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An integral homological characterization of finite groups ☆

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Abstract

We show that a group G is finite if and only if every injective $\mathbb{Z}G$-module has projective length one.

Keywords: spli $\mathbb{Z}G$; Finitistic dimension of $\mathbb{Z}G$; $H^n(G,\mathbb{Z}G)$; Groups acting on trees

1. Introduction

Let G be a group and $\mathbb{Z}G$ its integral group ring. For a $\mathbb{Z}G$-module M, let $\operatorname{pd}_{\mathbb{Z}G} M$ and $\operatorname{id}_{\mathbb{Z}G} M$ denote the projective dimension and injective dimension of M respectively. The algebraic invariants of $\mathbb{Z}G$, $\operatorname{silp}\mathbb{Z}G = \sup\{\operatorname{id}_{\mathbb{Z}G} P \mid P$ a projective $\mathbb{Z}G$-module$\}$ and $\operatorname{spli}\mathbb{Z}G = \sup\{\operatorname{pd}_{\mathbb{Z}G} I \mid I$ an injective $\mathbb{Z}G$-module$\}$, were studied in [4] in connection with the existence of complete cohomological functors on groups. In [4] it was shown that for any group G $\operatorname{silp}\mathbb{Z}G \leqslant \operatorname{spli}\mathbb{Z}G$ and that if $\operatorname{spli}\mathbb{Z}G < \infty$ then $\operatorname{silp}\mathbb{Z}G = \operatorname{spli}\mathbb{Z}G$. It is not known whether there is a group G with $\operatorname{silp}\mathbb{Z}G$ finite and $\operatorname{spli}\mathbb{Z}G$ infinite.

The invariants $\operatorname{silp}\mathbb{Z}G$ and $\operatorname{spli}\mathbb{Z}G$ also appeared in the study of $\mathbf{H}\mathfrak{F}$-groups in [2], where it was shown that if G is in $\mathbf{H}\mathfrak{F}$ then

$$\operatorname{findim}\mathbb{Z}G = \operatorname{silp}\mathbb{Z}G = \operatorname{spli}\mathbb{Z}G = \kappa(\mathbb{Z}G)$$

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where $\operatorname{findim} \mathbb{Z}G = \sup\{\operatorname{pd}_{\mathbb{Z}G} M \mid \operatorname{pd}_{\mathbb{Z}G} M < \infty\}$ and $\kappa(\mathbb{Z}G) = \sup\{\operatorname{pd}_{\mathbb{Z}G} M \mid \operatorname{pd}_{\mathbb{Z}H} M < \infty$ for every finite $H \leqslant G\}$. Actually it was proved in [9] that for any group G, if $\kappa(\mathbb{Z}G)$ is finite then

$$\operatorname{findim} \mathbb{Z}G = \operatorname{silp} \mathbb{Z}G = \operatorname{spli} \mathbb{Z}G = \kappa(\mathbb{Z}G).$$

The invariants $\operatorname{silp} \mathbb{Z}G$ and $\operatorname{spli} \mathbb{Z}G$ are also related to the study of periodicity in group cohomology. If a group G has periodic cohomology after some steps, i.e. there are integers q, k with $q > 0$ so that the functors $H^i(G, \_)$ and $H^{i+q}(G, \_)$ are naturally equivalent for all $i > k$, then it was shown in [8] that $\operatorname{silp} \mathbb{Z}G$ is finite if and only if $\operatorname{spli} \mathbb{Z}G$ is finite. It was also shown that if a group G has periodic cohomology after some steps, then the periodicity isomorphism is given by cup product with an element in $H^q(G, \mathbb{Z})$ if and only if $\operatorname{spli} \mathbb{Z}G$ is finite. Note that a group G admits a finite dimensional free G-CW-complex homotopy equivalent to a sphere if and only if G has periodic cohomology after some steps and the periodicity isomorphisms are induced by cup product with an element in $H^q(G, \mathbb{Z})$ for some $q > 0$. This was proved in [6] for a certain class of groups and in [1] for any group.

Here we show:

Theorem. *A group G is finite if and only if* $\operatorname{spli} \mathbb{Z}G = 1$.

This characterization of finite groups supports Conjecture A in [9] which states that a group G admits a finite dimensional model for its classifying space for proper actions, $\underline{E}G$, if and only if $\operatorname{spli} \mathbb{Z}G$ is finite.

2. Proof of the theorem

Proposition 1. *(See [4].)*

(a) *For any group G* $\operatorname{silp} \mathbb{Z}G \leqslant \operatorname{spli} \mathbb{Z}G$, *and if* $\operatorname{spli} \mathbb{Z}G < \infty$ *then* $\operatorname{silp} \mathbb{Z}G = \operatorname{spli} \mathbb{Z}G$.
(b) *If G is a group and H a subgroup of G then*
  (b_1) $\operatorname{spli} \mathbb{Z}H \leqslant \operatorname{spli} \mathbb{Z}G$.
  (b_2) *If* $|G : H| < \infty$ *then* $\operatorname{spli} \mathbb{Z}H = \operatorname{spli} \mathbb{Z}G$.

Since the injective $\mathbb{Z}$-modules are the divisible abelian groups, an immediate corollary of Proposition 1(b_2) is

Corollary 2. *If G is a finite group then* $\operatorname{spli} \mathbb{Z}G = 1$.

Theorem 3. *If G is a group with* $\operatorname{spli} \mathbb{Z}G = 1$ *then G is finite.*

Proof. Assume that $\operatorname{spli} \mathbb{Z}G = 1$. By Theorem 2.2 of [8] $\operatorname{spli} \mathbb{Z}G < \infty$ if and only if there is a $\mathbb{Z}$-split $\mathbb{Z}G$-exact sequence $0 \to \mathbb{Z} \to A$ with A $\mathbb{Z}$-free and $\operatorname{pd}_{\mathbb{Z}G} A < \infty$. Since $\operatorname{silp} \mathbb{Z}G = \operatorname{spli} \mathbb{Z}G$ it follows that $\operatorname{pd}_{\mathbb{Z}G} A \leqslant 1$. By the corollary of [7], which is proved using the Almost Stability Theorem of Dicks and Dunwoody [3], it follows that G acts on a tree with finite vertex stabilizers. Hence, by a theorem of Tits (cf. [3, Theorem I. 4.12]), it follows that either G contains an element of infinite order or G is a countable locally finite group. Propositions 5 and 6 now complete the proof of the theorem. □

To prove Propositions 5 and 6 we will use the following lemma:

Lemma 4. *If G is a group with* $\operatorname{spli}\mathbb{Z}G \leqslant n+1$ *and $H^n(G,P)$ is not a divisible abelian group for some projective $\mathbb{Z}G$-module P, then* $\operatorname{spli}\mathbb{Z}G = n+1$.

Proof. Since $H^n(G,P)$ is not a divisible abelian group, there is a prime p such that multiplication by $p: H^n(G,P) \to H^n(G,P)$ is not an epimorphism. The short exact sequence of trivial $\mathbb{Z}G$-modules

$$0 \to \mathbb{Z} \xrightarrow{\pi_p} \mathbb{Z} \longrightarrow \mathbb{Z}_p \to 0$$

where π_p is multiplication by p, gives rise to the following exact sequence of abelian groups

$$H^n(G,P) \xrightarrow{\pi_p^*} H^n(G,P) \longrightarrow \operatorname{Ext}^{n+1}_{\mathbb{Z}G}(\mathbb{Z}_p, P)$$

where π_p^* is multiplication by p. It follows that $\operatorname{Ext}^{n+1}_{\mathbb{Z}G}(\mathbb{Z}_p, P) \neq 0$, hence $\operatorname{silp}\mathbb{Z}G \geqslant n+1$. By Proposition 1(a) $\operatorname{spli}\mathbb{Z}G \geqslant n+1$. The result now follows. □

Proposition 5. *If a group G contains an element x of infinite order then* $\operatorname{spli}\mathbb{Z}G \geqslant 2$.

Proof. By Proposition 1(b_1) it is enough to show that $\operatorname{spli}\mathbb{Z}K = 2$ where $K = \langle x \rangle$. Since $H^i(K, \_) = 0$ for $i > 1$ it follows that $\operatorname{spli}\mathbb{Z}K \leqslant 2$. Since $H^1(K, \mathbb{Z}K) \simeq \mathbb{Z}$, the result follows from Lemma 4. □

Proposition 6. *If a group G is an infinite countable locally finite group then* $\operatorname{spli}\mathbb{Z}G = 2$.

For the proof of Proposition 6 we need the following lemma:

Lemma 7. *If a group G is an infinite countable locally finite group then $H^1(G,\mathbb{Z}G)$ has a direct summand isomorphic to $\frac{\prod_{n\in\mathbb{N}}\mathbb{Z}}{\bigoplus_{n\in\mathbb{N}}\mathbb{Z}}$. In particular, $H^1(G,\mathbb{Z}G)$ is not a divisible abelian group.*

Proof. Let $G = \bigcup_{n\in\mathbb{N}} G_n$ where $\{G_n\}_{n\in\mathbb{N}}$ is a family of finite subgroups of G such that $G_n < G_{n+1}$ for all $n \in \mathbb{N}$. There is a $\mathbb{Z}$-split short exact sequence of $\mathbb{Z}G$-modules

$$0 \to \bigoplus_{n\in\mathbb{N}} \mathbb{Z}[G/G_n] \xrightarrow{\sigma} \bigoplus_{n\in\mathbb{N}} \mathbb{Z}[G/G_n] \xrightarrow{\pi} \mathbb{Z} \to 0 \qquad (1)$$

where $\sigma(gG_n) = gG_{n+1} - gG_n$ and $\pi(gG_n) = 1$ for all $n \in \mathbb{N}$, which gives rise to the following long exact sequence of abelian groups

$$\prod_{n\in\mathbb{N}} H^0(G_n, \mathbb{Z}G) \xrightarrow{\sigma^*} \prod_{n\in\mathbb{N}} H^0(G_n, \mathbb{Z}G) \to H^1(G, \mathbb{Z}G) \to \prod_{n\in\mathbb{N}} H^1(G_n, \mathbb{Z}G)$$

where $\prod_{n\in\mathbb{N}} H^1(G_n, \mathbb{Z}G) = 0$ since G_n is finite for all $n \in \mathbb{N}$. So the above exact sequence becomes

$$\prod_{n\in\mathbb{N}} \mathbb{Z}G^{G_n} \xrightarrow{\sigma^*} \prod_{n\in\mathbb{N}} \mathbb{Z}G^{G_n} \to H^1(G, \mathbb{Z}G) \to 0$$

and $H^1(G,\mathbb{Z}G) \simeq \operatorname{coker}\sigma^*$, where $\sigma^*(x_n)_{n\in\mathbb{N}} = (\operatorname{res} x_{n+1} - x_n)_{n\in\mathbb{N}}$ and $\operatorname{res} : \mathbb{Z}G^{G_{n+1}} \to \mathbb{Z}G^{G_n}$ are the restriction maps for all $n \in \mathbb{N}$.

Let T_n be a right transversal of G_n in G_{n+1}, such that $1_G \in T_n$ for $n \in \mathbb{N}$. Then the set $A_n = \{t_n t_{n+1} \cdots t_{n+k} \mid t_i \in T_i,\ n \leqslant i \leqslant n+k,\ k \geqslant 0\}$ is a right transversal of G_n in G, such that $A_n = \{ta \mid t \in T_n,\ a \in A_{n+1}\}$ and $1_G \in A_n$ for all $n \in \mathbb{N}$. The group $\mathbb{Z}G^{G_n}$ is a free abelian with basis $\{N_n a \mid a \in A_n\}$ where $N_n = \sum_{g\in G_n} g$. It turns out that $\operatorname{res}(N_{n+1}a) = \sum_{t\in T_n} N_n t a$ for all $a \in A_{n+1}$. So, if we denote by x^a the $(N_{n+1}a)$th coordinate of an element $x \in \mathbb{Z}G^{G_{n+1}}$ we have that $(\operatorname{res} x)^{ta} = x^a$ for all $a \in A_{n+1},\ t \in T_n$.

Consider the map $\phi : \prod_{n\in\mathbb{N}} \mathbb{Z} \to \prod_{n\in\mathbb{N}} \mathbb{Z}G^{G_n}$ with $\phi(m_n)_{n\in\mathbb{N}} = (m_n N_n)_{n\in\mathbb{N}}$. It is easy to show that if $m_n = 0$ for almost all $n \in \mathbb{N}$ then $(m_n N_n)_{n\in\mathbb{N}} \in \operatorname{im}\sigma^*$. Conversely, let $(m_n N_n)_{n\in\mathbb{N}} = \sigma^*(x)$ for some $x = (x_n)_{n\in\mathbb{N}} \in \prod_{n\in\mathbb{N}} \mathbb{Z}G^{G_n}$. Then $m_n N_n = \operatorname{res} x_{n+1} - x_n$ for all $n \in \mathbb{N}$. Thus for all $n \in \mathbb{N}$ and $a \in A_n$ we have that

$$(\operatorname{res} x_{n+1})^a - x_n^a = \begin{cases} 0, & \text{if } a \neq 1_G; \\ m_n, & \text{if } a = 1_G. \end{cases}$$

Since $x_1^a \neq 0$ for finitely many $a \in A_1$, there is an $r \in \mathbb{N}$ such that for any $n > r$ there is an element $t_1 t_2 \cdots t_n = a \in A_1$ with $t_i \neq 1_G$ and $x_1^a = 0$. Then $x_1^a = (\operatorname{res} x_2)^a = x_2^{t_2\cdots t_n} = (\operatorname{res} x_3)^{t_2\cdots t_n} = \cdots = x_n^{t_n} = (\operatorname{res} x_{n+1})^{t_n} = x_{n+1}^{1_G}$. It follows that $x_{n+1}^{1_G} = 0$ for all $n > r$. Since $x_{n+1}^{1_G} - x_n^{1_G} = (\operatorname{res} x_{n+1})^{1_G} - x_n^{1_G} = m_n$ we have that $m_n = 0$ for $n > r+1$.

Thus ϕ induces an injection $\bar{\phi} : \frac{\prod_{n\in\mathbb{N}} \mathbb{Z}}{\bigoplus_{n\in\mathbb{N}} \mathbb{Z}} \hookrightarrow \operatorname{coker}\sigma^*$. We show that $\bar{\phi}$ is a pure injection. Let $(m_n N_n)_{n\in\mathbb{N}} + \operatorname{im}\sigma^* \in k(\operatorname{coker}\sigma^*)$ for some integer $k > 1$. Then there is an element $(x_n)_{n\in\mathbb{N}} \in \prod_{n\in\mathbb{N}} \mathbb{Z}G^{G_n}$ such that $(m_n N_n - k x_n)_{n\in\mathbb{N}} \in \operatorname{im}\sigma^*$. Repeating the above argument modulo k we find that k divides m_n for almost all $n \in \mathbb{N}$, so $(m_n)_{n\in\mathbb{N}} \in k(\frac{\prod_{n\in\mathbb{N}} \mathbb{Z}}{\bigoplus_{n\in\mathbb{N}} \mathbb{Z}})$. Since $\frac{\prod_{n\in\mathbb{N}} \mathbb{Z}}{\bigoplus_{n\in\mathbb{N}} \mathbb{Z}}$ is pure injective [5, 38.1 and 42.2], it follows that $\bar{\phi}$ splits. Since $\frac{\prod_{n\in\mathbb{N}} \mathbb{Z}}{\bigoplus_{n\in\mathbb{N}} \mathbb{Z}}$ is not a divisible abelian group, the result follows. □

We return now to the

Proof of Proposition 6. Tensoring the $\mathbb{Z}$-split short exact sequence (1) of Lemma 7, with an injective $\mathbb{Z}G$-module I yields the short exact sequence of $\mathbb{Z}G$-modules

$$0 \to \bigoplus_{n\in\mathbb{N}} \mathbb{Z}G \otimes_{G_n} I \to \bigoplus_{n\in\mathbb{N}} \mathbb{Z}G \otimes_{G_n} I \to I \to 0$$

where $\operatorname{pd}_{\mathbb{Z}G} \mathbb{Z}G \otimes_{G_n} I = 1$ from Corollary 2. It follows that $\operatorname{pd}_{\mathbb{Z}G} I \leqslant 2$ for every injective $\mathbb{Z}G$-module, so $\operatorname{spli}\mathbb{Z}G \leqslant 2$. Applying Lemma 4 and Lemma 7 gives $\operatorname{spli}\mathbb{Z}G = 2$. □

The theorem follows from Corollary 2 and Theorem 3.

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