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Graph Minors. I. Excluding a Forest

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The path-width of a graph is the minimum value of k such that the graph can be obtained from a sequence of graphs $G_1,..., G_r$ each of which has at most $k+1$ vertices, by identifying some vertices of G_i pairwise with some of G_{i+1} $(1 \leqslant i < r)$. For every forest H it is proved that there is a number k such that every graph with no minor isomorphic to H has path-width $\leqslant k$. This, together with results of other papers, yields a "good" algorithm to test for the presence of any fixed forest as a minor, and implies that if P is any property of graphs such that some forest does not have property P, then the set of minor-minimal graphs without property P is finite.

1. INTRODUCTION

Let G be a graph. (All graphs in this paper are finite, and may have loops or multiple edges unless we state otherwise.) A sequence $X_1,..., X_r$ of subsets of $V(G)$ (the vertex set of G) is a *path-decomposition* of G if the following conditions are satisfied.

(W1) *For every edge e of G, some X_i $(1 \leqslant i \leqslant r)$ contains both ends of e.*

(W2) *For $1 \leqslant i \leqslant i' \leqslant i'' \leqslant r$, $X_i \cap X_{i''} \subseteq X_{i'}$.*

The *path-width* of G is the minimum value of $k \geqslant 0$ such that G has a path-decomposition $X_1,..., X_r$ with $|X_i| \leqslant k+1$ $(1 \leqslant i \leqslant r)$.

H is a *minor* of G if H can be obtained from G by deleting some vertices and/or edges, and/or contracting some edges. The main theorem of this paper is the following:

(1.1). *For every forest H there is an integer w such that every graph with no minor isomorphic to H has path-width $\leqslant w$.*

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There are three attractive features of the theorem which serve as its motivation. First, the theorem is, in a sense, sharp, for it can be reformulated as follows:

(1.2). *Let $\mathscr{F}$ be a set of graphs. Then the following are equivalent:*

(i) *for some integer w, every $G \in \mathscr{F}$ has path-width $\leqslant w$;*

(ii) *there is a forest H such that no $G \in \mathscr{F}$ has a minor isomorphic to H.*

It is easy to prove the equivalence of (1.2) with (1.1) by making use of two observations: (a) if H is a minor of G, then its path-width is no greater than the path-width of G, and (b) there are trees with arbitrarily large path-width (e.g., the tree $\mathbf{Y}_\lambda$ defined later has path-width $\lceil \frac{1}{2}(\lambda + 1) \rceil$, for any integer $\lambda \geqslant 1$, where $\lceil x \rceil$ is the least integer not less than x).

The second attractive feature of (1.1) is that it dovetails nicely with the main theorem of [3]; the two results together yield the following:

(1.3). *Let $G_1, G_2, \ldots,$ be a countable sequence of graphs, such that G_1 is a forest. Then there exist $j > i \geqslant 1$ such that G_i is isomorphic to a minor of G_j.*

(This implies the final result mentioned in the abstract, as may easily be seen.) Theorem (1.3) is relevant to an interesting conjecture of Wagner (unpublished), that (1.3) is true even without the hypothesis that G_1 is a forest. See [3] for a more complete discussion.

And third (see [1]), (1.1) yields a "good" algorithm to test if an arbitrary graph G has a minor isomorphic to a fixed forest H. ("Good" here is in its technical sense of "polynomially bounded." It is not a practical algorithm; the exponent of the polynomial, although constant, is enormous.)

For the purposes of this paper we need the following three facts about path-width (the proofs are easy).

(1.4). *If every connected component of G has path-width $\leqslant k$, then G has path-width $\leqslant k$.*

(1.5). *If $X \subseteq V(G)$ and $G \backslash X$ has path-width $\leqslant k$, then G has path-width $\leqslant k + |X|$.*

($G \backslash X$ denotes the result of deleting X.)

(1.6). *If $X_1, \ldots, X_r \subseteq V(G)$ is a path-decomposition of G, and for $1 \leqslant i \leqslant r-1$, $|X_i \cap X_{i+1}| \leqslant k$, and for $1 \leqslant i \leqslant r$,*

$$G \backslash \bigcup (X_j \colon 1 \leqslant j \leqslant r, j \neq i)$$

has path-width $\leqslant k'$, then G has path-width $\leqslant k' + 2k$.

Our proof of (1.1) is basically by induction on the "complexity" of the forest H. This is explained in detail in Section 3, and the induction argument is performed in Sections 4 and 5. Section 2 contains some crucial lemmas about "grids."

Our terminology is mostly standard, but a few terms need explanation. If X, Y are sets, $X - Y$ denotes

$$\{x : x \in X, x \notin Y\}.$$

If G is a graph, $V(G)$ and $E(G)$ denote its vertex-set and edge-set, respectively. A graph is *simple* if it has no loops or multiple edges. If $X \subseteq V(G)$, $G \backslash X$ denotes the result of deleting from G all the vertices in X, and $G \mid X$ denotes $G \backslash (V(G) - X)$. A graph G is a *subdivision* of a graph H if H can be obtained from G by (repeatedly) choosing a vertex of valency 2 and contracting an edge incident with it which is not a loop. Two subgraphs of a graph are *disjoint* if they have no common vertices, and a set of subgraphs is disjoint if every pair of its members are disjoint. Every *path* has at least one vertex, and no "repeated" vertices. It has an *initial* and a *terminal* vertex (which are equal for a one-vertex path) called its *ends*, and any other vertices are called *internal* vertices. A path *avoids* $X \subseteq V(G)$ if it has no vertex in X. $X \subseteq V(G)$ *separates* Y, $Z \subseteq V(G)$ if no path in G from Y to Z avoids X. (Thus $Y \cap Z \subseteq X$.) We say that X separates y, $z \in V(G)$ if it separates $\{y\}$, $\{z\}$.

We shall require the following lemma:

(1.7). *Let ρ be a vertex of a graph G, and let $Y_1, Y_2, \ldots, Y_T$ be disjoint subsets of $V(G)$, where $T \geqslant 1$. Suppose that for $1 < i, i' \leqslant T$, and for all $y \in Y_{i'}$, Y_i separates y and ρ if and only if $i \leqslant i'$. For $1 \leqslant i < T$, let W_i be the set of all vertices of G separated from ρ by Y_i, and not separated from ρ by Y_{i+1}. Let W_0 be the set of vertices not separated from ρ by Y_1, and let W_T be the set of all vertices separated from ρ by Y_T. Then the sets $W_0, W_1, \ldots, W_T$ are disjoint and partition $V(G)$, and $Y_i \subseteq W_i$ $(1 \leqslant i \leqslant T)$, and the sequence*

$$W_0 \cup Y_1, W_1 \cup Y_2, \ldots, W_{T-1} \cup Y_T, W_T$$

is a path-decomposition of G.

Proof. For every vertex $v \in V(G)$, there is a unique value of i with $0 \leqslant i \leqslant T$ such that

(i) either $i = 0$ or Y_i separates v and ρ, and

(ii) either $i = T$ or Y_{i+1} does not separate v and ρ.

For it is clear that there is at least one such value of i; and if i, i' are two values, with $i < i'$ say, then $i < T$ and $i' > 0$, and we have that Y_{i+1} does not separate v and ρ, and $Y_{i'}$ separates v and ρ. Let P be a path in G from v to ρ avoiding Y_{i+1}. Then P meets $Y_{i'}$, and so Y_{i+1} does not separate $Y_{i'}$ and ρ, contrary to our hypothesis.

It follows that for every vertex v there is a unique value of i with $v \in W_i$, and so $W_0, W_1, \ldots, W_T$ partition $V(G)$. Clearly, $Y_i \subseteq W_i$ $(1 \leqslant i \leqslant T)$. It remains to prove that the sequence

$$W_0 \cup Y_1, W_1 \cup Y_2, \ldots, W_{T-1} \cup Y_T, W_T$$

is a path-decomposition of G. We need only show (W1), since (W2) is obvious.

Let e be an edge of G, with ends u, v say. Let $u \in W_i$, $v \in W_j$, where $i \leqslant j$ say. If $i = j$, then ρ has both ends in some set in the sequence, as required; we assume therefore that $i < j$. Thus Y_{i+1} separates v and ρ. It does not separate u and ρ, since $u \in W_i$, and yet u, v are adjacent. It follows that $v \in Y_{i+1}$, and so e has both ends in $W_i \cup Y_{i+1}$, as required.

2. Grids

If $\theta \geqslant 2$, the *θ-grid* is the simple graph with vertices v_{ij} $(1 \leqslant i, j \leqslant \theta)$ in which v_{ij} and $v_{i'j'}$ are adjacent if $|i - i'| + |j - j'| = 1$. $\mathscr{F}_\theta$ denotes the set of all graphs with no minor isomorphic to the θ-grid. For the remainder of the paper, $\theta \geqslant 6$ will be fixed and, for convenience, even. In this section we prove some connectivity results about the graphs in $\mathscr{F}_\theta$.

(2.1). *If $P_1, \ldots, P_m$ are disjoint paths of a graph $G \in \mathscr{F}_\theta$, and $B_1, \ldots, B_n$ are also disjoint paths, and each P_i has exactly one vertex in common with each B_j, and each P_i meets $B_1, \ldots, B_n$ in that order, then either $m < \theta$ or $n < \frac{1}{2}\theta \cdot \theta!$.*

Proof. We assume that $m \geqslant \theta$. Since there are only $\frac{1}{2}(\theta!)$ different orderings of $P_1, \ldots, P_\theta$ up to reversal, it follows that there exists $J \subseteq \{1, \ldots, n\}$ with $|J| \geqslant n/\frac{1}{2}(\theta!)$ such that for $j, j' \in J$, B_j and $B_{j'}$ meet $P_1, \ldots, P_\theta$ in the same order or the reverse. If $n \geqslant \frac{1}{2}\theta \cdot \theta!$, then $|J| \geqslant \theta$ and so G has a minor isomorphic to the θ-grid, which is impossible. Thus $n < \frac{1}{2}\theta \cdot \theta!$.

(2.2). *If $P_1, \ldots, P_m$ are disjoint paths of $G \in \mathscr{F}_\theta$, and $B_1, \ldots, B_n$ are disjoint stars, and each P_i has exactly one vertex in common with each B_j, and no P_i contains the centre of any star B_j, and each P_i meets $B_1, \ldots, B_n$ in that order, then either $m < \theta$ or $n < \theta(\theta - 1)$.*

(A *star* is a tree in which one vertex, called the *centre*, is adjacent to all others.)

Proof. We may assume, by deletion, that every vertex and edge of G is either in some P_i or in some B_j; and by contraction, that every vertex is in some B_j. For $1 \leqslant j \leqslant n$ let $v(j)$ be the centre of the star B_j, and let the common vertex of P_i and B_j be $v(i, j)$ $(1 \leqslant i \leqslant m)$. We assume, for a contradiction, that $m \geqslant \theta$ and $n \geqslant \theta(\theta-1)$. For $1 \leqslant k \leqslant \theta$, let Q_k be the subgraph of G induced by

$$\bigcup_{1 \leqslant l \leqslant \theta-1} \{v(l, (k-1)(\theta-1)+l),$$

$$v((k-1)(\theta-1)+l), v(l+1, (k-1)(\theta-1)+l)\}.$$

We see that for $1 \leqslant k \leqslant \theta$, Q_k is a path, meeting $P_1, \ldots, P_\theta$ in that order, and each P_i meets $Q_1, \ldots, Q_\theta$ in that order. But then G has a minor isomorphic to the θ-grid, a contradiction.

(2.3). *If $P_1, \ldots, P_m$ are disjoint paths of $G \in \mathscr{F}_\theta$, and $B_1, \ldots, B_n$ are disjoint connected subgraphs of G, and each P_i meets each B_j, and for $1 \leqslant j < j' \leqslant n$ all vertices of B_j on P_i occur before all vertices of $B_{j'}$, then either $m < \frac{1}{2}\theta^2$ or $n < \theta^{2\theta-2}$.*

Proof. If possible, choose a counterexample with $|E(G)| + m$ minimum. Then evidently we have $m = \frac{1}{2}\theta^2$.

If some P_i has more than one vertex in common with some B_j, we may contract the edges of P_i between the two vertices of B_j and produce a smaller counterexample, a contradiction. Thus each P_i meets each B_j in exactly one vertex, and in particular no edge is both in some P_i and in some B_j. Each B_j is a tree, for if some B_j is not, we may delete an edge from it and produce a smaller counterexample. Every vertex v of each B_j is on some P_i; for if not, we may contract some edge of B_j incident with v and produce a smaller counterexample. It follows that each B_j has exactly $\frac{1}{2}\theta^2$ vertices.

It is elementary that for $1 \leqslant j \leqslant n$, B_j has a subgraph Q_j which is either a path with θ vertices or a tree with θ end-vertices.

For $1 \leqslant j \leqslant n$, let $I_j \subseteq \{1, \ldots, \frac{1}{2}\theta^2\}$ be $\{i: P_i$ meets Q_j in some vertex which, if Q_j is not a path, is an end-vertex of $Q_j\}$. Then $|I_j| = \theta$ for all j, and so there exists $J \subseteq \{1, \ldots, n\}$ with

$$|J| \geqslant n \bigg/ \binom{\frac{1}{2}\theta^2}{\theta}$$

such that $I_j = I_{j'}$ for $j, j' \in J$.

Let $J_1 \subseteq J$ be $\{j \in J: Q_j$ is a path$\}$ and let $J_2 = J - J_1$. By (2.1), $|J_1| < \frac{1}{2}\theta \cdot \theta!$, and by (2.2) $|J_2| < \theta(\theta-1)$. Thus

$$|J| < \tfrac{1}{2}\theta \cdot \theta! + \theta(\theta-1)$$

and so

$$n < \binom{\frac{1}{2}\theta^2}{\theta}\left(\tfrac{1}{2}\theta\cdot\theta! + \theta(\theta-1)\right)$$
$$\leqslant \theta^{2\theta-2}$$

since $\theta \geqslant 6$, a contradiction, as required.

(2.4). *If $P_1,\ldots,P_m$ are disjoint paths of $G \in \mathscr{F}_\theta$, and $B_1,\ldots,B_n$ are disjoint connected subgraphs of G, and each B_j meets at least $\frac{1}{2}\theta^2$ of $P_1,\ldots,P_m$, and for $1 \leqslant i \leqslant m$ and for $1 \leqslant j < j' \leqslant n$, all vertices of B_j on P_i occur before all vertices of $B_{j'}$, then $n < \theta^{2\theta-2}\binom{m}{\frac{1}{2}\theta^2}$.*

Proof. For $1 \leqslant j \leqslant n$, choose $I_j \subseteq \{1,\ldots,m\}$ with $|I_j| = \frac{1}{2}\theta^2$ such that P_i meets B_j for each $i \in I_j$. Now there exists $J \subseteq \{1,\ldots,n\}$ with $|J| \geqslant n\binom{m}{\frac{1}{2}\theta^2}^{-1}$, such that for $j,\ j' \in J$, $I_j = I_{j'} = I$ say. Then $|I| = \frac{1}{2}\theta^2$. Now each path P_i ($i \in I$) meets each B_j ($j \in J$) in order, and so by (2.3), $|J| < \theta^{2\theta-2}$. The result follows.

(2.5). *If $A_1, A_2 \subseteq V(G)$, and there is a unique set $\{P_1,\ldots,P_m\}$ of m disjoint paths in G from A_1 to A_2, and every vertex of G is in one of these paths, then there is an integer-valued function μ defined on $V(G)$, such that*

(i) *if v' is after v on P_i, then $\mu(v') > \mu(v)$ $(1 \leqslant i \leqslant m)$;*

(ii) *for any integer t, Z_t separates $\{v : \mu(v) < t\}$ from $\{v : \mu(v) \geqslant t\}$, where $Z_t = \{v$: for some i, v is the first vertex on P_i with $\mu(v) \geqslant t\}$.*

Proof. Step 1. *There is no sequence $u_1, v_1, u_2, v_2,\ldots, u_r, v_r$ of distinct vertices of G, where $r \geqslant 1$, and for $1 \leqslant j \leqslant r$ v_{j-1}, u_j are adjacent, joined by an edge not in any of $P_1,\ldots,P_m$ (setting $v_0 = v_r$), and $u_1, u_2,\ldots, u_r$ are all on different P_i's, and for $1 \leqslant j \leqslant r$, v_j occurs after u_j on one of $P_1,\ldots,P_m$.*

For if there is such a sequence, let us renumber $P_1,\ldots,P_m$ for simplicity so that u_i lies on P_i $(1 \leqslant i \leqslant r)$. For $1 \leqslant i \leqslant r$, let e_i be an edge of G joining v_{i-1} and u_i. Let Q_i, R_i, S_i be the subpaths of P_i from A to u_i, from u_i to v_i, and from v_i to B, respectively. Then R_i has at least one edge, since $u_i \neq v_i$. Setting $S_0 = S_r$, we have Q_i and S_{i-1} are disjoint $(1 \leqslant i \leqslant r)$ and so Q_i, e_i, and S_{i-1} form a path P'_i say from A_1 to A_2. Then $P'_1,\ldots,P'_r, P_{r+1},\ldots,P_m$ are disjoint paths of G from A_1 to A_2; and this set is different from $P_1,\ldots,P_m$, since none of $P_1,\ldots,P_m$ uses the edge e_1. This contradicts our hypothesis that $P_1,\ldots,P_m$ are unique.

Step 2. *There is no sequence $u_1, v_1,\ldots, u_r, v_r$ of vertices of G, where $r \geqslant 1$, such that for $1 \leqslant i \leqslant r$, v_{i-1}, u_i are adjacent, joined by an edge not in any of $P_1,\ldots,P_m$ (setting $v_0 = v_r$), and for $1 \leqslant i \leqslant r$, v_i occurs after u_i on one of $P_1,\ldots,P_m$.*

If possible, choose such a sequence with r minimum. If $u_1, u_2,..., u_r$ are all on distinct paths from $P_1,..., P_m$, we contradict the result of step 1. Thus we may assume that for some $j \neq j'$ with $1 \leqslant j, j' \leqslant r$, u_j and $u_{j'}$ are both on P_1 say, and by symmetry we may assume that either $u_j = u_{j'}$ or u_j occurs before $u_{j'}$ on P_1. Consider the sequence

$$u_{j'+1}, v_{j'+1},..., u_{j-1}, v_{j-1}, u_j, v_{j'}$$

reading the subscripts modulo r if $j < j'$. The sequence has $2(j-j')$ terms if $j > j'$, and $2r-2(j'-j)$ terms if $j < j'$. In either case it has fewer than $2r$ terms, and hence contradicts the minimality of r.

Step 3. *For $u, v \in V(G)$, we say $u < v$ if there is a sequence $u_0, v_0,..., u_r, v_r$ with $r \geqslant 0$, such that $u = u_0$, $v_r = v$, and for $1 \leqslant i \leqslant r$, v_{i-1}, u_i are adjacent, joined by an edge not in any of $P_1,..., P_m$, and for $1 \leqslant i \leqslant r-1$, v_i occurs after u_i on one of $P_1,..., P_m$, and v_0 occurs after u_0 on one of $P_1,..., P_m$, and either $v_r = u_r$ or v_r occurs after u_r on one of $P_1,..., P_m$. Then $(V(G), <)$ is a strict partial ordering.*

We must check that if $u < v < w$, then $u < w$; and that $v \nless v$. First, if $u < v < w$, let

$$u = u_0, v_0,..., u_r, v_r = v,$$

$$v = u'_0, v'_0,..., u'_s, v'_s = w,$$

be the corresponding sequences. Then the sequence

$$u = u_0, v_0,..., u_r, v'_0, u'_1,..., u'_s, v'_s = w$$

demonstrates that $u < w$. Second, if $v < v$, then there is a sequence

$$v = u_0, v_0,..., u_r, v_r = v;$$

we have $r \geqslant 1$, since $u_0 \neq v_0$, and so the sequence $u_1, v_1, u_2, v_2,..., u_{r-1}, v_{r-1}, u_r, v_0$ contradicts the result of step 2. Hence $<$ is a strict partial ordering.

Step 4. *For each $v \in V(G)$, let $\mu(v)$ be an integer, chosen so that for v, $v' \in V(G)$, if $v < v'$, then $\mu(v) < \mu(v')$. (This is possible since $<$ is a strict partial ordering of $V(G)$.) Then μ satisfies our requirements.*

Certainly if v' occurs on some P_i after v, then $v < v'$ and so $\mu(v) < \mu(v')$; hence condition (i) of (2.5) is satisfied. For any integer t, let $Z_t = \{v \in V(G)$: for some i $(1 \leqslant i \leqslant m)$, v is the first vertex of P_i with $\mu(v) \geqslant t\}$. Let $X_t = \{v \in V(G): \mu(v) < t\}$, and $Y_t = V(G) - (X_t \cup Z_t)$. We must show that Z_t separates X_t and Y_t.

But $X_t \cup Y_t \cup Z_t = V(G)$, and so it is sufficient to prove no edge has one end in X_t and the other in Y_t. Suppose then that $u \in X_t$ and $v \in Y_t$ are adjacent. Let v be a vertex of P_1 say. Since $\mu(v) \geqslant t$, there is a first vertex u' say of P_1 with $\mu(u') \geqslant t$. We have $u' \in Z_t$ and so $v \neq u'$, and u' occurs on P_1 before v. But u and v are adjacent, and are joined by an edge not in any of $P_1, \ldots, P_m$; and so the sequence

$$u', v, u, u$$

demostrates that $u' < u$. Yet $\mu(u') \geqslant t > \mu(u)$, a contradiction. This completes the proof of (2.5).

(2.6). *Suppose that $G \in \mathscr{F}_\theta$, and $A_1, A_2 \subseteq V(G)$, and $m > 0$ is an integer, and the following conditions are satisfied*:

(i) *there is a unique set $\{P_1, \ldots, P_m\}$ of m disjoint paths in G from A_1 to A_2*;

(ii) *every vertex of G is in one of $P_1, \ldots, P_m$*;

(iii) *$B_1, \ldots, B_n$ are disjoint connected subgraphs of G, and each B_j meets at least $\frac{1}{2}\theta^2$ of $P_1, \ldots, P_m$.*

Then $n < 2m\theta^{2\theta-2}\binom{m}{\frac{1}{2}\theta^2}$.

Proof. Define $s = \lceil n/2m \rceil$. Choose an integer-valued function μ on $V(G)$ as in (2.5). Define a sequence of integers $k_1, \ldots, k_s$ as follows:

$$k_1 = \min(k\text{: for some } j\ (1 \leqslant j \leqslant n), \mu(v) < k \text{ for all } v \in V(B_j)),$$

$$k_l = \min(k\text{: for some } j\ (1 \leqslant j \leqslant n), k_{l-1} \leqslant \mu(v) < k \text{ for all } v \in V(B_j)) \qquad (2 \leqslant l \leqslant s).$$

We must show that this is well defined. Let us suppose inductively that $k_{l'}$ is well defined for $1 \leqslant l' < l$, where $1 \leqslant l \leqslant s$, and we shall show that k_l is well defined. If $l = 1$ this is clear, and we assume $l > 1$. For each integer t, let $Z_t = \{v \in V(G)$: for some i, v is the first vertex of P_i with $\mu(v) \geqslant t\}$. For $1 \leqslant l' < l$, let

$$J_{l'} = \{j\text{: } 1 \leqslant j \leqslant n, \text{ and for some } v \in V(B_j), \mu(v) < k_{l'}\}.$$

Then clearly for $l' \geqslant 2$, we have $J_{l'-1} \subset J_{l'}$; we claim that

$$|J_{l'} - J_{l'-1}| \leqslant 2m.$$

To prove that it is sufficient to show that for every $j \in J_{l'} - J_{l'-1}$, B_j meets $(Z_{k_{l'-1}} \cup Z_{k_{l'}})$, because $|Z_t| \leqslant m$ for every integer t, and $B_1, \ldots, B_n$ are disjoint. Suppose then that $j \in J_{l'} - J_{l'-1}$. There exists $u \in V(B_j)$ such that $\mu(u) < k_{l'}$. If there exists $u' \in V(B_j)$ such that $\mu(u') \geqslant k_{l'}$, then B_j uses a vertex from $Z_{k_{l'}}$, because $Z_{k_{l'}}$ separates $\{v: \mu(v) < k_{l'}\}$ and $\{v: \mu(v) \geqslant k_{l'}\}$. We may assume therefore that $k_{l'-1} \leqslant \mu(v) < k_{l'}$ for every vertex v of B_j. By the minimality of $k_{l'}$, there is a vertex v of B_j with $\mu(v) \geqslant k_{l'} - 1$; and so $\mu(v) = k_{l'} - 1$. But then $v \in Z_{k_{l'-1}}$, and again our claim is true. Thus for $2 \leqslant l' < l$ we have $|J_{l'} - J_{l'-1}| \leqslant 2m$. A similar argument shows that $|J_1| \leqslant 2m$. It follows that

$$|J_{l-1}| \leqslant 2(l-1)m$$

and hence that $J_{l-1} \neq \{1,..., n\}$, since $l \leqslant s = \lceil n/2m \rceil$. Thus k_l is well defined. Hence by induction, $k_1,..., k_s$ are well defined.

Choose j_1 with $1 \leqslant j_1 \leqslant n$ so that $\mu(v) < k_1$ for every vertex v of B_{j_1}; and for $2 \leqslant l \leqslant s$, choose j_l with $1 \leqslant j_l \leqslant n$ so that $k_{l-1} \leqslant \mu(v) < k_l$ for every vertex v of B_{j_l}. Put $C_l = B_{j_l}$ $(1 \leqslant l \leqslant s)$. Then $C_1,..., C_s$ are disjoint connected subgraphs of G, and each meets at least $\frac{1}{2}\theta^2$ of $P_1,..., P_m$: and moreover, for $1 \leqslant l < l' \leqslant s$, all vertices of C_l on P_i occur before all vertices of $C_{l'}$. By (2.4), $s < \theta^{2\theta-2}\binom{m}{\frac{1}{2}\theta^2}$, and so $n < 2m\theta^{2\theta-2}\binom{m}{\frac{1}{2}\theta^2}$, as required.

(2.7). *Suppose that there are m disjoint paths in $G \in \mathscr{F}_\theta$ from A_1 to A_2, where $A_1, A_2 \subseteq V(G)$; and that $B_1,..., B_n$ are disjoint connected subgraphs of G, and for $1 \leqslant j \leqslant n$, $V(B_j)$ separates A_1 and A_2. Then either $m < \frac{1}{2}\theta^2$ or $n < \theta^{2\theta}$.*

Proof. If possible, choose a counterexample with $|V(G)| + |E(G)| + m$ minimum. Then evidently we have $m = \frac{1}{2}\theta^2$.

Let $\{P_1,..., P_m\}$ be a set of m disjoint paths in G from A_1 to A_2. If some P_i and some B_j have an edge in common, we can contract that edge and produce a smaller counterexample. Thus no edge belongs to both some P_i and some B_j. Moreover, every edge belongs either to some P_i or to some B_j, for otherwise we could delete it.

It follows that

$$E(P_1) \cup \cdots \cup E(P_m) = E(G) - (E(B_1) \cup \cdots \cup E(B_n)).$$

If $\{P'_1,..., P'_m\}$ is another set of m disjoint paths from A_1 to A_2, we have, by the same argument,

$$E(P'_1) \cup \cdots \cup E(P'_m) = E(G) - (E(B_1) \cup \cdots \cup E(B_n))$$

and hence

$$E(P'_1) \cup \cdots \cup E(P'_m) = E(P_1) \cup \cdots \cup E(P_m).$$

It follows that $\{P'_1,..., P'_m\} = \{P_1,..., P_m\}$, and so the set $\{P_1,..., P_m\}$ is unique.

We claim that every vertex v of G is in some P_i. For if not, and v is an isolated vertex, we can delete v, and if v is not an isolated vertex, we can contract some edge incident with v_0 in either case we produce a smaller counterexample, which is impossible.

For $1 \leqslant j \leqslant n$, each P_i meets B_j, by our hypothesis. By (2.6),

$$n < 2m\theta^{2\theta-2}\binom{m}{\frac{1}{2}\theta^2} = \theta^{2\theta},$$

a contradiction, as required.

(2.8). *Let $A_1, A_2, X \subseteq V(G)$, where $G \in \mathscr{F}_\theta$, and let $B_1, ..., B_n$ be disjoint subgraphs of G, each with at most d components. Suppose that the following conditions are satisfied*:

(i) $|X| \leqslant rn$, *for some integer* $r \geqslant 0$;

(ii) *for* $1 \leqslant j \leqslant n$, $X \cup V(B_j)$ *separates* A_1 *and* A_2;

(iii) *there are m disjoint paths in G from A_1 to A_2, each avoiding X.*

Let $s = d + r(\theta^2 - 1)$. Then either $m < \frac{1}{2}s\theta^2$ or $n < 2^{s\theta^2/2}(\theta^2 + 2s\theta^{2\theta})$.

Proof. If possible, choose a counterexample with $|V(G)| + |E(G)| + m$ minimum. Then evidently we have $m = \frac{1}{2}s\theta^2$.

Let $P_1, ..., P_m$ be disjoint paths of G from A_1 to A_2, avoiding X. If some edge of G in both in some P_i and in some B_j, we may contract it, and if some edge is neither in some P_i nor in some B_j, we may delete it, in either case producing a smaller counterexample. It follows that

$$E(P_1) \cup \cdots \cup E(P_m) = E(G) - (E(B_1) \cup \cdots \cup E(B_n)),$$

and so $\{P_1, ..., P_m\}$ is the unique set of m disjoint paths of G from A_1 to A_2 avoiding X.

Every vertex v of $V(G) - X$ is in one of $P_1, ..., P_m$; for if not, and v is isolated, we can delete it, and if v is not isolated, we can contract some edge incident with it, in either case producing a smaller counterexample.

Let $J_1 \subseteq \{1, ..., n\}$ be defined by $j \in J_1$ of and only if some component of $B_j \backslash X$ meets at least $\frac{1}{2}\theta^2$ of the paths $P_1, ..., P_m$. By setting $A_1' = A_1 - X$, $A_2' = A_2 - X$, and applying (2.6) to $G \backslash X$, we deduce that $|J_1| < 2m\,\theta^{2\theta-2}2^m$.

Now let $J_2 \subseteq \{1, ..., n\}$ be defined by $j \in J_2$ if and only if $|X \cap V(B_j)| > 2r$. Since $B_1, ..., B_n$ are disjoint, we have

$$rn \geqslant |X| \geqslant \sum_{1 \leqslant j \leqslant n} |X \cap V(B_j)| \geqslant 2r\,|J_2|.$$

If $r = 0$, then $X = \varnothing$ and so $J_2 = \varnothing$ by definition. If $r > 0$, then $n \geqslant 2\,|J_2|$. Thus in either case $|J_2| \leqslant \frac{1}{2}n$, and so there exists $J_3 \subseteq \{1, ..., n\}$ with

$$|J_3| \geqslant \tfrac{1}{2}n - 2m\,\theta^{2\theta-2}2^m$$

such that for every $j \in J_3$, $|X \cap V(B_j)| \leqslant 2r$ and every component of $B_j \backslash X$ meets fewer than $\frac{1}{2}\theta^2$ different P_i's.

Let $j \in J_3$. Let $X \cap V(B_j) = Y_j$, and let $N(Y_j)$ be the set of those vertices which are adjacent in G to at least one element of Y_j. A component of $B_j \backslash X$ which includes no element of $N(Y_j)$ is a component of B_j, and hence $B_j \backslash X$ has at most d such components. Each of these components meets fewer than $\frac{1}{2}\theta^2$ paths P_i. Hence at least $m - \frac{1}{2} d\theta^2 \geqslant \frac{1}{2}r\theta^2(\theta^2 - 1)$ paths P_i meet at least one component of $B_j \backslash X$ which includes an element of $N(Y_j)$. Since $|Y_j| \leqslant 2r$, there exists $v_j \in Y_j$ such that at least $\frac{1}{4}\theta^2(\theta^2 - 1)$ paths P_i meet a component of $B_j \backslash X$ which includes a vertex adjacent to v_j. Let $\{D_j^1, ..., D_j^{\alpha(j)}\}$ be a minimal set of components of $B_j \backslash X$ such that each D_j^k includes a vertex adjacent to v_j and at least $\frac{1}{4}\theta^2(\theta^2 - 1)$ paths P_i meet $D_j^1 \cup \cdots \cup D_j^{\alpha(j)}$. Then $\alpha(j) \geqslant \frac{1}{2}\theta^2$ since each component of $B_j \backslash X$ meets fewer than $\frac{1}{2}\theta^2$ paths P_i. By the minimality of $\{D_j^1, ..., D_j^{\alpha(j)}\}$ there exist $P_{i(1)}, ..., P_{i(\alpha(j))}$ such that $P_{i(u)}$ meets D_j^u but meets no D_j^t $(t \neq u)$, for $u = 1, ..., \alpha(j)$. Then $i(1), ..., i(\alpha(j))$ are distinct: define $I(j) = \{i(1), ..., i(\frac{1}{2}\theta^2)\}$.

Then $I(j) \subseteq \{1, ..., m\}$, for all $j \in J_3$, and so there exists $J_4 \subseteq J_3$ with $|J_4| \geqslant 2^{-m} |J_3|$, such that for j, $j' \in J_4$, $I(j) = I(j') = I$ say. Renumber, for simplicity, so that $I = \{1, 2, ..., \frac{1}{2}\theta^2\}$, and for $j \in J_4$, P_i meets D_j^i but does not meet any $D_j^{i'}$ for distinct i, $i' \in I$. For each $i \in I$, let Q_i be the subgraph of G consisting of the vertices and edges in the graphs P_i and $D_j^i (j \in J_4)$. Then $Q_1, ..., Q_{\theta^2/2}$ are all connected, and are disjoint, and do not intersect X. Moreover, if $1 \leqslant i \leqslant \frac{1}{2}\theta^2$ and $j \in J_4$, v_j is adjacent to a vertex of Q_i. It follows that G has a minor isomorphic to the complete bipartite graph

$$K_{|J_4|, \theta^2/2}.$$

But $K_{\theta^2/2, \theta^2/2}$ has a minor isomorphic to the θ-grid, and $G \in \mathscr{F}_\theta$; thus $|J_4| < \frac{1}{2}\theta^2$. Hence $|J_3| < 2^m \frac{1}{2}\theta^2$, and so

$$\tfrac{1}{2}n - 2m\theta^{2\theta - 2}2^m < 2^m \tfrac{1}{2}\theta^2,$$

that is,

$$n < 2^m(\theta^2 + 4m\theta^{2\theta - 2}).$$

The result follows.

(2.9). *If $B_1, ..., B_n$ are disjoint connected subgraphs of a graph $G \in \mathscr{F}_\theta$, and $V' \subseteq V(G)$ and $r \geqslant 0$ is an integer, then one of the following is true*:

(i) *there exists $J \subseteq \{1, ..., n\}$ with $|J| = r$ such that for each $j \in J$ there is a path P_j in G from B_j to V', and the paths P_j $(j \in J)$ are disjoint, and each has no internal vertex in $V' \cup \bigcup_{j \in J} V(B_j)$*;

(ii) *there exist* $X \subseteq V(G)$ *and* $J \subseteq \{1,..., n\}$ *with* $|X| + |J| \leqslant 2^{r\theta^4}$, *such that* X *separates* V' *from every* B_j ($j \in \{1,..., n\} - J$).

Proof. We assume (i) is false. Define $\nu = 2^{r\theta^4/2}(\theta^2 + 2r\theta^{2\theta+2})$. Let $X_0 = J_0 = \varnothing$. We define inductively a sequence $X_0, X_1,..., X_\nu$ of subsets of $V(G)$ and a sequence $J_0, J_1,..., J_\nu$ of subsets of $\{1,..., n\}$, as follows: Suppose that $X_0,..., X_{i-1}, J_0,..., J_{i-1}$ are defined. Put $J' = \{1,..., n\} - (J_0 \cup \cdots \cup J_{i-1})$. Now no $J \subseteq J'$ satisfies (i), and so (by a form of Menger's theorem) there exist $X_i \subseteq V(G)$ and $J_i \subseteq J'$ with $|X_i| + |J_i| < r$, such that every path in G from any B_j ($j \in J'$) to V' uses either some vertex of X_i or some vertex of $\bigcup_{j \in J_i} V(B_j)$. This completes our inductive definition of X_i, J_i.

Put $X' = X_1 \cup \cdots \cup X_\nu$, $J = J_1 \cup \cdots \cup J_\nu$. For $1 \leqslant i \leqslant \nu$, let B_i' be the union of the graphs B_j ($j \in J_i$). Then B_i' has at most r components. Put

$$A_1 = \bigcup (V(B_j): j \in \{1,..., n\} - J)$$

and $A_2 = V'$. Now for $1 \leqslant i \leqslant \nu$, $X' \cup V(B_i')$ separates A_1 and A_2; for every path from A_1 to A_2 uses either a vertex of X_i or a vertex of B_i', by definition of X_i, J_i. Moreover, for $1 \leqslant i \leqslant \nu$, $|X_i| + |J_i| < r$, and so $|X'| < r\nu$. By (2.8), the maximum number of disjoint paths in G from A_1 to A_2, avoiding X', is less than $\frac{1}{2}r\theta^4$ (to apply (2.8), we set $d = r$). By Menger's theorem, there exists $X'' \subseteq V(G)$ with

$$|X''| \leqslant \tfrac{1}{2}r\theta^4$$

such that $X' \cup X''$ separates A_1 and A_2. Put $X = X' \cup X''$; then X separates V' from every B_j ($j \in \{1,..., n\} - J$); and

$$\begin{aligned} |X| + |J| &\leqslant |X''| + |X'| + |J| \\ &\leqslant |X''| + \sum_{1 \leqslant i \leqslant \nu} (|X_i| + |J_i|) \\ &\leqslant \tfrac{1}{2}r\theta^4 + r\nu \\ &\leqslant 2^{r\theta^4} \end{aligned}$$

(after some arithmetic) and so (ii) is true.

Incidentally, the following extension of (2.7) will appear in [4].

(2.10). *There is an integer* $\theta' > 0$ *such that if* $A_1,..., A_{\theta'}$ *are disjoint connected subgraphs of a graph* $G \in \mathscr{F}_\theta$, *and* $B_1,..., B_{\theta'}$ *are also disjoint connected subgraphs of* G, *then some* A_i *is disjoint from some* B_j.

3. The Main Theorem

In order to prove our main theorem (1.1), it is only necessary to prove a special case, as we shall now see. Let $\mathbf{Y}_1$ be the complete bipartite graph $K_{1,3}$. For $\lambda \geqslant 2$, we define $\mathbf{Y}_\lambda$ inductively by taking a copy of $\mathbf{Y}_{\lambda-1}$, and to each vertex of valency 1 in this graph making adjacent two new vertices. (See Fig. 1.)

In order to prove (1.1), it is only necessary to prove the following:

(3.1). *For $\lambda \geqslant 1$, odd, there is a number $w(\lambda)$ such that every graph with no minor isomorphic to $\mathbf{Y}_\lambda$ has path-with $\leqslant w(\lambda)$.*

(The "odd" condition is introduced for future technical convenience.)

Proof of (1.1) (assuming (3.1)). For any forest H there is an odd value of λ such that $\mathbf{Y}_\lambda$ has a minor isomorphic to H (e.g., any odd $\lambda \geqslant |V(H)|$ will do, although this is usually extravagent). If G is a graph with no minor isomorphic to H, then it certainly has no minor isomorphic to $\mathbf{Y}_\lambda$, and so, by (3.1), it has path-width $\leqslant w(\lambda)$. Thus (1.1) is true.

We now introduce a second class of graphs. $\mathbf{H}_0$ is the graph with just one vertex and no edges, and $\mathbf{H}_1$ is $K_{1,2}$. For $\lambda \geqslant 2$, we define $\mathbf{H}_\lambda$ inductively by taking a copy of $\mathbf{H}_{\lambda-1}$, and to each vertex of valency 1 in this graph making adjacent two new vertices. (See Fig. 2.)

(3.2). *For any even $\lambda \geqslant 0$,* the $(2^{(\lambda+1)/2} - 1)$*-grid has a minor isomorphic to $\mathbf{H}_\lambda$.*

The proof by induction is left to the reader. We hope that a "proof by diagram" is convincing. Figure 3 shows a subdivision of $\mathbf{H}_6$ drawn as a subgraph of a 15-grid.

(3.3). *For any $\lambda \geqslant 1$, $\mathbf{Y}_\lambda$ is isomorphic to a minor of $\mathbf{H}_{\lambda+1}$.*

The proof is clear.

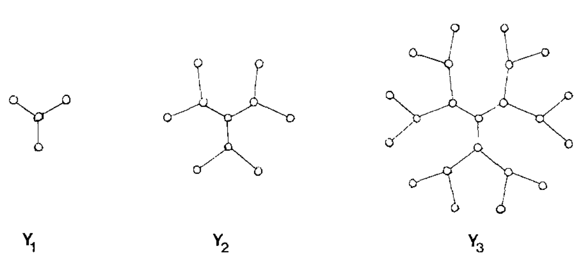

Figure 1

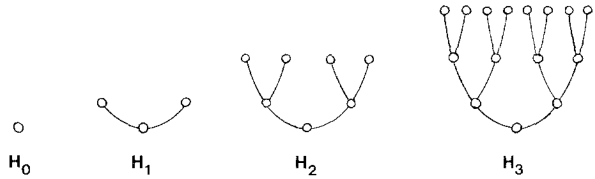

FIGURE 2

(3.4). *For any odd* $\lambda \geqslant 1$, *the* $(2^{(\lambda+3)/2} - 1)$*-grid has a minor isomorphic to* $\mathbf{Y}_\lambda$.

This follows from (3.2) and (3.3).

(3.5). *For odd* $\lambda \geqslant 1$, *let* $\theta = \max(6, 2^{(\lambda+3)/2})$. *For any graph* G, *if* G *has no minor isomorphic to* $\mathbf{Y}_\lambda$, *then* $G \in \mathscr{F}_\theta$.

This follows from (3.4).

Thus to study the graphs with no minor isomorphic to $\mathbf{Y}_\lambda$, we can confine ourselves to $\mathscr{F}_\theta$, and hence we can apply the results of Section 2.

A *rooted* graph is a graph with one vertex distinguished, called the *root*. We denote the root of G by $\rho(G)$.

If G_1, G_2 are rooted graphs we say that G_2 has a *rooted minor isomorphic to* G_1 if for each vertex v of G_1 there is a subset $Y(v)$ of the vertices of G_2, satisfying conditions (M1)–(M4):

(M1) *For distinct* $v, v' \in V(G_1)$, $Y(v) \cap Y(v') = \varnothing$.

(M2) *For each* $v \in V(G_1)$, $G_2 \mid Y(v)$ *is connected.*

(M3) *There is an injection* $f: E(G_1) \to E(G_2)$ *such that for every edge*

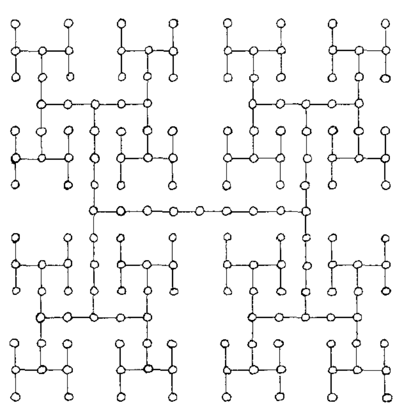

FIGURE 3

e of G_1, *if e has ends* v, v' *say, then* $f(e)$ *has one end in* $Y(v)$ *and the other in* $Y(v')$.

(M4) $\rho(G_2) \in Y(\rho(G_1))$.

We now introduced two further classes of trees. If $\gamma \geqslant 1$, $\delta \geqslant 1$ are integers, the rooted tree $P(\gamma, \delta)$ is defined as follows: Take δ disjoint copies of $\mathbf{H}_{\gamma-1}$. Each of these copies has a unique vertex of even valency; let these be $u_1,..., u_\delta$, say. Take a path of δ new vertices $v_1,..., v_\delta$ say, and for $1 \leqslant i \leqslant \delta$, add an edge joining u_i and v_i. The resultant graph is rooted at v_1, and is $P(\gamma, \delta)$. We also define $P(0, \delta)$ to be the path of δ vertices, rooted at one end, for $\delta \geqslant 1$.

Second, if $\gamma \geqslant 1$, $\delta \geqslant 2$ are integers, we define a rooted tree $Q(\gamma, \delta)$ as follows: Take a copy of $P(\gamma - 1, \delta - 1)$, with root v_1 say, and a disjoint copy of $\mathbf{H}_{\gamma-1}$. Let v_2 be the (unique) vertex of the second graph with even valency. Take a new vertex v, and two new edges, joining v to v_1, v_2, respectively. The resultant graph is $Q(\gamma, \delta)$, and it is rooted at v. (See Fig. 4.)

Now let $\lambda \geqslant 1$ be odd. Define $\theta = \max(6, 2^{(\lambda+3)/2})$, $\alpha = 2^{\theta^4}$, and $\beta = 2^{\theta^6}$. For integers x, $i \geqslant 0$, define $f_i(x)$ inductively, by

$$f_0(x) = x, \qquad f_i(x) = f_{i-1}(\beta^x)(i > 0).$$

For integers γ, δ with $\gamma \geqslant 0$, and $\delta \geqslant 2$, define $p(\gamma, \delta) = f_\gamma(\delta)$; and if $\gamma \geqslant 1$, define $q(\gamma, \delta) = f_{\gamma-1}(\beta + 2\delta)$.

(3.6). *With the above definitions, the following inequalities hold*:

(i) $p(0, \delta) \geqslant \delta - 2$ *for all* $\delta \geqslant 2$;

(ii) $q(\gamma, \delta) \geqslant \theta^2 + p(\gamma - 1, 2 + 4\theta^{2\theta} + 2\delta)$ *for* $1 \leqslant \gamma$, $2 \leqslant \delta$;

(iii) $p(\gamma, \delta) \geqslant q(\gamma, 4)$ *for* $1 \leqslant \gamma$, $2 \leqslant \delta$;

(iv) $p(\gamma, \delta) \geqslant \alpha^{\delta \cdot 2^\gamma} + \max(p(\gamma, \delta - 1), q(\gamma, 3\alpha^{\delta \cdot 2^\gamma} + 6))$ *for* $1 \leqslant \gamma \leqslant \lambda$, $3 \leqslant \delta$.

Proof. This is routine, and is left to the reader.

The aim of the next two sections is to prove the following:

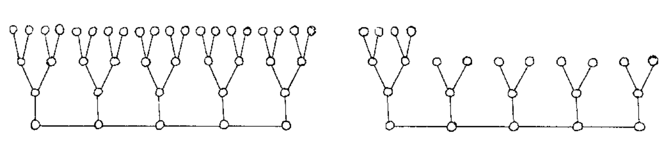

FIGURE 4

(3.7). (i) *If* $G \in \mathscr{F}_\theta$, *and is rooted and connected, and has no rooted minor isomorphic to* $P(\gamma, \delta)$, *then* G *has path-width at most* $p(\gamma, \delta)$, *for* $0 \leqslant \gamma \leqslant \lambda$, $\delta \geqslant 2$.

(ii) *If* $G \in \mathscr{F}_\theta$, *and is rooted and connected, and has no rooted minor isomorphic to* $Q(\gamma, \delta)$, *then* G *has path-width at most* $q(\gamma, \delta)$, *for* $1 \leqslant \gamma \leqslant \lambda$, $\delta \geqslant 2$.

Proof of (3.1) (assuming (3.7)). Let G be a graph with no minor isomorphic to $\mathbf{Y}_\lambda$, where $\lambda \geqslant 1$ and is odd.

Let C be any component of G, and let C be assigned a root, arbitrarily. Now C has no rooted minor isomorphic to $P(\lambda, 3)$, because $P(\lambda, 3)$ has a minor isomorphic to $\mathbf{Y}_\lambda$. Moreover, $C \in \mathscr{F}_\theta$, by (3.5). Hence, by (3.7)(i), C has path-width $\leqslant p(\lambda, 3)$, and so by (1.4), G has path-width $\leqslant p(\lambda, 3)$.

Thus, to prove our main theorem, it suffices to prove (3.7).

The proof of (3.7) is divided into three parts, as follows: In Part 1 we prove that (3.7)(i) is true when $\gamma = 0$. In Part 2 we show that for γ with $1 \leqslant \gamma \leqslant \lambda$, if (3.7)(i) is true for $\gamma - 1$ and all $\delta \geqslant 2$, then (3.7)(ii) is true for γ and all $\delta \geqslant 2$. Finally, in Part 3 we show that for any γ with $1 \leqslant \gamma \leqslant \lambda$, if (3.7)(ii) is true for γ and all $\delta \geqslant 2$, then so is (3.7)(i). These three parts combined yield that (3.7) is true.

Part 1 ($\gamma = 0$). $P(0, \delta)$ is a δ-vertex path, rooted at one end. We show, by induction on δ, that for $\delta \geqslant 2$ every connected rooted graph with no rooted minor isomorphic to $P(0, \delta)$ has path-width $\leqslant \delta - 2$. If $\delta = 2$ this is clear. We suppose then that $\delta > 2$. Let $G_1, \ldots, G_r$ be the components of $G \backslash \{\rho(G)\}$. For $1 \leqslant i \leqslant r$, let v_i be a vertex of G_i adjacent to $\rho(G)$ in G. We define $\rho(G_i) = v_i$ $(1 \leqslant i \leqslant r)$. Then certainly for $1 \leqslant i \leqslant r$, G_i has no rooted minor isomorphic to $P(0, \delta - 1)$, and so by induction has path-width $\leqslant \delta - 3$. Hence, by (1.4), $G \backslash \{\rho(G)\}$ has path-width $\leqslant \delta - 3$, and so by (1.5), G has path-width $\leqslant \delta - 2$. This completes the inductive argument that for $\delta \geqslant 2$ every rooted connected graph with no rooted minor isomorphic to $P(0, \delta)$ has path-width $\leqslant \delta - 2$. But by (3.6)(i), $p(0, \delta) \geqslant \delta - 2$, and so (3.7)(i) is true if $\gamma = 0$. This completes the argument for Part 1.

4. Part 2: The Reduction of $Q(\gamma, \delta)$

We now assume that $1 \leqslant \gamma \leqslant \lambda$, and that for $\delta \geqslant 2$, every rooted connected graph in $\mathscr{F}_\theta$ with no rooted minor isomorphic to $P(\gamma - 1, \delta)$ has path-width $\leqslant p(\gamma - 1, \delta)$. We shall prove that for all $\delta \geqslant 2$, every rooted connected graph in $\mathscr{F}_\theta$ with no rooted minor isomorphic to $Q(\gamma, \delta)$ has path-width $\leqslant q(\gamma, \delta)$.

Suppose then that $G \in \mathscr{F}_\theta$, and is rooted and connected, and has no rooted minor isomorphic to $Q(\gamma, \delta)$. Define $v = \theta^{2\theta}$, and $\mu = 2 + 4v + 2\delta$. If G has no rooted minor isomorphic to $P(\gamma - 1, \mu)$, then by hypothesis it has path-width $\leqslant p(\gamma - 1, \mu) \leqslant q(\gamma, \delta)$, by (3.6)(ii). We assume then that G has a rooted minor isomorphic to $P(\gamma - 1, \mu)$. Choose N maximum such that G has a rooted minor isomorphic to $P(\gamma - 1, N)$, and then $N \geqslant \mu$. Choose an integer T maximum so that $2vT \leqslant N - 2\delta$. Then $T \geqslant 2$, since $\mu \geqslant 4v + 2\delta$. Put $n - 1 = vT$; and then

$$N - 2\delta - 2v < 2n - 2 \leqslant N - 2\delta$$

by the maximality of T.

Now G has a rooted minor isomorphic to $P(\gamma - 1, N)$; and so, since all vertices of this graph have valency $\leqslant 3$, G has a subgraph H' which is isomorphic to a subdivision of $P(\gamma - 1, N)$, with a vertex v say corresponding to the root of $P(\gamma - 1, N)$, and G has a path from v to $\rho(G)$, with no vertex except v in common with H'. Let H be the subgraph of G consisting of the vertices and edges in H' together with those in the path.

Now $P(\gamma - 1, N)$ is formed by taking a path of N vertices $v'_1, \ldots, v'_N$ say, and if $\gamma \geqslant 2$, taking N disjoint copies of $\mathbf{H}_{\gamma - 2}$, and joining them to $v'_1, \ldots, v'_N$ appropriately; and then letting v'_1 be the root. Let $v_1, \ldots, v_N$ be the vertices of H' which correspond to $v'_1, \ldots, v'_N$. For $1 \leqslant j \leqslant N - 1$, let R_j be the set of vertices of H which may be joined to v_N by a path of H which avoids v_j, and put $R_0 = V(H)$, $R_N = \varnothing$. For $1 \leqslant j < n$, let $S_j = R_{2j-2} - R_{2j}$; and put $S_n = R_{2n-2}$. Then $S_1, \ldots, S_n$ are disjoint, and have union $V(H)$. For $1 \leqslant j \leqslant n$, let B_j be $H \mid S_j$. Then each B_j is a connected subgraph of G, and $B_1, \ldots, B_n$ are disjoint.

(4.1). *For $1 \leqslant j \leqslant n$, S_j separates $\bigcup_{1 \leqslant j' \leqslant j} S_{j'}$ from $\bigcup_{j \leqslant j' \leqslant n} S_{j'}$ in G.*

For suppose not; then there is a path of G from S_{j_1} to S_{j_2} say, where $1 \leqslant j_1 < j < j_2 \leqslant n$, with no interior vertices in H. Thus there is a path P of G from S_{j_1} to S_n, avoiding $S_1 \cup \cdots \cup S_{j_1 - 1} \cup S_{j_1 + 1}$. But then G has a rooted minor isomorphic to $Q(\gamma, \delta)$, as can be seen by deleting all vertices and edges of G except those in H and P, deleting all the vertices of $S_{j_1 + 2} \cup \cdots \cup S_{n-1}$ except those in P, and contracting all edges in $B_1, \ldots, B_{j_1}$ and P (note that B_n has a minor isomorphic to $P(\gamma - 1, 2\delta)$ since $N - 2n + 2 \geqslant 2\delta$).

(4.2). *For $v \leqslant j \leqslant n$, there exists $X_j \subseteq V(G)$ with $|X_j| < \frac{1}{2}\theta^2$, separating $\bigcup_{1 \leqslant j' \leqslant j - v + 1} S_{j'}$ and $\bigcup_{j \leqslant j' \leqslant n} S_{j'}$.*

Suppose not. Then by Menger's theorem there are $\frac{1}{2}\theta^2$ disjoint paths of G from $\bigcup_{1 \leqslant j' \leqslant j - v + 1} S_{j'}$ to $\bigcup_{j \leqslant j' \leqslant n} S_{j'}$, and yet these two sets are separated by each of $S_{j-v+1}, S_{j-v+2}, \ldots, S_j$, by (4.1). Then we have a contradiction from (2.7).

For $v \leqslant j \leqslant n$, choose X_j as in (4.2), minimal.

For $1 \leqslant j \leqslant n$, let Z_j denote the set of all vertices of G which can be joined to a vertex of S_j by a path of G which avoids $(S_1 \cup \cdots \cup S_n) - S_j$. Then clearly $Z_j \cap (S_1 \cup \cdots \cup S_n) = S_j$ and $Z_1 \cup \cdots \cup Z_n = V(G)$. By (4.1), $Z_j \cap Z_{j'} \neq \varnothing$ only if $|j - j'| \leqslant 1$.

(4.3). *For $v \leqslant j \leqslant n$, $X_j \subseteq S_{j-v+1} \cup Z_{j-v+2} \cup \cdots \cup Z_{j-1} \cup S_j$.*

For if $v \in X_j$, then by the minimality of X_j there is a path P from $\bigcup_{1 \leqslant j' \leqslant j-v+1} S_{j'}$ to $\bigcup_{j \leqslant j' \leqslant n} S_{j'}$ which avoids $X_j - \{v\}$ and uses v, and which has no interior vertices in

$$\bigcup_{1 \leqslant j' \leqslant j-v+1} S_{j'} \cup \bigcup_{j \leqslant j' \leqslant n} S_{j'}.$$

Since $S_{j-v+2} \cup \cdots \cup S_{j-1}$ separates $\bigcup_{1 \leqslant j' \leqslant j-v+1} S_{j'}$ and $\bigcup_{j \leqslant j' \leqslant n} S_{j'}$ (by (4.1)), P has an interior vertex u in $S_{j-v+2} \cup \cdots \cup S_{j-1}$. Choose u such that the subpath P' of P from v to u has minimum length. Then no vertices of P' except v, u are in $S_1 \cup \cdots \cup S_n$. If $v \notin S_1 \cup \cdots \cup S_n$, then $v \in Z_{j-v+2} \cup \cdots \cup Z_{j-1}$, because of P'. If $v \in S_1 \cup \cdots \cup S_n$, then

$$v \in S_{j-v+1} \cup S_{j-v+2} \cup \cdots \cup S_{j-1} \cup S_j$$

by (4.1), since no interior vertices of P' are in S_{j-v+1} or S_j. Thus in either case,

$$v \in S_{j-v+1} \cup Z_{j-v+2} \cup \cdots \cup Z_{j-1} \cup S_j$$

as claimed.

(4.4). *For $v \leqslant i \leqslant n$, $1 \leqslant j \leqslant n$, if $s \in S_j$, then*

(i) *X_i separates s and $\rho(G)$ if $j \geqslant 1$, and*

(ii) *X_i does not separate s and $\rho(G)$ if $j \leqslant i - v$.*

Statement (i) follows from the definition of X_i, since $\rho(G) \in S_1$. To show (ii), we observe that from (4.3), the path of H from s to $\rho(G)$ does not meet X_i if $j \leqslant i - v$.

Let $Y_i = X_{vi}$ for $1 \leqslant i \leqslant T$. From (4.3), $Y_1, \ldots, Y_T$ are disjoint.

(4.5). *For $1 \leqslant i, i' \leqslant T$ and for every $y \in Y_{i'}$, Y_i separates y and $\rho(G)$ if and only if $i \leqslant i'$.*

If $i = i'$ the result is true, and so we assume that $i \neq i'$. Define

$$Z = S_{vi'-v+1} \cup Z_{vi'-v+2} \cup \cdots \cup Z_{vi'-1} \cup S_{vi'}$$

and

$$S = S_{vi'-v+1} \cup S_{vi'-v+2} \cup \cdots \cup S_{vi'-1} \cup S_{vi'}.$$

By (4.3), $y \in Z$ and there is a path of G from y to some $s \in S$ within Z. Now $Z_j \cap Z_{j'} \neq \varnothing$ only if $|j - j'| \leqslant 1$, and so this path avoids

$$S_{vi-v+1} \cup Z_{vi-v+2} \cup \cdots \cup Z_{vi-1} \cup S_{vi}$$

since $i \neq i'$. Hence by (4.3) it avoids Y_i. Thus Y_i separates y and $\rho(G)$ if and only if it separates s and $\rho(G)$. The result follows from (4.4).

For $1 \leqslant i \leqslant T-1$, let W_i be the set of all vertices v of G such that Y_i separates v and $\rho(G)$, and Y_{i+1} does not separate v and $\rho(G)$. Let W_0 be the set of vertices not separated from $\rho(G)$ by Y_1, and let W_T be the set of vertices separated from $\rho(G)$ by Y_T. Then the sequence

$$W_0 \cup Y_1,\ W_1 \cup Y_2,\ W_2 \cup Y_3, \ldots,\ W_{T-1} \cup Y_T,\ W_T$$

is a path-decomposition of G by (4.5) and (1.7), and the intersection of any consecutive pair of terms of this sequence has cardinality $\leqslant \frac{1}{2}\theta^2$. In order to complete step 2, it suffices to show that $G \mid W_0, \ldots, G \mid W_T$ each have path-width $\leqslant p(\gamma - 1, \mu)$, by (1.6) and (3.6)(ii). By (1.4), it suffices to show that for $0 \leqslant i \leqslant T$, every component of $G \mid W_i$ has path-width $\leqslant p(\gamma - 1, \mu)$.

Thus, take $0 \leqslant i \leqslant T$, and let C be a component of $G \mid W_i$. Assign a root to C, arbitrarily. By our initial hypothesis it is sufficient to show that C has no rooted minor isomorphic to $P(\gamma - 1, \mu)$. Suppose for a contradiction, that it does; then C has a subgraph D which is isomorphic to a subdivision of $P(\gamma - 1, \mu)$.

Let

$$A = \bigcup_{v(i+1) \leqslant j \leqslant n} S_j,\quad B = \bigcup_{1 \leqslant j \leqslant v(i-1)} S_j.$$

(4.6). *A, B and W_i are disjoint.*

For every vertex of A is separated from $\rho(G)$ by Y_i (if $i > 0$) and by Y_{i+1} (if $i < T$), from (4.4). Every vertex of W_i is separated from $\rho(G)$ by Y_i (if $i > 0$) but not by Y_{i+1} (if $i < T$), by the definition of W_i. Every vertex of B is separated from $\rho(G)$ neither by Y_i (if $i > 0$) nor by Y_{i+1} (if $i < T$), from (4.4). The result follows.

(4.7). *There is a path in G from $V(D)$ to $V(H)$ avoiding $A \cup B$.*

This follows from (4.5) and the definition of W_i when $i = 0$. When $1 \leqslant i \leqslant T$ there is a path P within W_i from D to Y_i, by definition of W_i. P certainly avoids $A \cup B$, by (4.6). Let y be a vertex of Y_i on P. By (4.3) there is a path P' from y to $S_{\nu i-\nu+1} \cup \cdots \cup S_{\nu i}$ which avoids $A \cup B$. The union of P and P' yields a path satisfying (4.7).

If $i \geqslant 2$, let w_1 be the vertex of $S_{\nu i-\nu}$ which is adjacent in H to a vertex of $S_{\nu i-\nu+1}$. If $i = 0$ or 1, let w_1 be $\rho(G)$.

(4.8). *There is a path P_1 in G from D to w_1, avoiding $A \cup B - \{w_1\}$.*

This is clear from (4.7). Let P_1 be a minimal such path, so that it uses only one vertex of D, d_1 say. Let P_0 be the path of H from w_1 to $\rho(G)$.

Now D is a subdivision of $P(\gamma-1, 4\nu+2\delta+2)$. But $P(\gamma-1, 4\nu+2\delta+2)$ consists of a $(4\nu+2\delta+2)$-vertex path with vertices $p(1),\ldots, p(4\nu+2\delta+2)$ say, in order, and (if $\gamma > 1$) $4\nu+2\delta+2$ copies of $\mathbf{H}_{\gamma-2}$, $H(1),\ldots, H(4\nu+2\delta+2)$ say, in order, and an edge from $p(j)$ to $H(j)$ $(1 \leqslant j \leqslant 4\nu+2\delta+2)$. By the *first end* of D we mean the subgraph of D corresponding to $H(1)$, $H(2)$ and the path of D joining them. By the *second end* we mean the subgraph corresponding to $H(4\nu+2\delta+1)$, $H(4\nu+2\delta+2)$ and the path joining them.

(4.9). *d_1 is in one of the ends of D.*

Otherwise, since $4\nu+2\delta+2 \geqslant 2\delta$, G has a rooted minor isomorphic to $Q(\gamma, \delta)$, as can be seen by deleting all vertices and edges of G not in D, P_0, or P_1, and contracting all edges in P_0 and P_1.

Without loss of generality we may assume that d_1 is in the first end of D.

(4.10). *$i \neq T$.*

If $i = T$, then G has a rooted minor isomorphic to $P(\gamma-1,\ 2\nu(T-1) + 4\nu+2\delta)$ by (4.8) and (4.9), as can be seen by deleting all vertices and edges of G except those of H within B and those of P_1 and D, and by contracting the edges in P_1 and those in the first end of D. But

$$2\nu(T-1) + 4\nu + 2\delta > N$$

by the maximality of T, contrary to the maximality of N.

Let w_2 be the vertex of $S_{\nu i+\nu}$ adjacent in H to a vertex of $S_{\nu i+\nu-1}$.

(4.11). *There is a path P_2 from D to w_2 avoiding $A \cup B - \{w_2\}$.*

This follows from (4.7) and (4.10).

Choose a minimal path P_2 satisfying (4.11) so that P_2 only contains one vertex of D, d_2 say. Let P_3 be the path of H from w_2 to S_n, with no internal vertex in S_n.

(4.12). *d_2 is in the second end of D.*

If not, then the second end of D is disjoint from P_1, P_2, A and B. But then G has a rooted minor isomorphic to $Q(\gamma, \delta)$, as can be seen by deleting all vertices and edges of G not in D, P_0, P_1, P_2, P_3, and S_n and contracting the edges of P_0, P_1, P_2, P_3, and those of D not incident with vertices of the second end.

(4.13). *P_1 and P_2 are disjoint.*

If not, then the union of P_1 and P_2 contains a path P_2' from d_1 to w_2 avoiding A and B. P_2' then satisfies the defining conditions for P_2, and so from (4.12) we deduce d_1 is in the second end of D, which is impossible because the ends of D are disjoint (since $4\nu + 2\delta + 2 \geqslant 4$).

(4.14). *Conclusion.*

From (4.12) and (4.13) we deduce that G has a rooted minor isomorphic to $P(\gamma - 1), (N - 4\nu) + (4\nu + 2\delta + 2 - 4)) = P(\gamma - 1, N + 2\delta - 2)$, as can be seen by deleting all vertices of G except those in A, B, D, P_1 and P_2, and contracting the edges in P_1, P_2 and the ends of D. This contradicts the maximality of N, and completes the argument for Part 2.

5. Part 3: The Reduction of $P(\gamma, \delta)$ When $\gamma \geqslant 1$

We now assume that $1 \leqslant \gamma \leqslant \lambda$, and that for all $\delta \geqslant 2$, every rooted connected graph in $\mathscr{F}_\theta$ with no rooted minor isomorphic to $Q(\gamma, \delta)$ has path-width $\leqslant q(\gamma, \delta)$. We shall show, by induction on $\delta \geqslant 2$, that every rooted connected graph in $\mathscr{F}_\theta$ with no rooted minor isomorphic to $P(\gamma, \delta)$ has path-width $\leqslant p(\gamma, \delta)$.

Suppose then that $G \in \mathscr{F}_\theta$ is a rooted connected graph with no rooted minor isomorphic to $P(\gamma, \delta)$. If G has no rooted minor isomorphic to $Q(\gamma, 4)$, then by hypothesis, its path-width is $\leqslant q(\gamma, 4) \leqslant p(\gamma, \delta)$ by (3.6)(iii). Thus we may assume that G has a rooted minor isomorphic to $Q(\gamma, 4)$.

Now $P(\gamma, 2)$ is isomorphic to a rooted minor of $Q(\gamma, 4)$, and so G has a rooted minor isomorphic to $P(\gamma, 2)$. It follows that $\delta > 2$. By induction on δ, we have that every rooted connected graph in $\mathscr{F}_\theta$ with no rooted minor isomorphic to $P(\gamma, \delta - 1)$ has path-width $\leqslant p(\gamma, \delta - 1)$.

Let H be $\mathbf{H}_\gamma$, rooted at its vertex of valency 2. Now H is isomorphic to a rooted minor of $Q(\gamma, 4)$ and so G has a rooted minor isomorphic to H.

Choose an integer $N \geqslant 0$, maximum such that there exist $X_0, X_1, \ldots, X_N \subseteq V(G)$, disjoint, with the following properties:

(X1) *$\rho(G) \in X_0$, and $G \mid X_0$ rooted at $\rho(G)$ has a rooted minor isomorphic to H;*

(X2) *for $1 \leqslant i \leqslant N$, $G \mid X_i$ has a minor isomorphic to $\mathbf{Y}_{\gamma-1}$.*

Now all vertices of $\mathbf{Y}_{\gamma-1}$ have valency $\leqslant 3$, and so for $1 \leqslant i \leqslant N$, $G \mid X_i$ has a subgraph B_i which is isomorphic to a subdivision of $\mathbf{Y}_{\gamma-1}$. Moreover, $G \mid X_0$ has a subgraph B_0 which consists of a subgraph H' isomorphic to a subdivision of H, together with a path from $\rho(G)$ to the vertex of H' corresponding to the root of H, such that the path has only this vertex in common with H'.

Clearly we may assume that $X_i = V(B_i)$ $(0 \leqslant i \leqslant N)$. Suppose that there exists $J \subseteq \{1, \ldots, N\}$ with $|J| = \delta \cdot 2^\gamma$, and disjoint paths $P_j (j \in J)$ of G, such that

(i) for each $j \in J$, P_j has one end in X_j and the other in X_0;

(ii) for each $j \in J$, P_j has no interior vertex in $X_0 \cup \bigcup_{j' \in J} X_{j'}$.

Now B_0 has only 2^γ vertices distinct from $\rho(G)$ of valency 1, and B_0 is a tree; and so there are paths $Q_1, \ldots, Q_{2^\gamma}$ of B_0, each from $\rho(G)$ to some vertex of valency 1 distinct from $\rho(G)$, such that every vertex of B_0 is used by at least one of $Q_1, \ldots, Q_{2^\gamma}$. It follows that there exists $J' \subseteq J$ with $|J'| = \delta$ $(=2^{-\gamma}|J|)$ such that for some i $(1 \leqslant i \leqslant 2^\gamma)$, all the paths P_j $(j \in J')$ have their terminal vertices on Q_i. But then G has a rooted minor isomorphic to $P(\gamma, \delta)$, as can be seen by deleting all vertices except those in Q_i and P_j and B_j $(j \in J')$, and for each $j \in J'$ contracting all edges of P_j except one. Thus there there is no such J.

By (2.9) (setting $V' = X_0$, $r = \delta \cdot 2^\gamma$, and $n = N$) we deduce that there exists $X \subseteq V(G)$ and $J \subseteq \{1, \ldots, N\}$ with $|X| + |J| \leqslant \Omega$, where $\Omega = \alpha^{\delta \cdot 2^\gamma}$ (and $\alpha = 2^{\theta^4}$, as before), such that X separates X_0 from X_j for every $j \in \{1, \ldots, N\} - J$. To complete the argument for Part 3, it suffices to show that every component of $G \backslash X$ has path-width $\leqslant \max(p(\gamma, \delta - 1), q(\gamma, 3\Omega + 6))$, by (3.6)(iv), (1.4), and (1.5). Thus, let C be a component of $G \backslash X$. There are two cases.

(5.1). *If X_0 does not contain any vertex of C, then C has path-width $\leqslant p(\gamma, \delta - 1)$.*

Let P be a minimal path in G from $V(C)$ to X_0. Then P has at least one edge, and no interior vertex in $V(C) \cup X_0$. Let v be the vertex of C on P. Root C at v. Suppose that C has a rooted minor isomorphic to $P(\gamma, \delta - 1)$. Then G has a rooted minor isomorphic to $P(\gamma, \delta)$, as can be seen by contracting all edges of P except one and contracting one "half" of B_0, suitably chosen. This is a contradiction, and so C has no such rooted minor. By induction, (5.1) is true.

(5.2). *If* X_0 *contains a vertex of* C, *then* C *has path-width at most* $q(\gamma, 3\Omega + 6)$.

In this case C contains no vertices of $\bigcup(X_j : j \in \{1,..., N\} - J)$ since X separates X_0 and this set. Let P be a minimal path of B_0 from $V(C)$ to $\rho(G)$. Let v be the vertex of C on P. Root C at v. Suppose that C has a rooted minor isomorphic to $Q(\gamma, 3\Omega + 6)$. Then there exist $X'_0,..., X'_{\Omega+1} \subseteq V(C)$, disjoint, such that $C \mid X'_0$, rooted at v, has a rooted minor isomorphic to H, and for $1 \leqslant i \leqslant \Omega + 1$, $C \mid X'_i$ has a minor isomorphic to $\mathbf{Y}_{\gamma-1}$. But then the sets

$$X'_0 \cup V(P), X'_1,..., X'_{\Omega+1} \qquad \text{and} \qquad X_j (j \in \{1,..., N\} - J)$$

satisfy (X1) and (X2), and there are $\Omega + 1 + N - |J| > N$ of them. This contradicts the maximality of N, and hence proves (5.2). This completes the argument for Part 3, and so proves (3.7).

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