Extension of solutions of convolution equations in spaces of holomorphic functions with polynomial growth in convex domains

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Abstract

In this paper we consider a problem of extension of solutions to homogeneous convolution equations defined by operators acting from a space $A^{-\infty}(D + K)$ of holomorphic functions with polynomial growth near the boundary of $D + K$ into another space of such a type $A^{-\infty}(D)$ ($D$ and $K$ being a bounded convex domain and a convex compact set in $\mathbb{C}$, respectively). We show that under some exact conditions each such solution can be extended as $A^{-\infty}(\Omega + K)$-solution, where $\Omega \supset D$ is a certain convex domain.

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Résumé

Dans cet article, nous considérons le problème de prolongement des solutions d’une équation homogène de convolution définie dans l’espace $A^{-\infty}(D + K)$ des fonctions holomorphes à croissance polynomiale près du bord de $D + K$ à valeur dans l’espace $A^{-\infty}(D)$ de même type (où $D$ et $K$ étant respectivement un domaine convexe borné et un ensemble convexe compact de $\mathbb{C}$). Nous montrons que sous certaines conditions exactes, chaque solution se prolonge comme $A^{-\infty}(\Omega + K)$-solution, où $\Omega \supset D$ est un certain domaine convexe.

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1. Introduction

Let $D$ be a bounded convex domain in $\mathbb{C}$ and $O(D)$ the space of all holomorphic functions in $D$, equipped with the topology of uniform convergence on compact subsets of $D$. Given a non-trivial analytic functional $\mu$ on $\mathbb{C}$, carried by a compact convex set $K$, consider the convolution operator

$$\mu * f \in O(D + K) \mapsto (\mu * f)(z) := \langle \mu_w, f(z + w) \rangle \in O(D).$$

Here we note that following some authors (see, e.g., [18, Chapter 9, (9.1)]), we prefer not to use the reflection of $K$ with respect to the origin. This leads only to a sign change in the usual formula for convolution.

Let $\hat{\mu}(\zeta) := \langle \mu_z, e^{i\zeta} \rangle$, $\zeta \in \mathbb{C}$, be the Fourier–Borel (or Laplace) transformation of $\mu$ and $N_{\hat{\mu}}$ the zeros set of $\hat{\mu}$. Denote by $V^\infty_{\hat{\mu}}$ the projection on the unit circle $S := \{ \zeta : |\zeta| = 1 \}$ of zero accumulation points of $\hat{\mu}$, that is

$$V^\infty_{\hat{\mu}} := \left\{ \eta \in S : \exists \zeta_k \in N_{\hat{\mu}} \text{ with } |\zeta_k| \to \infty \text{ and } \frac{\zeta_k}{|\zeta_k|} \to \eta \text{ as } k \to \infty \right\}.$$

Obviously, $V^\infty_{\hat{\mu}}$ is a closed subset of $S$ and its complement is represented as a union of either finite or countable number of open arcs

$$W_{\hat{\mu}} := (V^\infty_{\hat{\mu}})^c = \bigcup_{k \in I} (e^{i\alpha_k}, e^{i\beta_k}),$$

which are pairwise disjoint.

As it follows from [15, Proposition 2.6], provided that the radial indicator of the entire function of exponential type $\hat{\mu}$ coincides with the supporting function of $K$ and $\hat{\mu}$ has a completely regular growth in $\mathbb{C}$, each solution $f \in O(D + K)$ of the homogeneous convolution equation $\mu * f = 0$ can be analytically continued into $D_I + K$, where

$$D_I := \bigcap_{\zeta \notin W_{\hat{\mu}}} \left\{ z \in \mathbb{C} : \Re z \zeta < H_D(\zeta) \right\}.$$

It should be noted that the problem of analytic continuation of the solutions of homogeneous linear partial differential equations, as well as of convolution equations has been studied by many authors, such as C.A. Berenstein, R. Gay and A. Vidras [6], J.M. Bony and P. Schapira [7], T. Kawai [13], C.O. Kiselman [14], A. Sébbar [23], M. Zerner [25], etc. for the first equations, and I.F. Krasičkov-Ternovskiǐ [15], A.S. Krivosheev [16], R. Ishimura and Y. Okada [12], etc. for the second ones.

In particular, for the convolution operator with the hyperfunction kernel of compact support case in [12] it was proved that the analytic solutions of a homogeneous equation are extended to the set determined by the directions to which the zeros of the symbol, i.e. the Fourier–Laplace transformation of the kernel, accumulate at infinity. The problem of extension for convolution equations in a more general framework has been studied in [16].

Our main goal is to study a similar problem of extension for spaces $A^{-\infty}$ of holomorphic functions in convex domains of the complex plane with polynomial growth near the boundary. So far as we know, this topic is never treated before.
Recall that the space $A^{-\infty}(D)$ is defined as
\[
A^{-\infty}(D) := \left\{ f \in \mathcal{O}(D) : \exists p > 0, \sup_{z \in D} |f(z)|^p < +\infty \right\}
\]
and endowed with its natural inductive limit topology, where $d_D(z) := \inf_{w \in \partial D} |w - z|$ is the distance from $z \in D$ to the boundary $\partial D$.

By [3, Lemma 2.1] (see also [4, Proposition 2.1]), the operator $\mu \ast$ acts from $A^{-\infty}(D + K)$ into $A^{-\infty}(D)$ if and only if $\hat{\mu}$ belongs to the space
\[
A_K^+ := \left\{ g \in \mathcal{O}(\mathbb{C}) : \sup_{\xi \in \mathbb{C}} \frac{|g(\xi)|}{(1 + |\xi|)^s e^{H_K(\xi)}} < +\infty, \text{ for some } k \in \mathbb{N} \right\}.
\]
Here and below $H_M(\xi) := \sup_{\zeta \in M} \Re \zeta, \xi \in \mathbb{C}$, denotes the supporting function of a set $M$.

Recall also that an entire function $\sigma$ of exponential type is said to satisfy the condition $(S^a)$ (see [3]), if
\[
\exists s, N > 0 \forall \xi \in \mathbb{C}, |\xi| > N \exists \xi' \in \mathbb{C}, |\xi' - \xi| < \log(1 + |\xi|) : \log |\sigma(\xi')| \geq h_\sigma(\xi') - s \log(1 + |\xi'|),
\]
where $h_\sigma(\xi) := \lim_{t \to +\infty} \frac{\log \|\sigma(t \xi)\|}{t}$, the radial indicator of $\sigma$. Note that from [13] (see also [11]) it follows that if the function $\sigma$ satisfies condition $(S^a)$, then it has a completely regular growth in $\mathbb{C}$. It is clear that the converse is not true. Indeed, consider a subharmonic function $u$ in the complex plane of the form
\[
u(z) = \begin{cases} 0 & \text{for } |z| < 1, \\ |z| - \sqrt{|z|} & \text{for } |z| \geq 1. \end{cases}
\]
Then, by [24, Theorem 5], there exists an entire function $f$ for which $|\log |f(z)|| - u(z)| \leq C \log(1 + |z|)$ outside a countable set of disks with a finite sum of radii (here $C$ is an absolute constant). In this case, $f$ is an entire function of exponential type having a completely regular growth in $\mathbb{C}$ (its radial indicator equals $|z|$), which obviously does not satisfy $(S^a)$.

For any $J \subseteq I$, we denote
\[
W_{\hat{\mu}, J} := \bigcup_{k \in J} (e^{\imath \alpha_k}, e^{\imath \beta_k}), \quad G_J := \bigcup_{k \in J} \{ \xi \in \mathbb{C} : \alpha_k < \arg \xi < \beta_k \}
\]
and
\[
D_J := \bigcap_{\zeta \notin W_{\hat{\mu}, J}} \{ z \in \mathbb{C} : \Re z \xi < H_D(\xi) \}.
\]
Note that the convex domain $D_J$ is bounded if and only if $\beta_k - \alpha_k < \pi$ for each $k \in J$. Given nontrivial $\xi$ in $G_J$, we take $k$ so that $\alpha_k < \arg \xi < \beta_k$ and put $\delta(\xi) := \min\{\arg \xi - \alpha_k, \beta_k - \arg \xi\}$.

The main result of the present paper is stated as follows.

**Theorem 1.1.** Let $D$ be a bounded convex domain in $\mathbb{C}$ and $\mu$ an analytic functional in $\mathbb{C}$ with $\hat{\mu} \in A_K^{+\infty}$. Suppose that $h_{\hat{\mu}} = H_K$ and $\mu$ satisfies condition $(S^a)$. If $D_J$ is bounded and
\[
\lim_{\xi \to +\infty} \sup_{\xi \notin G_J \setminus \partial G_J} \frac{|\xi| \delta(\xi)}{\log |\xi|} < +\infty, \tag{1.1}
\]
then every $A^{-\infty}$-solution $f \in A^{-\infty}(D + K)$ of the homogeneous convolution equation $\mu \ast f = 0$ is continued as $A^{-\infty}$-solution to $D_J + K$. 

Remark 1.2. Obviously, Theorem 1.1 has sense only when $D_J \supsetneq D$. In this case, as will be seen in the last section of the paper, condition (1.1) is also necessary for the extension.

Remark 1.3. We remark that for any convex bounded domain $D$, the space $A^{-\infty}(D)$ is nothing but the space of holomorphic functions on $D$ which allow an extension as distributions across the boundary $\partial D$ to a neighbourhood of $D$ ([20, Chapter III, Proposition 2]; see also [5, Proposition 4.3]). Then our continuation problem in Theorem 1.1 is also understood as follows: Let $S$ and $T$ be any distributions with compact support in $\mathbb{C}$ such that $S|_{D+K} = f$, $T|_{D} = 0$ and we have $\mu * S = T$ as analytic functionals in $D$. The question is to prove that we can take distributions $S$ and $T$ so that $S$ is holomorphic in the larger set $D_j + K$ and supp $T \subset \mathbb{C}D_j$.

The structure of the present paper is as follows. Section 2 is devoted to the criterion of extension of solutions in terms of the problem of division with a remainder. The equivalent assertions of this criterion are applied in the sections that follow. The main part of the paper is Section 3, where some sufficient conditions for validity of the criterion are obtained. Note that our method presented in this section is based on solving of the appropriate $\bar{\partial}$-problem satisfying not a single, but a family of weight estimations. This method is quite different from those of other works on the same topic. It allows us to prove the main result of the paper, Theorem 1.1, at the end of Section 3. In Section 4 we study the necessity of condition (1.1). We prove that for a reasonable case this condition is also necessary for the validity of Theorem 1.1.

2. Criterion of extension

As is well known, for spaces of all holomorphic functions in a domain the extension theorems are closely connected with the theorems of division with a remainder. The equivalent assertions of this criterion are applied in the sections that follow. The main part of the paper is Section 3, where some sufficient conditions for validity of the criterion are obtained. Note that our method presented in this section is based on solving of the appropriate $\bar{\partial}$-problem satisfying not a single, but a family of weight estimations. This method is quite different from those of other works on the same topic. It allows us to prove the main result of the paper, Theorem 1.1, at the end of Section 3. In Section 4 we study the necessity of condition (1.1). We prove that for a reasonable case this condition is also necessary for the validity of Theorem 1.1.

Assuming that $\hat{\mu} \in A_K^{+\infty}$, we denote by $Z^{\infty}\mu(D+K)$ the kernel of the convolution operator $\mu *: A^{-\infty}(D+K) \to A^{-\infty}(D)$ endowed with the induced topology from $A^{-\infty}(D+K)$. It is obvious that if $\lambda$ is a zero point of $\hat{\mu}$ with multiplicity $k$, then $\zeta^k e^{\lambda \zeta} \in Z^{\infty}\mu(D+K)$ for each $\ell = 0, 1, \ldots, k - 1$.

Let $\mathcal{N}_{\hat{\mu}} = \{\lambda_j\}$ and $k_j$ be the multiplicity of $\lambda_j$. We see that the set

$$E_{\mu} := \left\{ f(z) = \sum_{j=1}^{m} P_j(z) e^{\lambda_j z}, \ P_j \text{ is a polynomial, } \deg P_j \leq k_j - 1, \ m \in \mathbb{N} \right\}$$
is always contained in $\mathcal{Z}_\mu^{-\infty}(D + K)$. Following the case of holomorphic functions in domains, we say that $\mathcal{Z}_\mu^{-\infty}(D + K)$ admits spectral synthesis if the space $E_\mu$ is dense in $\mathcal{Z}_\mu^{-\infty}(D + K)$.

**Lemma 2.1.** Suppose that $\hat{\mu} \in A_k^{+\infty}$, $h_{\hat{\mu}} = H_K$ and $\hat{\mu}$ satisfies condition $(S^\mu)$. Then the kernel $\mathcal{Z}_\mu^{-\infty}(D + K)$ admits spectral synthesis.

**Proof.** Let $v \in (A^{-\infty}(D + K))^\prime$ and $v = 0$ on $E_\mu$. Since

$$\hat{v}^{(\ell)}(\lambda) = \langle v_z, e^{\lambda z} \rangle, \quad \text{for all } \ell \geq 0 \text{ and } \lambda \in \mathbb{C},$$

we see that $\hat{v}^{(\ell)}(\lambda_j) = 0$ for every $\lambda_j \in \mathcal{N}_\mu$ and $\ell = 0, 1, \ldots, k_j - 1$. Then $\hat{v}/\hat{\mu}$ is an entire function and, by [3, Lemma 2.4] (see also [4]), $\hat{v}/\hat{\mu} \in H_D^{-\infty}$.

By the above-mentioned duality, we can find an analytic functional $\varphi \in (A^{-\infty}(D))^\prime$ such that $\hat{\varphi} = \hat{v}/\hat{\mu}$. Note that for every $\lambda \in \mathbb{C}$ there always holds

$$\langle v_z, e^{\lambda z} \rangle = \hat{v}(\lambda) = \hat{\varphi}(\lambda)$$

Furthermore, the system $\mathcal{E} := \{e^{\lambda z} : \lambda \in \mathbb{C}\}$ is complete in $A^{-\infty}(D + K)$ (see [2, Proposition 2.3]), and so

$$v(f) = \langle \varphi, \mu \ast f \rangle, \quad \forall f \in A^{-\infty}(D + K).$$

Therefore,

$$v(g) = \langle \varphi, \mu \ast g \rangle = 0, \quad \forall g \in \mathcal{Z}_\mu^{-\infty}(D + K).$$

It remains to apply the Hahn–Banach theorem to complete the proof. □

**Proposition 2.2.** Let $\hat{\mu}$ be as in Lemma 2.1 and $G \supset D$ a bounded convex domain in $\mathbb{C}$. The following two assertions are equivalent:

1. Each $f \in \mathcal{Z}_\mu^{-\infty}(D + K)$ can be extended as $A^{-\infty}$-solution to $G + K$.
2. Each $p \in H_{G + K}^{-\infty}$ can be represented in the form $p = \hat{\mu}q + r$ with some $q \in H_G^{-\infty}$ and $r \in H_D^{-\infty} + K$.

**Proof.** Consider the linear continuous restriction mapping $R : A^{-\infty}(G + K) \to A^{-\infty}(D + K)$. By the uniqueness theorem for holomorphic functions, this mapping is injective. It is clear that condition (1) is equivalent to the surjectivity of $R : \mathcal{Z}_\mu^{-\infty}(G + K) \to \mathcal{Z}_\mu^{-\infty}(D + K)$.

Note that both $A^{-\infty}(G + K)$ and $A^{-\infty}(D + K)$ are (DFS)-spaces, and so $\mathcal{Z}_\mu^{-\infty}(G + K)$ and $\mathcal{Z}_\mu^{-\infty}(D + K)$, as their closed subspaces, have the same topological structure. Then by the open mapping theorem, each linear continuous operator from $\mathcal{Z}_\mu^{-\infty}(G + K)$ onto $\mathcal{Z}_\mu^{-\infty}(D + K)$ is open. So we can conclude that condition (1) holds if and only if $R : \mathcal{Z}_\mu^{-\infty}(G + K) \to \mathcal{Z}_\mu^{-\infty}(D + K)$ is a topological isomorphism.

Note again that the spaces $\mathcal{Z}_\mu^{-\infty}(G + K)$ and $\mathcal{Z}_\mu^{-\infty}(D + K)$, as (DFS)-spaces, are the strong dual of the (FS)-spaces, and moreover, each (FS)-space is a reflexive Fréchet space, applying [8, Corollary 8.6.18], we get that condition (1) is equivalent to the fact that the conjugate operator $R' : (\mathcal{Z}_\mu^{-\infty}(D + K))^\prime \to (\mathcal{Z}_\mu^{-\infty}(G + K))^\prime$ is bijective. In addition, by the same reasons about (DFS)-spaces, the following natural isomorphisms hold

$$(\mathcal{Z}_\mu^{-\infty}(G + K))^\prime \simeq (A^{-\infty}(G + K))^\prime / (\mathcal{Z}_\mu^{-\infty}(G + K))^\circ$$
and
\[(Z_{\mu}^{-\infty}(D + K))^\prime \simeq (A^{-\infty}(D + K))^\prime/(Z_{\mu}^{-\infty}(D + K))^\circ,\]
where \((Z_{\mu}^{-\infty}(G + K))^\circ\) and \((Z_{\mu}^{-\infty}(D + K))^\circ\) are the polar sets of \(Z_{\mu}^{-\infty}(G + K)\) and \(Z_{\mu}^{-\infty}(D + K)\), respectively.

So condition (1) is equivalent to the following statement: for each \(v \in (A^{-\infty}(G + K))^\prime\) there exists a unique \([\varphi] \in (A^{-\infty}(D + K))^\prime/(Z_{\mu}^{-\infty}(D + K))^\circ\) such that
\[
\langle v, g \rangle = \langle \varphi, Rg \rangle, \quad \forall g \in Z_{\mu}^{-\infty}(G + K).
\]

In its turn, this condition holds, due to Lemma 2.1, if and only if
\[
\hat{\varphi}^{(\ell)}(\lambda_j) = \{v_z, z^{\ell} e^{\lambda_j z}\} = \{\varphi_z, z^{\ell} e^{\lambda_j z}\} = \hat{\varphi}^{(\ell)}(\lambda_j), \quad 0 \leq \ell \leq k_j - 1, \ \forall \lambda_j \in \mathcal{N}_{\hat{\mu}}.
\]

Next notice that by Lemma 2.1 and [22, Proposition 4], \((Z_{\mu}^{-\infty}(D + K))^\circ\) can be identified, via the Fourier–Borel transformation, with
\[
\mathcal{I}_{D+K}^{-\infty} := \{g \in H_{D+K}^{-\infty}: \ g^{(\ell)}(\lambda_j) = 0, \ 0 \leq \ell \leq k_j - 1, \ \forall \lambda_j \in \mathcal{N}_{\hat{\mu}}\}.
\]

Therefore the Fourier–Borel transformation
\[
[\varphi] \in (A^{-\infty}(D + K))^\prime/(Z_{\mu}^{-\infty}(D + K))^\circ \mapsto [\hat{\varphi}] \in H_{D+K}^{-\infty}/\mathcal{I}_{D+K}^{-\infty}
\]
is well-defined and isomorphic. The same is valid if we replace \(D\) with \(G\).

From all discussions said above we arrive to the fact that condition (1) holds if and only if for each function \(p \in H_{G+K}\) there exists a unique \([r] \in H_{D+K}^{-\infty}/\mathcal{I}_{D+K}^{-\infty}\) such that
\[
p^{(\ell)}(\lambda_j) - r^{(\ell)}(\lambda_j) = 0, \quad 0 \leq \ell \leq k_j - 1, \ \forall \lambda_j \in \mathcal{N}_{\hat{\mu}}.
\]

This fact is equivalent to that \((p - r)/\hat{\mu}\) is an entire function. In its turn, this holds, by [3, Lemma 2.4] (see also [4, Proposition 3.4]), if and only if \(p - r = \hat{\mu}q\), where \(q \in H_{G}^{-\infty}\).

It remains to notice that if \(q_1 \in H_{G}^{-\infty}\) and \(r_1 \in H_{D+K}^{-\infty}\) are some other functions for which \(p = \hat{\mu}q_1 + r_1\), then \(r - r_1 = \hat{\mu}(q_1 - q)\). Consequently,
\[
r^{(\ell)}(\lambda_j) - r_1^{(\ell)}(\lambda_j) = 0, \quad 0 \leq \ell \leq k_j - 1, \ \forall \lambda_j \in \mathcal{N}_{\hat{\mu}}.
\]

From this and \(r - r_1 \in H_{D+K}^{-\infty}\) it follows that \(r - r_1 \in \mathcal{I}_{D+K}^{-\infty}\). This implies \([r_1] = [r]\), which shows the uniqueness of \([r]\). The proposition is proved completely. \(\square\)

3. Division with a remainder and extension

In this section we establish some sufficient conditions under which the statement of type (2) in Proposition 2.2 holds. Following the method of [12], we reduce our study to appropriate \(\bar{\partial}\)-problem and use the well-known Hörmander’s type result to solve this problem in our weight classes.

The difference from [12], as well as from other papers on the same topic (see, e.g., [16,17]), and the main difficulty is that we need a solution for the \(\bar{\partial}\)-problem satisfying not a single, but a family of weight estimation. The lemma below helps us to overcome this difficulty.

Denote by \(W_{D}^{-\infty}\) the set of all measurable functions \(w : \mathbb{C} \to \mathbb{C}\) satisfying the condition
\[
\forall m \exists C_m > 0: \ |w(\xi)| \leq H_D(\xi) - m \log(1 + |\xi|) + C_m, \quad \forall \xi \in \mathbb{C}.
\]
We identify \(\mathbb{C}\) with \(\mathbb{R}^2\) and consider the Lebesgue measure \(\lambda\) in \(\mathbb{C} \simeq \mathbb{R}^2\). Notice that \(H_{D}^{-\infty} = W_{D}^{-\infty} \cap \mathcal{O}(\mathbb{C})\).
Lemma 3.1. We have $\bar{\partial}W^{-\infty}_D \supset W^{-\infty}_D$, that is for each $w \in W^{-\infty}_D$ there always exists $v \in W^{-\infty}_D$ such that $\frac{\partial v}{\partial \zeta} = w$ in the weak sense.

Proof. We will use [9, Theorem 3] which is a version of the well-known Hörmander theorem (see [10, Theorem 4.4.2]) on a solvability of $\bar{\partial}$-equation satisfying a countable number of weighted estimates. In [9], there were obtained some necessary and sufficient conditions on weights under which $\bar{\partial}$-equation is solvable in the corresponding projective limit of weighted Banach spaces. Our proof is only checking of these conditions.

By $M(C)$ denote the set of all functions $\phi$ that are represented in the form $\phi(\zeta) = \sup \{|f(\zeta)|: f \in F\}$, $\zeta \in C$ with some locally bounded in $C$ family $F$ of entire functions.

W.l.o.g. we can assume that $0 \in D$. For each $m \in \mathbb{N}$ define $H_{D,m}(\zeta) := \sup_{z \in D} (\text{Re} \zeta z + m \log d_D(z))$, $\zeta \in C$.

Obviously, $v_m(\zeta) := e^{H_{D,m}(\zeta)} = \sup_{z \in D} \{|e^{\zeta z}[d_D(z)]^m\}$, $\zeta \in C$, $m \in \mathbb{N}$, and hence $v_m \in M(C)$ for every $m \in \mathbb{N}$.

As shown in the proof of [2, Lemma 2.2],

$$c_m \leq H_{D,m}(\zeta) - H_D(\zeta) + m \log (1 + |\zeta|) \leq C_m, \quad \forall \zeta \in C, \ m \in \mathbb{N},$$

(3.1)

where $c_m := m \log \min\{\frac{d_D}{e}, \frac{m d_D}{e R_D}\}$, $C_m := m \log \frac{m}{e} + R_D$, $R_D := \max_{\zeta \in \partial D} |\zeta|$.

Also note, by the proof of [4, Proposition 2.5], that the space $H^{-\infty}_D$ is dense in $A^{-m}_D$ with respect to the norm $|\cdot|_{m-1}$.

Let

$$d_1 := 1, \quad d_m := \frac{1}{2^{m-1}} \exp \left( \sum_{k=1}^{m-1} c_k - \sum_{k=2}^{m} C_k \right), \quad m \geq 2.$$

Consider the functions $k_m(\zeta) := d_m v_m(\zeta) \in M(C)$. By (3.1) we have

$$W^{-\infty}_D = \{ w: w: C \rightarrow C \text{ is measurable, } w(\zeta) = O(k_m(\zeta)) \text{ in } C, \forall m \in \mathbb{N} \},$$

$$H^{-m}_D = \left\{ g \in \mathcal{O}(C): |g|_m = \sup_{\zeta \in C} \frac{|g(\zeta)|(1 + |\zeta|)^m}{e^{H_D(\zeta)}} < \infty \right\},$$

and

$$e^{c_m} |g|_m \leq |g|_m \leq e^{C_m} |g|_m, \quad \forall g \in \mathcal{O}(C), \ m \in \mathbb{N},$$

which shows that $A^{-\infty}_D$ is dense in $A^{-m}_D$ with respect to the norm $|\cdot|_{m-1}$, and, in addition,

$$\frac{k_{m+1}(\zeta)}{k_m(\zeta)} = \frac{1}{2} e^{c_m} \frac{v_{m+1}(\zeta)}{v_m(\zeta)} \leq \frac{1}{2(1 + |\zeta|)}, \quad \forall \zeta \in C.$$

Applying [9, Theorem 3, condition 1)] we obtained the desired result. \qed
Lemma 3.2. Let $\sigma$ be an entire function of exponential type with indicator $h_\sigma$, satisfying condition $(S^\alpha)$ and

$$\exists n \exists A: \log|\sigma(\zeta)| \leq h_\sigma(\zeta) + n \log(1 + |\zeta|) + A, \quad \forall \zeta \in \mathbb{C}. \quad (3.2)$$

There exist $s_0$ and $N_0 > 0$ such that if $\xi \in \mathbb{C}$, $|\xi| > N_0$, and the disk

$$U_\xi := \{ w \in \mathbb{C}: |w - \xi| \leq 3 \log(1 + |\xi|) \}$$

does not contain any zero of $\sigma$, then

$$\log|\sigma(\xi)| \geq h_\sigma(\xi) - s_0 \log(1 + |\xi|).$$

Proof. Let $s, N$ be chosen by condition $(S^\alpha)$. For $\xi \in \mathbb{C}$ with $|\xi| > N_0 = N$, take $\xi'$ with $|\xi' - \xi| < \log(1 + |\xi|)$ and

$$\log|\sigma(\xi')| \geq h_\sigma(\xi) - s \log(1 + |\xi|). \quad (3.3)$$

Denote

$$K_1 := \{ w: |w - \xi| \leq \log(1 + |\xi|) \}, \quad K_2 := \{ w: |w - \xi'| \leq 2 \log(1 + |\xi|) \}.$$

It is easy to see that $K_1 \subset K_2 \subset U_\xi$. Note that, for $w \in U_\xi$,

$$\log(1 + |w|) \leq \log(1 + |\xi|) + \log \left(1 + \frac{|w - \xi|}{1 + |\xi|}\right) \leq \log(1 + |\xi|) + \log 4$$

and

$$|h_\sigma(w) - h_\sigma(\xi)| \leq \Delta_\sigma |w - \xi| \leq 3 \Delta_\sigma \log(1 + |\xi|), \quad \text{where } \Delta_\sigma := \max_{|\xi| \leq 1} h_\sigma(\xi).$$

By (3.2), we get

$$\log|\sigma(w)| \leq h_\sigma(\xi) + (3\Delta_\sigma + n) \log(1 + |\xi|) + A + n \log 4, \quad \forall w \in U_\xi.$$ 

Since $\sigma(w) \neq 0$ in $U_\xi$, $\log|\sigma(\xi)|$ is a harmonic function in some neighbourhood of $U_\xi$. Combining this, (3.3) and the last estimate yields

$$h_\sigma(\xi) - s \log(1 + |\xi|)$$

$$\leq \log|\sigma(\xi')| = \frac{1}{4\pi \log^2(1 + |\xi|)} \int_{K_1} \log|\sigma(w)|d\lambda_w$$

$$= \frac{1}{4\pi \log^2(1 + |\xi|)} \int_{K_1} \log|\sigma(w)|d\lambda_w + \frac{1}{4\pi \log^2(1 + |\xi|)} \int_{K_2 \setminus K_1} \log|\sigma(w)|d\lambda_w$$

$$\leq \frac{1}{4} \log|\sigma(\xi)| + \frac{1}{4\pi \log^2(1 + |\xi|)}$$

$$\times \left[h_\sigma(\xi) + (3\Delta_\sigma + n) \log(1 + |\xi|) + A + n \log 4\right] \lambda(K_2 \setminus K_1)$$

$$= \frac{1}{4} \log|\sigma(\xi)| + \frac{3}{4} \left[h_\sigma(\xi) + (3\Delta_\sigma + n) \log(1 + |\xi|) + A + n \log 4\right].$$

Consequently,

$$\log|\sigma(\xi)| \geq h_\sigma(\xi) - (4s + 9\Delta_\sigma + 3n) \log(1 + |\xi|) - 3(A + n \log 4).$$
Taking
\[ s_0 \geq 4s + 9\Delta + 3n + 3\frac{A + n\log 4}{\log(1 + N)}, \]
which is independent of \( \xi \), we arrive to the desired estimation. \( \square \)

Now we are ready to prove our main result.

**Proof of Theorem 1.1.** We will make use of Proposition 2.2.

Let \( p \in H^∞_{D_j + K} \). Consider a function \( \phi \in C^∞(\mathbb{C}) \) with \( 0 \leq \phi(\xi) \leq 1 \) for all \( \xi \in \mathbb{C} \),
\[
\phi(\xi) = \begin{cases} 1, & |\xi| \leq \frac{1}{2}, \\ 0, & |\xi| \geq 1, \end{cases}
\]
and let
\[
\tilde{\phi}(\xi) := \frac{1}{c_0} \phi(\xi), \quad \text{where} \quad c_0 := \int_{\mathbb{C}} \phi(\xi) d\lambda_\xi.
\]

Setting \( H^*_{K} := \tilde{\phi} * H_K \), we note that \( H^*_{K} \in C^∞(\mathbb{C}) \) and, for any \( \xi \in \mathbb{C} \),
\[
\left| H^*_{K}(\xi) - H_K(\xi) \right| \leq \int_{|\eta| \leq 1} \left| H_K(\xi - \eta) - H_K(\xi) \right| \tilde{\phi}(\eta) d\lambda_\eta \leq R_K := \max_{\xi \in K} H_K(\xi).
\]

This makes it possible to replace \( H_K \) with \( H^*_{K} \). Moreover, since
\[
\log(1 + |\xi|) \leq \log(1 + |\xi|^2) \leq 2\log(1 + |\xi|), \quad \forall |\xi| \geq 1,
\]
we can use \( \log(1 + |\xi|^2) \) instead of \( \log(1 + |\xi|) \), when it is needed.

By the assumptions of the theorem, \( \hat{\mu} \in A^∞_{K}, \) \( h_\hat{\mu} = H_K \) and \( \hat{\mu} \) satisfies \( S^a \), we have
\[
\begin{align*}
\exists n & \geq 1, \quad A > 0: \quad \log|\hat{\mu}(\xi)| \leq H^*_{K}(\xi) + n\log(1 + |\xi|^2) + A, \quad \forall \xi \in \mathbb{C}; \\
\exists s > 0, \quad N > 0 & \forall \xi \in \mathbb{C}, \quad |\xi| > N \exists \xi' \in \mathbb{C}, \quad |\xi' - \xi| < \log(1 + |\xi|): \\
\log|\hat{\mu}(\xi')| & \geq H^*_{K}(\xi) - s\log(1 + |\xi|^2). \quad (3.4)
\end{align*}
\]

Then, by Lemma 3.2, we can choose \( s_0 \) and \( N_0 \) so that, if \( |\xi| > N_0 \) and the disk
\[
U_\xi := \{ w \in \mathbb{C}: \ |w - \xi| \leq 3\log(1 + |\xi|) \}
\]
do not contain any zero of \( \hat{\mu} \), then
\[
\log|\hat{\mu}(\xi)| \geq H^*_{K}(\xi) - s_0\log(1 + |\xi|^2). \quad (3.5)
\]

Furthermore, from (1.1) it follows that there exist \( s_1 \) and \( N_1 \) such that
\[
\delta(\xi) \leq s_1 \frac{\log(1 + |\xi|)}{|\xi|}, \quad \text{for all} \ \xi \in \mathcal{N}_\hat{\mu} \cap \Gamma_j, \ |\xi| > N_1. \quad (3.6)
\]

Put \( \ell := s_0 + s, \ L := N + N_0 + N_1, \) \( \alpha(\xi) := \ell \log(1 + |\xi|^2) (\xi \in \mathbb{C}), \) and construct functions \( q \) and \( r \) as follows:
\[
\begin{align*}
q(\xi) & := \frac{p(\xi)}{\hat{\mu}(\xi)} (1 - \phi(\tau(\xi))) + v(\xi), \\
r(\xi) & := p(\xi)\phi(\tau(\xi)) - \hat{\mu}(\xi)v(\xi),
\end{align*}
\]
where \( \tau(\xi) := e^{\alpha(\xi) - H^*_{K}(\xi)}\hat{\mu}(\xi) \) and the function \( v \) will be chosen a bit later.
Obviously, $p = \hat{\mu}q + r$ and $q, r$ are well-defined in $\mathbb{C}$. The condition for the functions $q$ and $r$ to be entire is given by the equality $\frac{\partial w}{\partial \xi} = w$, where

$$w(\xi) = \frac{p(\xi)}{\hat{\mu}(\xi)} e^{\alpha(\xi) - H_K^*(\xi)} \left[ \left( \frac{\partial \phi}{\partial \tau}(\tau(\xi))\hat{\mu}(\xi) + \frac{\partial \phi}{\partial \xi}(\tau(\xi))\frac{\partial \hat{\mu}}{\partial \xi}(\xi) \right) \right] \cdot$$

$$\times \left( \ell + \frac{\xi}{1 + \xi^2} - \frac{\partial H_K^*(\xi)}{\partial \xi} + \frac{\partial \phi}{\partial \xi}(\tau(\xi))\frac{\partial \hat{\mu}}{\partial \xi}(\xi) \right].$$

Let’s now estimate $|w(\xi)|$.

(1) If either $|\hat{\mu}(\xi)| < \frac{1}{2} e^{H_K^*(\xi) - \alpha(\xi)}$, or $|\hat{\mu}(\xi)| > e^{H_K^*(\xi) - \alpha(\xi)}$, then $\phi(\tau(\xi))$ is constant there and hence, $w(\xi) = 0$.

(2) Let now $\frac{1}{2} e^{H_K^*(\xi) - \alpha(\xi)} \leq |\hat{\mu}(\xi)| \leq e^{H_K^*(\xi) - \alpha(\xi)}.$

Denote $M_1 := \max_{\tau \in \mathbb{C}} (|\frac{\partial \phi}{\partial \tau}(\tau)| + |\frac{\partial \phi}{\partial \xi}(\tau)|)$ and notice that

$$\left| \frac{\partial \phi}{\partial \tau}(\tau) \right| = \frac{1}{c_0} \left| \frac{\partial \phi}{\partial \tau}(\tau) \right| \leq \frac{M_1}{c_0}, \quad \forall \tau \in \mathbb{C},$$

from

$$\frac{\partial H_K^*}{\partial \xi}(\xi) = \int_{\mathbb{C}} H_K(\eta) \frac{\partial \phi}{\partial \xi}(\xi - \eta) d\lambda_\eta = \int_{|\eta| \leq 1} H_K(\xi - \eta) \frac{\partial \phi}{\partial \xi}(\eta) d\lambda_\eta,$$

it follows that

$$\left| \frac{\partial H_K^*}{\partial \xi}(\xi) \right| \leq \frac{M_1}{c_0} \int_{|\eta| \leq 1} H_K(\xi - \eta) d\lambda_\eta \leq M_2 \left( 1 + \xi^2 \right), \quad \forall \xi \in \mathbb{C},$$

where $M_2 := 2\pi M_1 R_K c_0^{-1}$.

Furthermore, by the Cauchy integral formula, from (3.4) we get

$$\log \left| \frac{\partial \hat{\mu}}{\partial \xi}(\xi) \right| \leq \log \max_{|\eta| \leq 1} |\hat{\mu}(\xi + \eta)| \leq \max_{|\eta| \leq 1} H_K^*(\xi + \eta) + n \log(2 + |\xi^2|) + A$$

$$\leq H_K^*(\xi) + n \log(1 + |\xi^2|) + A_0,$$

where $A_0 := 2R_K + n \log 2 + A$. Also since $p \in H_{D_{\xi} + K}^{-\infty}$,

$$\forall m \exists C_m: \quad \log|p(\xi)| \leq H_{D_{\xi} + K}(\xi) - m \log(1 + |\xi^2|) + C_m, \quad \forall \xi \in \mathbb{C}. \quad (3.8)$$

From all these estimations we have, for every $m \in \mathbb{N}$,

$$\log|w(\xi)| \leq H_{D_{\xi} + K}(\xi) - m \log(1 + |\xi^2|) + C_m + \log 2 + 2(\alpha(\xi) - H_K^*(\xi))$$

$$+ \log \left[ M_1 e^{H_K^*(\xi) - \alpha(\xi)} \left( \frac{\ell}{2} + M_2 \left( 1 + |\xi^2| \right) \right) + \frac{M_1}{c_0} e^{H_K^*(\xi) + n \log(1 + |\xi^2|) + A_0} \right]$$

$$\leq H_{D_{\xi}}(\xi) - (m - 2\ell - n) \log(1 + |\xi^2|) + \tilde{C}_m,$$

where $\tilde{C}_m := C_m + R_K + \log M_1 + \log(2M_2 + \ell + 2/c_0) + A_0$.

Notice that if $\xi \notin \Gamma_f$, then $H_{D_{\xi}}(\xi) = H_D(\xi)$. Let now $\xi \in \Gamma_f$ and $|\xi| > L$. Then there exists a unique $k$ such that $\alpha_k < \arg \xi < \beta_k$. Consider three possibilities:
The first case, if
\[ \alpha_k + (s_1 + 4) \frac{\log(1 + |\zeta|)}{|\zeta|} \leq \arg \zeta \leq \beta_k - (s_1 + 4) \frac{\log(1 + |\zeta|)}{|\zeta|}, \]
then, by (3.6), the disk \( U_\varepsilon := \{ w: |w - \zeta| < 3 \log(1 + |\zeta|) \} \) does not contain any zero of \( \hat{\mu} \). This implies, by (3.5), that \( |\tau(\zeta)\hat{\mu}(\zeta)| > 1 \), and hence, \( w(\zeta) = 0 \).

The second case, if
\[ \alpha_k < \arg \zeta < \alpha_k + (s_1 + 4) \frac{\log(1 + |\zeta|)}{|\zeta|}, \]
then, denoting \( \Delta_{D_j} := \sup_{z \in D_j} |z| \), we get
\[
H_{D_j}(\zeta) = H_{D_j}(\zeta|e^{i\arg \zeta}|) \leq H_{D_j}(\zeta|e^{i\alpha_k}| + \Delta_{D_j}|\arg \zeta - \alpha_k|)
\leq H_D(\zeta) + \Delta_{D_j}|\arg \zeta - \alpha_k| + \Delta_{D_j} \log(1 + |\zeta|).
\]

The third case, if
\[ \beta_k - (s_1 + 4) \frac{\log(1 + |\zeta|)}{|\zeta|} < \arg \zeta < \beta_k, \]
then, by a similar argument, the same estimation is valid for all such \( \zeta \).

So we can state that
\[ \forall m \exists E_m > 0: \log |w(\zeta)| \leq H_D(\zeta) - m \log(1 + |\zeta|^2) + E_m, \forall \zeta \in \mathbb{C}. \tag{3.9} \]

This is the desired estimation for \( |w(\zeta)| \) we need in the sequel.

Similarly, we have
\[ \forall m \exists F_m > 0: \log |v(\zeta)| \leq H_D(\zeta) - m \log(1 + |\zeta|^2) + F_m, \forall \zeta \in \mathbb{C}. \tag{3.10} \]

By Lemma 3.1, from (3.9) it follows that there exists a solution \( v \) of the equation \( \frac{\partial v}{\partial \zeta} = w \) such that
\[ \forall m \exists B_m > 0: \log |v(\zeta)| \leq H_D(\zeta) - m \log(1 + |\zeta|^2) + B_m, \forall \zeta \in \mathbb{C}. \tag{3.11} \]

Furthermore, we have
\[ |q(\zeta)| \leq |v(\zeta)| + \left| \frac{\rho(\zeta)}{\hat{\mu}(\zeta)} \right| \left| 1 - \varphi(\tau(\zeta)\hat{\mu}(\zeta)) \right|, \forall \zeta \in \mathbb{C}. \]

From this estimate we have that if \( |\tau(\zeta)\hat{\mu}(\zeta)| \leq \frac{1}{2} \), then \( \varphi(\tau(\zeta)\hat{\mu}(\zeta)) = 1 \), by the construction of \( \varphi \). This gives \( |q(\zeta)| \leq |v(\zeta)| \).

On the other hand, if \( |\tau(\zeta)\hat{\mu}(\zeta)| > \frac{1}{2} \), then using (3.8) and (3.11), we obtain that for every \( m \in \mathbb{N} \) and all \( \zeta \in \mathbb{C} \),
\[
|q(\zeta)| \leq |v(\zeta)| + \left| \frac{\rho(\zeta)}{\hat{\mu}(\zeta)} \right| \leq e^{H_D(\zeta) - m \log(1 + |\zeta|^2)} + B_m + 2e^{H_D(K) - m \log(1 + |\zeta|^2)} + C_m + \alpha(\zeta) - H_K(\zeta),
\]
where \( \tilde{B}_m := B_m + C_m + R_K + \log 3. \) Thus \( q \in \mathcal{H}^{-\infty}_{D_j}. \)
Next using (3.4), (3.10) and (3.11), we get that for every $m \in \mathbb{N}$ and all $\zeta \in \mathbb{C}$,
\[
|r(\zeta)| \leq |p(\zeta)|\varphi(\tau(\zeta)) + |\hat{\mu}(\zeta)||v(\zeta)|
\leq e^{H_{D+k}(\zeta)-m \log(1+|\zeta|^2) + F_m} + e^{H^*_K(\zeta)+n \log(1+|\zeta|^2) + A} e^{H_D(\zeta)-m \log(1+|\zeta|^2) + B_m}
\leq e^{H_{D+k}(\zeta)-(m-n) \log(1+|\zeta|^2) + \tilde{F}_m},
\]
where $\tilde{F}_m := F_m + B_m + R_K + \log 2$, which shows that $r \in H^{-\infty}_{D+K}$.

Applying Proposition 2.2 completes the proof of the theorem. \qed

4. Necessary conditions

In this section we study the necessity of condition (1.1) for an extension. We show that, in the reasonable case $D_J \supseteq D$, this condition is really, in some sense, necessary. Note that $D_J \not= D$ if and only if there exists $k \in J$ such that $D_{[k]} \not= D$.

So take and fix $k \in I$ with $\beta_k - \alpha_k < \pi$. For the sake of simplicity of the notations we denote $\alpha := \alpha_k$, $\beta := \beta_k$, $G := D_{[k]}$, and $\Gamma_0 := \{\zeta \in \mathbb{C}: \alpha < \arg \zeta < \beta\}$.

**Proposition 4.1.** Let $D$, $\mu$ be as in Theorem 1.1 and $\alpha$, $\beta$, $G$, $\Gamma_0$ as said above. If $G \supseteq D$ and
\[
\limsup_{\zeta \to \infty, \zeta \in \Gamma_0 \cap N_\mu} \frac{|\zeta|\delta(\zeta)}{\log |\zeta|} = \infty,
\]
then there always exists an $A^{-\infty}$-solution $f \in A^{-\infty}(D + K)$ of the homogeneous convolution equation $\mu * f = 0$ that cannot be continued as $A^{-\infty}$-solution to $G + K$.

**Proof.** As is well known, $H_D(e^{i\theta})$ is a trigonometrically convex function of $\theta$ on $[\alpha, \beta]$. Moreover, by the definition of $G$ as well as of $H_G$, being the biggest trigonometrically convex function on $[\alpha, \beta]$ that coincides with $H_D$ at $\alpha$ and $\beta$ and is trigonometrical on this interval, we have
\[
H_G(e^{i\theta}) = H_D(e^{i\alpha}) \frac{\sin(\beta - \theta)}{\sin(\beta - \alpha)} + H_D(e^{i\beta}) \frac{\sin(\theta - \alpha)}{\sin(\beta - \alpha)}, \quad \theta \in [\alpha, \beta].
\]

Here and below we refer the reader to [19, Lectures 8 and 9] for having information in detail about trigonometrical and trigonometrically convex functions, as well as the connection between those functions and convex sets, which is needed in this part of the proof.

By the first assumption $G \supseteq D$, we have $H_G(e^{i\theta}) > H_D(e^{i\theta})$ for all $\theta \in (\alpha, \beta)$. Then, for $\gamma := \frac{\alpha + \beta}{2}$,
\[
\epsilon_0 := H_G(e^{i\gamma}) - H_D(e^{i\gamma}) > 0.
\]

From this fact and the definition of $H_G$, by the standard trigonometrical formulas, it follows that, for $\theta \in (\alpha, \gamma)$,
\[
H_G(e^{i\theta}) = H_G(e^{i\alpha}) \frac{\sin(\gamma - \theta)}{\sin(\gamma - \alpha)} + H_G(e^{i\gamma}) \frac{\sin(\theta - \alpha)}{\sin(\gamma - \alpha)}
\]
\[
= H_D(e^{i\alpha}) \frac{\sin(\gamma - \theta)}{\sin(\gamma - \alpha)} + H_D(e^{i\gamma}) \frac{\sin(\theta - \alpha)}{\sin(\gamma - \alpha)} + \epsilon_0 \frac{\sin(\theta - \alpha)}{\sin(\gamma - \alpha)}
\]
\[
\geq H_D(e^{i\theta}) + \epsilon_0 \frac{\sin(\theta - \alpha)}{\sin(\gamma - \alpha)}.
\]
Similarly, for $\theta \in (\gamma, \beta)$,

$$
H_G(e^{i\theta}) = H_G(e^{i\gamma}) \frac{\sin(\beta - \theta)}{\sin(\beta - \gamma)} + H_G(e^{i\beta}) \frac{\sin(\theta - \gamma)}{\sin(\beta - \gamma)}
$$

$$
= H_D(e^{i\gamma}) \frac{\sin(\beta - \theta)}{\sin(\beta - \gamma)} + \varepsilon_0 \frac{\sin(\beta - \theta)}{\sin(\beta - \gamma)} + H_D(e^{i\beta}) \frac{\sin(\theta - \gamma)}{\sin(\beta - \gamma)}
$$

$$
\geq H_D(e^{i\theta}) + \varepsilon_0 \frac{\sin(\beta - \theta)}{\sin(\beta - \gamma)}.
$$

Combining these two estimates yields

$$
H_G(e^{i\theta}) \geq H_D(e^{i\theta}) + \varepsilon_1 \min\{\theta - \alpha, \beta - \theta\}, \quad \forall \theta \in (\alpha, \beta),
$$

where $\varepsilon_1 := \frac{2\varepsilon_0}{\pi \sin \frac{\beta - \gamma}{2}}$. Therefore,

$$
H_G(\xi) \geq H_D(\xi) + \varepsilon_1 |\xi| \delta(\xi), \quad \forall \xi \in \Gamma_0. \quad (4.2)
$$

Next, from the second assumption \((4.1)\) it follows that there exists a sequence \((\xi_n)_{n=1}^\infty \subset \Gamma_0 \cap \mathcal{N}_{\mu}\) such that

$$
0 < |\xi_n| \uparrow \infty \quad \text{and} \quad \delta(\xi_n) \geq n \frac{\log(1 + |\xi_n|)}{|\xi_n|}, \quad n \in \mathbb{N}. \quad (4.3)
$$

W.l.o.g. we can assume that $|\xi_{n+1}| - |\xi_n| > n^2$ for all $n \in \mathbb{N}$ and

$$
\lim_{n \to \infty} \frac{\log(1 + |\xi_n|)}{n} = \infty. \quad (4.4)
$$

Combining \((4.2)\) and \((4.3)\) yields

$$
H_G(\xi_n) \geq H_D(\xi_n) + \varepsilon_1 n \log(1 + |\xi_n|), \quad n \in \mathbb{N}. \quad (4.5)
$$

Now consider the function $w$ defined by

$$
\log w(\xi) := H_G(\xi) + H_K(\xi) - \varepsilon_1 n \log(1 + |\xi|),
$$

$$
|\xi_{n-1}| + n - 1 \leq |\xi| < |\xi_n| + n, \quad n \in \mathbb{N} \text{ (here } \xi_0 = 0). \text{ Obviously, } w \in W^\infty_G. \text{ By Lemma 3.1 and } [9, \text{Theorem 3, condition 3}], \text{ there exists a function } h \text{ such that } e^h \in \mathcal{M}(\mathbb{C}) \cap W^\infty_D \text{ and } \log w(\xi) \leq h(\xi) \text{ for all } \xi \in \mathbb{C}.
$$

From $e^h \in \mathcal{M}(\mathbb{C}) \cap W^\infty_D$ it follows that $h$ is subharmonic in $\mathbb{C}$ and

$$
\forall m \exists C_m > 0: \quad h(\xi) \leq H_G(\xi) + H_K(\xi) - m \log(1 + |\xi|) + C_m, \quad \forall \xi \in \mathbb{C}. \quad (4.6)
$$

Then, using \([24, \text{Theorem 5}],\) we can find an entire function $p_0$ with

$$
|\log p_0(\xi)| - h(\xi) \leq C \log(1 + |\xi|), \quad \xi \in \mathbb{C} \setminus E_0, \quad (4.7)
$$

where $C$ is some positive constant and $E_0$ is some set in $\mathbb{C}$, which can be covered by a sequence of disks $\{\zeta: |\zeta - z_k| \leq r_k\}$ with $\sum_{k=1}^\infty r_k < \infty$.

Let $R := (\sum_{k=1}^\infty r_k^2)^{1/2}$. Take the circle $U := \{\tau \in \mathbb{C}: |\tau| \leq 2R\}$ and consider its subsets

$$
U_n := \{\tau \in U: \xi_n + \tau \in E_0\}, \quad n \in \mathbb{N}.
$$

Choose $n_0 \in \mathbb{N}$ so large that $\xi_n + U \subset \Gamma_0$ and $|\xi_{n+1}| > |\xi_n| + 4R + n$, for all $n \geq n_0$. Then

$$
\bigcup_{n \geq n_0} (\xi_n + U_n) \subset E_0
$$
and the sets $\zeta_n + U_n, \ n \geq n_0$, are pairwise disjoint. In this case, as before, denoting by $\lambda$ the Lebesgue measure in $\mathbb{C} \cong \mathbb{R}^2$, we get

$$\lambda\left(\bigcup_{n \geq n_0} U_n\right) \leq \sum_{n \geq n_0} \lambda(U_n) = \lambda(\zeta_n + U_n) = \lambda\left(\bigcup_{n \geq n_0} (\zeta_n + U_n)\right) \leq \pi R^2.$$  

Therefore, there exists $\tau \in U$ such that $\zeta_n + \tau \notin E_0$ for all $n \geq n_0$.

Consider the function

$$p(\zeta) = p_0(\zeta + \tau).$$

Since $|\tau| \leq 2R$, for all $\zeta \in \mathbb{C}$ there hold

$$|H_G(\zeta + \tau) - H_G(\zeta)| \leq 2RR_G,$$
$$|H_K(\zeta + \tau) - H_K(\zeta)| \leq 2RR_K,$$
$$|\log(1 + |\zeta + \tau|) - \log(1 + |\zeta|)| \leq \log(1 + 2R),$$

where $R_G := \sup_{\zeta \in G} |\zeta|$, which together with (4.6) and (4.7) gives that $p \in H^{-\infty}_{G+k}$.

To finish our proof, we apply Proposition 2.2 by claiming that $p$ cannot be represented in the form $p = \hat{\mu}q + r$ with $q \in H^{-\infty}_D$ and $r \in H^{-\infty}_{D+k}$.

Expecting a contrary, we let the equation $p = \hat{\mu}q + r$ hold, which implies that

$$r(\zeta_n) = p(\zeta_n) = p_0(\zeta_n + \tau), \ \ \forall n \in \mathbb{N}.$$  

Then noticing that $h(\zeta) \geq \log w(\zeta)$, for all $\zeta \in \mathbb{C}$, and $\zeta_n + \tau \in \Gamma_0 \setminus E_0$, for $n \geq n_0$, by (4.5) and (4.7), we get

$$\log|r(\zeta_n)| = \log|p_0(\zeta_n + \tau)| \geq \log h(\zeta_n + \tau) - C \log(1 + |\zeta_n + \tau|)
\geq h_G(\zeta_n + \tau) + h_K(\zeta_n + \tau) - (\epsilon_1 n + C) \log(1 + |\zeta_n + \tau|)
\geq H_G(\zeta_n) + H_K(\zeta_n) - (\epsilon_1 n + C) \log(1 + |\zeta_n|) - B_n
\geq H_D(\zeta_n) + H_K(\zeta_n) - C \log(1 + |\zeta_n|) - B_n,$$

for all $n \geq n_0$, where $B_n := 2RR_G + 2RR_K + (\epsilon_1 n + C) \log(1 + 2R)$.

Since $\lim_{n \to \infty} \frac{B_n}{n} < \infty$, from the last estimate and (4.4) it then follows that, for $m > C$,

$$\sup_{\zeta \in \mathbb{C}} \frac{|r(\zeta)|(1 + |\zeta|)^m}{e^{H_D(\zeta) + H_K(\zeta)}} \sup_{n \geq n_0} \frac{|r(\zeta_n)|(1 + |\zeta_n|)^m}{e^{H_D(\zeta_n) + H_K(\zeta_n)}} \sup_{n \geq n_0} \exp((m - C) \log(1 + |\zeta_n|) - B_n) = +\infty,$$

which shows $r \notin H^{-\infty}_{D+k}$. This contradiction completes the proof of the proposition.  

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