = 0$ the terms $3xh + h^2$ disappear, and that through a correct mathematical operation. That is, they are now removed without sleight of hand" [Marx 1968, 172].

This is the improvement over Leibniz' method. Nevertheless, the derivative here also appears already as the coefficient of the first power of h. "The derivative is therefore essentially the same as with Newton and Leibniz, but the ready-made derivative $3x^2$ is separated in a strictly algebraic way from its former context. It is not a development [Entwicklung], but rather a separation [Loswicklung], ..." [Marx 1968, 172]. The only development, as Marx points out, takes place on the left side of the equation where the differential coefficient dy/dx is obtained.

How, then, does Marx explain differentiation as a dialectical process, and precisely as a negation of the negation? First of all he treats the variable as truly *variable*. Thus, instead of adding h or Δx to x, he considers the increase of x to x_1. Using the function $f(x)$ or $u = x^3 + ax^2$ to point up the difference between the method of D'Alembert and his 'algebraic' method, he forms the difference $f(x_1) - f(x)$ or $u_1 - u = x_1^3 + ax_1^2 - (x^3 + ax^2)$. "Here it is in no way a question of again removing the original function, since $x_1^3 + ax_1^2$ contains x^3 and ax^2 in no form. On the contrary, the first difference equation yields us a moment of development, ..." [Marx 1968, 234]. This is the first negation, i.e. x is negated to become x_1. Dividing by $x_1 - x$, he obtains:

$$[f(x_1) - f(x)]/(x_1 - x) \text{ or } (u_1 - u)/(x_1 - x) = x_1^2 + x_1x + x^2) + a(x_1 + x).$$

"By this division we have obtained the preliminary derivative" [Marx 1968, 234]. Now, by setting $x_1 = x$, we obtain $3x^2 + 2ax$ on the right hand side and $0/0$ on the left. This is the second negation, or the negation of the negation. By it we are brought, not back to our starting point, but to a new function. The real development has taken place on the right side of the equation, producing the derived function $f'(x) = 3x^2 + 2ax$. This process is reflected on the left side of the equation in $0/0$, which is replaced by the symbol dy/dx, to show that this is the result of a definite process.

V

Engels, too, saw a dialectical process in differentiation, and in the beginning he was prepared to find it even in the infinitesimal method of Leibniz. Or rather, he found analogues of infinitesimals in the real world. ("Nature offers prototypes of all these imaginary magnitudes" [MEW 20, 530; Engels 1940, 314].) In *Anti-Dühring* he wrote: "The negation of the negation is even more strikingly obvious in the higher analyses, in those 'summations of indefinitely small magnitudes' which Herr Dühring himself declares are the highest operations of mathematics, and in ordinary language are known as the differential and integral calculus. How are these forms of calculus used? In a given problem, for example, I have two variable magnitudes x and y, neither of which can vary without the other also varying in a relation determined by the conditions of the case. I differentiate x and y, i.e. I take x and y as so infinitely small that in comparison with any real magnitude, however small, they disappear, so that nothing is left of x and y but their reciprocal relation without any, so to speak, material basis, a quantitative relation in which there is no quantity. Therefore, dy/dx, the relation between the differentials of x and y, is equal to $0/0$ as the expression of y/x" [MEW 20, 128; Engels 1939, 150-151].

Unlike Marx, Engels was prepared to accept mathematics as he found it. He continues: "I only mention in passing that this relation between two magnitudes which have disappeared, caught at the moment of their disappearance, is a contradiction; it cannot disturb us any more than it has disturbed the whole of mathematics for almost two hundred years. And yet what have I done but negate x and y, though not in such a say that I need not bother about them any more, not in the way that metaphysics negates, but in the way that corresponds with the facts of the case? In place of x and y, therefore, I have their negation, dx and dy in the formulae of equations before me. I continue then to operate with these formulae, treating dx and dy as magnitudes which are real, though subject to certain exceptional laws, and at a certain *point I negate the negation*, i.e., I integrate the differential formula, and in place of dx and dy again get the real magnitudes x and y, and am not then where I was at the beginning, but by using this method I have solved the problem on which ordinary geometry and algebra might perhaps have broken their teeth in vain" [MEW 20, 128; Engels 1939, 151].

Thus, while Engels was willing to accept the view of dy/dx as a ratio of infinitely small quantities, for Marx the differentiation was completed only when Δx and Δy became zero. Marx would probably have been amused by Berkeley's jibe at Newton's fluxions as "ghosts of departed quantities." He certainly would have appreciated the verses in Samuel Butler's mock romance *Hudibras*, first published in 1663, from which (according to Wolfgang Breidert [private communication]) Berkeley's expression was derived: "He could reduce all things to Acts/ And knew their Natures by Abstracts,/ Where Entity and Quiddity,/ The Ghosts of defunct Bodies, flie" [Butler 1967, 5].

But after reading Marx' exposition, Engels was immediately converted to his viewpoint, as we have seen from his letter of 18 August 1881. Engels continued in that letter to set forth the view of Marx: "When we say that in $y = f(x)$ the x and y are variable, then this is, as long as we do not move on, a contention without all further consequences, and x and y still are, pro tempore, constants in fact. Only when they really change, that is *inside the function*, do they become variables in fact. Only in that case is it possible for the relation--not of both quantities as such, but of their variability--which still is hidden in the original equation, to reveal itself. The first derivative $\Delta y/\Delta x$ shows this relation as it occurs in the course of the real change, that is in every *given* change; the final derivative dy/dx shows it in its generality, pure. Hence we can come from dy/dx to every $\Delta y/\Delta x$, while this itself ($\Delta y/\Delta x$) only covers the special case. However, to pass from the special case to the general relationship the special case has to be liquidated as such [als solcher aufgehoben werden]. Hence, after the function has passed through the process from x to x_1 with all the consequences, x_1 can be quietly allowed to become x again, it is no longer the old x, which was only variable in name, it has passed through *real change*, and the *result* of the change remains, even if we liquidate it again itself [auch wenn wir sie selbst wieder aufheben]" [MEW 35, 24].

VI

Engels' letter continues: "We see here at last clearly, what many mathematicians have claimed for a long time, without being able to present rational reasons for it, that the differential *quotient* is the original, the differentials are derived" [MEW 35, 24]. This agrees with what Marx wrote in his article "On the differential": "In $0/0$ the numerator is inseparable from the denominator, but why? Because only unseparated do both express a relation, in this case the ratio

$$(y_1 - y)/(x_1 - x) = [f(x_1) - f(x)]/(x_1 - x)$$

which has been reduced to its minimum, where the numerator has become 0, because the denominator has. Separated both are 0, lose thereby their symbolic meaning, their sense.

"But as soon as $x_1 - x = 0$ has gained in dx a form that it unchangeably displays as a vanished difference of the independent variable x, thus also dy as vanished difference of the function of x or the dependent y, the separation of the denominator from the numerator becomes an entirely allowable operation. Wherever dx now stands, such a change of position leaves the relation of dy to it untouched. Thus $dy = f'(x)dx$ appears to us as another form of $dy/dx = f'(x)$ and is always replaceable by the latter" [Marx 1968, 62].

That is, the differentials dx and dy have their meaning from the symbol dy/dx. But Marx must still take into account the fact that in practice differentials are used in the calculation of derivatives. This he does by seeing them as symbols of operations to be carried out. "We know from this now a priori that if $y = f(x)$ and $dy = df(x)$, that if the differential operation signified by $df(x)$ is carried out, the result: $dy = f'(x)dx$, and that out of this finally comes: $dy/dx = f'(x)$.

"But also, only from the moment in which the differential functions as starting point for the calculation is the reversal of the algebraic differentiation method completed, and hence the differential calculus appears as a separate, specific way of reckoning with variable quantities" [Marx 1968, 64].

This last quotation shows two aspects of Marx' view of the differential and the derivative that have been pointed out by D. J. Struik: "his insistence on the operational character of the differential and on his search for the exact moment where the calculus springs from the underlying algebra as a new doctrine" [Struik 1948, 196]. The originality of Marx' view of the differential as an operational symbol was pointed out shortly after the publication of [Marx 1933]. K. A. Rybnikov has noted: "Already on the basis of the then published material V. I. Glivenko showed that Marx was the first to work out the concept of the differential as an operational symbol; later Fréchet extended the concept to functional analysis" [Rybnikov 1955, 197]. (Both Struik and Rybnikov refer to [Glivenko 1934]; I have not seen this article.)

The second idea of Marx mentioned by Struik shows up in what Marx called "the reversal of the method [Umschlag der Methode]." Consider the example: $y = x^3$. In order to find its derivative we let x increase to x_1, so that y increases to y_1, and write: $y_1 - y = x_1^3 - x^3$. Then dividing by $x_1 - x$ we have: $(y_1 - y)/(x_1 - x) = x_1^2 + x_1x + x^2$. We now let x_1 return to its minimum value x, so that *on the right side* we have $3x^2$, which is algebraic in Marx' sense that no differential symbols appear there, i.e. a real process has taken place that results in the derivative of the original function. But *on the left side* we have $0/0$ or dy/dx, i.e. operational symbols. Thus Marx distinguishes the two sides of the equation $dy/dx = 3x^2$: the left is the symbolic and the right is the algebraic. Viewing a mathematically variable magnitude as a reflection of a varying natural magnitude, we may investigate it by the 'algebraic' differentiation process that takes place on the right side of the equation. But this process is reflected symbolically on the left side of the equation and may in turn be investigated by the development of a calculus of those symbols. Thus the initiative, so to speak, passes from the right side of the equation to the left--in a "reversal of the method."

This reversal is seen already in a rudimentary form in in Marx' simplest example: $y = x$. Here the preliminary derivative is $\Delta y/\Delta x = 1$ and since 1 is constant, no further development can take place on the right side of the equation. Marx comments: "From the outset, as soon as we obtain [$\Delta y/\Delta x = 1$] we are forced to operate further on the left side, because the right is occupied by the constant 1. And with this, the *reversal in the method*, that throws the initiative from the right side to the left, appears in its nature [von Haus aus] once and for all proven, in fact the first word of the algebraic method itself" [Marx 1968, 68].

This idea is seen more clearly in Marx' investigation of $y = uz$, where u and z are each functions of x. Letting x increase to x_1, so that u increases to u_1, z to z_1, and y to y_1, we obtain, after dividing by $x_1 - x$:

$$\Delta y/\Delta x = z_1(\Delta u/\Delta x) + u(\Delta z/\Delta x).$$

Now, following the algebraic method, we let x_1 decrease to x or Δx to zero, to obtain $dy/dx = z(du/dx) + u(dz/dx)$. Here the right side is no longer algebraic, it contains symbolic differential coefficients. No 'real' functions have been operated on. In the earlier example, dy/dx was the symbolic equivalent of a derived function $3x^2$ and here the dy/dx plays the same role, but what of du/dx and dz/dx? They do not stand opposite any derived function whose double [Doppelgänger] they would be. Marx writes: "They have one-sidedly come into the world, shadow figures without bodies to cast them, symbolic differential coefficients without real differential coefficients, i.e. without corresponding equivalent 'derivatives'. The symbolic differential coefficient has become an *independent starting point*, whose real equivalent has first to be found. The initiative has been moved from the right hand pole, the algebraic, to the left hand one, the symbolic. With this, however, the differential calculus appears also as a specific kind of computation, operating already independently on its own territory. Its starting points du/dx, dz/dx are mathematical quantities which belong exclusively to this calculus and characterize it. And this reversal of the method resulted here from the algebraic differentiation of uz. The algebraic method changes automatically into its opposite, the differential method" [Marx 1968, 54-56]. This is what Struik meant by Marx' "search for the exact moment where the calculus springs from the underlying algebra as a new doctrine."

VII

While Marx' analysis of the derivative and differential had no immediate effect on the historical development of mathematics, Engels' claim that Marx made "independent discoveries" is certainly justified. It is interesting to note that Marx' operational definition of the differential anticipated 20th century developments in mathematics, and there is another aspect of the differential, that seems to have been seen by Marx, that has become a standard part of modern textbooks--the concept of the differential as the principal part of an increment. Yanovskaya writes: "This concept, which plays an essential role in mathematical analysis and especially in its applications, was introduced by Euler ..." [Marx 1968, 579] and "we have every reason to consider that Marx had at his disposal also a concept equivalent to the concept of the *differential* as *principal part of the increment of a function* (as with Euler ...)" [Marx 1968, 297].

But Marx' interest in differential calculus was perhaps primarily philosophical; certainly it was no mere pastime that brought him "quietness of mind." Indeed, Lombardo Radice has concluded: "More generally, there is no doubt that Marx gave so much attention and so much effort of thought in the last years of his life to the foundations of differential calculus because he found in it a decisive argument against a metaphysical interpretation of the dialectical law of the negation of the negation" [Lombardo Radice 1972, 275]. As Marx himself wrote: "here as everywhere it is important to strip the veil of secrecy from science" [Marx 1968, 192].

As we approach the 100th anniversary of Marx' death it is still true what Yanovskaya wrote at the time of the 50th anniversary: "Modern mathematics also defines the derivative in fact by means of a certain dialectical process, consisting at first of the positing of a finite difference, and then its 'removal', but which it carries out not in the form of a return to the equating of x_1 to x or Δx to zero, but in the form of a 'passage to the limit of Δx to zero" [Yanovskaya 1933, 97]. Nor can the recent justification of infinitesimals with the introduction of non-standard analysis by Abraham Robinson (or even the reintroduction of infinitesimals into the classroom [Keisler 1976]) take away the value of Marx' critique. Yanovskaya's prediction that "the publication by the Marx-Engels-Lenin Institute of the mathematical works of Marx will have for our mathematician-Marxists no less significance than the *Dialectics of Nature* for all the natural science front generally" [Yanovskaya 1933, 110] may have been a bit sanguine, but surely they "will always remain in the field of vision of mathematicians" [Gokieli 1947, 111]. Marx did not give us just another example of his philosophical approach. Rather, "the difficult task of the foundation of differential calculus became for K. Marx the touchstone [probnym kamnem] of the application of the method of materialistic dialectics to mathematics" [Rybnikov 1954, 496].

NOTES

1. The double reference here and later refers first to the original and then to the translation that I have used here. MEW = *Marx Engels Werke.*

2. I have used several of Struik's translations in this article.

3. All translations from Russian are mine.

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