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The Cayley Determinant of the Determinant Tensor and the Alon–Tarsi Conjecture

Paolo Zappa

Dipartimento di Matematica, Universitá di Perugia, Perugia, Italy

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Given a base on a vector space of dimension n, we can represent a tensor of order r with a hypermatrix of dimension n and order r. Then, the standard determinant tensor is represented by a hypermatrix H of order and dimension n. Gherardelli showed that the Cayley determinant of H, times $n!$, is equal to the number of even Latin squares of order n minus the number of odd Latin squares of order n. The Alon–Tarsi conjecture says that this difference is not zero, whenever n is even. If n is odd the difference is zero, but the conjecture can be extended to the odd case by computing the difference only for Latin squares which have the entries of the diagonal equal to 1. In this paper we use the Laplace rule in order to compute the Cayley determinant, and we prove that the difference between the number of even Latin squares and the number of odd Latin squares is nonnegative. We also prove the Alon–Tarsi conjecture for Latin squares of order $c2^r$, where r is a positive integer and either c is an even integer for which the Alon-Tarsi conjecture is true, or c is an odd integer such that the extended Alon-Tarsi conjecture is true for c and for $c + 1$.

1. INTRODUCTION AND NOTATIONS

1.1. *Notations*

Let I_n be equal to $(1, 2, \dots, n)$. A matrix M with entries in I_n is called a Latin square of order n, if its rows and columns are permutations of I_n. M is called *even* (*odd*), if the product of row and column permutations is even (odd). M is called *normalized* if the first row and the first column are equal to I_n, *semi-normalized* if the first column is equal to I_n, and *diagonal* if all the elements of the diagonal are equal to 1.

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In this paper we use capital letters to indicate sets of Latin squares or Latin rectangles and lowercase letters for their cardinalities. So we denote by

$\mathrm{LS}(n)$	the set of Latin squares of order n,
$\mathrm{ELS}(n)$	the set of even Latin squares of order n,
$\mathrm{OLS}(n)$	the set of odd Latin squares of order n,
$\mathrm{NLS}(n)$	the set of normalized Latin squares of order n,
$\mathrm{NELS}(n)$	the set of normalized even Latin squares of order n,
$\mathrm{NOLS}(n)$	the set of normalized odd Latin squares of order n,
$\mathrm{SNLS}(n)$	the set of semi-normalized Latin squares of order n,
$\mathrm{SNELS}(n)$	the set of semi-normalized even Latin squares of order n,
$\mathrm{SNOLS}(n)$	the set of semi-normalized odd Latin squares of order n
$\mathrm{DLS}(n)$	the set of diagonal Latin squares of order n,
$\mathrm{DELS}(n)$	the set of diagonal even Latin squares of order n,
$\mathrm{DOLS}(n)$	the set of diagonal odd Latin squares of order n,
$\mathrm{SNDLS}(n)$	the set of semi-normalized diagonal Latin squares of order n,
$\mathrm{SNDELS}(n)$	the set of semi-normalized diagonal even Latin squares of order n,
$\mathrm{SNDOLS}(n)$	the set of semi-normalized diagonal odd Latin squares of order n,

and by $\mathrm{ls}(n)$, $\mathrm{els}(n)$, $\mathrm{ols}(n)$, $\mathrm{nls}(n)$, $\mathrm{nels}(n)$, $\mathrm{nols}(n)$, $\mathrm{snls}(n)$, $\mathrm{snels}(n)$, $\mathrm{snols}(n)$, $\mathrm{dls}(n)$, $\mathrm{dels}(n)$, $\mathrm{dols}(n)$, $\mathrm{sndls}(n)$, $\mathrm{sndels}(n)$, $\mathrm{sndols}(n)$ their respective cardinalities.

Also, we use the notion of the Latin rectangle. A Latin rectangle of order $m \times n$ $(n > m)$ is a table $m \times n$, where rows are permutations of S_n and there are no repetitions in the columns. We define a Latin rectangle (and, generally, any table with no repetitions in rows and columns) to be even or odd, according to whether the product of the signs of the row permutations and the signs of the column permutations (with respect to the natural order), is even or odd.

Let $P(m, n)$ be the family of subsets of m elements of I_n and let $J_1, J_2, \ldots, J_n$ be elements of $P(m, n)$. We denote by

$\mathrm{LR}(J_1 \vert J_2 \vert \cdots \vert J_n)$	the set of Latin rectangles of order $m \times n$, such that the columns are permutations of $J_1 \vert J_2 \vert \cdots \vert J_n$,
$\mathrm{SNLR}(I_m \vert J_2 \vert \cdots \vert J_n)$	the set of semi-normalized Latin rectangles of order $m \times n$, i.e., a Latin rectangle with first column equal to I_m,
$\mathrm{SNELR}(I_m \vert J_2 \vert \cdots \vert J_n)$	the subset of semi-normalized even Latin rectangles in $\mathrm{LR}(I_m \vert J_2 \vert \cdots \vert J_n)$,
$\mathrm{SNOLR}(I_m \vert J_2 \vert \cdots \vert J_n)$	the subset of semi-normalized odd Latin rectangles in $\mathrm{LR}(I_m \vert J_2 \vert \cdots \vert J_n)$,

and by $\mathrm{lr}(J_1|J_2|\cdots|J_n)$, $\mathrm{snlr}(I_m|J_2|\cdots|J_n)$, $\mathrm{snelr}(I_m|J_2|\cdots|J_n)$, $\mathrm{snolr}(I_m|J_2|\cdots|J_n)$ their respective cardinalities.

1.2. *Signs of Latin Squares*

By switching two columns, it is trivial to check that, if n is odd and $n \neq 1$, $\mathrm{els}(n) = \mathrm{ols}(n)$. The Alon–Tarsi conjecture states [1]:

Conjecture (Alon–Tarsi). If n is even

$$\mathrm{els}(n) \neq \mathrm{ols}(n).$$

This conjecture is linked to other conjectures of combinatorics and linear algebra as shown in [4].

Although, n odd, $\mathrm{els}(n) = \mathrm{ols}(n)$ for we do not have the same equality if we consider only diagonal Latin squares, as is shown by the computations of the signs of Latin squares of low odd order.

For any integer n, define the Alon–Tarsi constant

$$\mathrm{AT}(n) := \frac{1}{(n-1)!}(\mathrm{dels}(n) - \mathrm{dols}(n)).$$

The sign of a Latin square does not change if one switches two numbers; hence, with an appropriate sequence of switches, one can associate to each diagonal Latin square a semi-normalized diagonal Latin square with the same sign. For this reason,

$$\mathrm{AT}(n) = \mathrm{sndels}(n) - \mathrm{sndols}(n), \tag{1}$$

and then $\mathrm{AT}(n)$ is an integer.

If n is even, switching two rows or two columns does not change the sign of a Latin square; then

$$\mathrm{els}(n) - \mathrm{ols}(n) = \begin{cases} n!(n-1)!\mathrm{AT}(n) & \text{if } n \text{ is even} \\ 0 & \text{if } n \text{ is odd.} \end{cases} \tag{2}$$

Conjecture (extended Alon–Tarsi). For every positive integer n,

$$\mathrm{AT}(n) \neq 0.$$

1.3. *The Cayley Determinant of a Hypermatrix*

A (hypercubic) hypermatrix M of dimension n and order r on a field $\mathbb{K}$ is an application from $(I_r)^n$ to $\mathbb{K}$; it can be represented by

$$(m_{i_1, i_2, \ldots, i_n})_{1 \le i_1 \le r, 1 \le i_2 \le r, \ldots, 1 \le i_n \le r}$$

or more simply by

$$(m_{i_1, i_2, \dots, i_n})_{1 \le i_1, i_2, \dots, i_n \le r}.$$

Given a base on a $\mathbb{K}$-vector space V, we can represent a tensor of order r in the same way.

The notion of determinant has been extended to hypermatrices in several ways. Perhaps one of the simplest is the Cayley determinant of M (as it has been called by Gherardelli in [3]); it is defined by

$$\det{}_{\mathrm{C}}(M) := \sum_{\sigma_2, \sigma_3, \dots, \sigma_n \in S_r} \operatorname{sgn}(\sigma_2 \sigma_3 \cdots \sigma_n) \times m_{1, \sigma_2(1), \dots, \sigma_n(1)} m_{2, \sigma_2(2), \dots, \sigma_n(2)} \cdots m_{r, \sigma_2(r), \dots, \sigma_n(r)}.$$

Only if the dimension of the hypermatrix is even is the definition of $\det_{\mathrm{C}}$ independent of the order of the dimensions, i.e., the order of the sequence of the indices in the entries of M.

The standard determinant of a square matrix of order n is a tensor of order n on the vector space $\mathbb{K}^n$, so it is represented by a hypermatrix $H_n = (h_{i_1, i_2, \dots, i_n})_{1 \le i_1, i_2, \dots, i_n \le n}$ of order and dimension n, with respect to the standard base of $\mathbb{K}^n$. The entries of H_n are

$$h_{i_1, i_2, \dots, i_n} = \begin{cases} 0 & \text{if } (i_1, i_2, \dots, i_n) \text{ is not a permutation} \\ \operatorname{sgn}((i_1, i_2, \dots, i_n)) & \text{if } (i_1, i_2, \dots, i_n) \text{ is a permutation.} \end{cases}$$

Gherardelli [3] remarked the relations between $\det_{\mathrm{C}}(H_n)$ and the signs of the Latin squares of order n. We have

$$\det{}_{\mathrm{C}}(H_n) = \begin{cases} 0 & \text{if } n \text{ is odd and } n > 1, \\ (n-1)!\,\mathrm{AT}(n) & \text{if } n \text{ is even.} \end{cases}$$

In this paper, interpreting the Cayley determinant of subhypermatrices of H_n from a combinatorial point of view, and using the Laplace rule, we prove the following statements:

PROPOSITION 1. *If n is even, $\det_{\mathrm{C}}(H_n)$ is a sum of square numbers. Hence, for n even,*

$$\mathrm{AT}(n) \ge 0.$$

PROPOSITION 2. *If n is even,*

$$\mathrm{AT}(n) \ne 0 \Rightarrow \mathrm{AT}(2n) \ne 0.$$

PROPOSITION 3. *If n is odd,*

$$\left.\begin{array}{r}\mathrm{AT}(n) \neq 0\\ \mathrm{AT}(n+1) \neq 0\end{array}\right\} \Rightarrow \mathrm{AT}(2n) \neq 0.$$

Previous propositions, joined to the results of Drisko [2], permit one to prove the Alon–Tarsi conjecture for many values of the order of the Latin squares.

2. SOME ELEMENTARY PROPERTIES OF THE CAYLEY DETERMINANT OF A HYPERMATRIX

Let M be a hypermatrix of dimension n and order r.

Let $J_1, J_2, \dots, J_n$ be n subsets of I_r of cardinality m. Define $M(J_1|J_2|\cdots|J_n)$ to be the superhypermatrix of M of dimension n and order m given by

$$m_{i_1, i_2, \dots, i_n} \in M\{J_1|J_2|\cdots|J_n\} \Leftrightarrow i_1 \in J_1, i_2 \in J_2, \dots, i_n \in J_n.$$

Call the kth j-slide the subhypermatrix $M(I_r|\cdots|\{k\}|\cdots|I_r)$; it is arranged by all the entries of M which have the j index equal to k.

Like the determinant of 2-dimensional matrices, the Cayley determinant of hypermatrices is a linear function of each slide. If the dimension is even, the Cayley determinant changes its sign, whenever two parallel slides are switched.

Define M^s to be the hypermatrix $(m^s_{i_1, i_2, \dots, i_n})_{1 \le i_1, i_2, \dots, i_n \le r}$, where

$$m^s_{i_1, i_2, \dots, i_n} = m_{r+1-i_1, r+1-i_2, \dots, r+1-i_n};$$

for n even, we have

$$\det_{\mathrm{C}}(M) = \det_{\mathrm{C}}(M^s). \tag{3}$$

Moreover, it is trivial that

$$\det_{\mathrm{C}}(M) = (-1)^r \det_{\mathrm{C}}(-M). \tag{4}$$

For hypermatrices M of even dimension, the Laplace development of $\det_{\mathrm{C}}(M)$ along the kth j-slide is given by

$$\begin{aligned}\det_{\mathrm{C}}(M) &= \sum_{1 \le i_1, \dots, i_{j-1}, i_{j+1}, \dots, i_n \le r} (-1)^{(i_1 + \cdots + i_{j-1} + k + i_{j+1} + \cdots + i_{j_n})} \\ &\quad \times m_{i_1, \dots, i_{j-1}, k, i_{j+1}, \dots, i_n} \\ &\quad \times \det_{\mathrm{C}}\Big(M\big(I_r - \{i_1\}|\cdots|I_r - \{i_{j-1}\}|I_r - \{k\}|I_r - \{i_{j+1}\}| \\ &\qquad \cdots|I_r - \{i_n\}\big)\Big). \end{aligned} \tag{5}$$

We can do the Laplace development also along a group of m j-slides. Let J_j be a fixed subset of I_r of cardinality m, and $\det_C(M)$ be equal to a sum, with the right signs, of the product of the Cayley determinant of all the subhypermatrices of order m which can be constructed with the fixed j-slides, times the Cayley determinant of their complementary subhypermatrices. In order to simplify the expression, we write the formula which represents this development, in the case $j = 1$, $J_1 = \{1, 2, \dots, m\}$, and n even:

$$\det_C(M) = \sum_{J_2 J_3, \dots, J_n \in P(m, I_r)} (-1)^{(\Sigma_{i_1 \in I_m} i_1 + \Sigma_{i_2 \in J_2} i_2 + \cdots + \Sigma_{i_n \in J_n} i_n)} \times \det_C\big(M(I_m|J_2|\cdots|J_n)\det_C\big(M(I_r - I_m|I_r - J_2|\cdots|I_r - J_n)\big). \tag{6}$$

Some other properties, like the Binet rule, can be found in [3].

3. THE CAYLEY DETERMINANT OF THE DETERMINANT

The Cayley determinant of the hypermatrix H_n, defined in the Introduction, is

$$\det_C(H_n) = \sum_{\sigma_2, \sigma_3, \dots, \sigma_n \in S_n} \operatorname{sgn}(\sigma_2 \sigma_3 \cdots \sigma_n) h_{1, \sigma_2(1), \dots, \sigma_n(1)} \times h_{2, \sigma_2(2), \dots, \sigma_n(2)} \cdots h_{r, \sigma_2(n), \dots, \sigma_n(n)}. \tag{7}$$

Consider the $n \times n$ table obtained from an addendum of (7), packing together the columns given by $e, \sigma_2, \sigma_3, \dots, \sigma_n$, where e represent the identity permutation. Call this the table associated to the addendum. The addendum is different from 0 if and only if its associated table is a semi-normalized Latin square of order n, and its value is 1 or -1 according to whether the associated Latin square is even or odd.

Observe that every semi-normalized Latin square is associated to an addendum. Hence, for (2),

$$\det_C(H_n) = \operatorname{snels}(n) - \operatorname{snols}(n) = \begin{cases} 0 & \text{if } n \text{ is odd and } n > 1, \\ (n-1)!\operatorname{AT}(n) & \text{if } n \text{ is even.} \end{cases}$$

In the case of Latin squares of order $2n$ even, it is quite natural to try to compute $\det_{\mathrm{C}}(H_{2n})$, applying the Laplace rule to a group of n slides.

For (6) we must compute terms of type

$$\det_{\mathrm{C}}\big(H_{2n}\big(I_n|J_2|\cdots|J_{2n}\big)\big)$$

and

$$\det_{\mathrm{C}}\big(H_{2n}\big(I_{2n}-I_n|I_{2n}-J_2|\cdots|I_{2n}-J_{2n}\big)\big),$$

where $J_j \in \mathsf{P}(n, 2n)$, for $2 \le j \le 2n$.

These Cayley determinants are linked to semi-normalized Latin rectangles.

PROPOSITION 4. *If there exist* $i \in I_{2n}$, *and* $K = \{k_1, k_2, \dots, k_r\} \subset I_{2n}$, *with* $r \neq n$, *such that*

$$i \in J_{k_1}, i \in J_{k_2}, \dots, i \in J_{k_r}, i \notin \bigcap_{s \in I-K} J_s$$

then $\det_{\mathrm{C}}(H_{2n}(I_n|J_2|\cdots|J_{2n})) = 0$.

Otherwise,

$$\det_{\mathrm{C}}\big(H_{2n}\big(I_n|J_2|\cdots|J_{2n}\big)\big) = \mathrm{snelr}\big(I_n|J_2|\cdots|J_{2n}\big) - \mathrm{snolr}\big(I_n|J_2|\cdots|J_{2n}\big);$$

i.e., it represents the difference between the number of even Latin rectangles and the number of odd Latin rectangles (*of order* $n \times 2n$), *which have the first column equal to* I_n *and the other columns equal to permutations of* $J_2, J_3, \dots, J_{2n}$, *respectively.*

Proof.

$$\det_{\mathrm{C}}\big(H_{2n}\big(I_n|J_2|\cdots|J_{2n}\big)\big) = \sum_{\substack{\sigma_2 \in S_{J_2}\\ \sigma_3 \in S_{J_3}\\ \sigma_{2n} \in S_{J_{2n}}}} \mathrm{sgn}(\sigma_2\sigma_3\cdots\sigma_{2n})h_{1,\sigma_2(1),\dots,\sigma_{2n}(1)}$$
$$\times h_{2,\sigma_2(2),\dots,\sigma_{2n}(2)} \cdots h_{n,\sigma_2(n),\dots,\sigma_{2n}(n)}. \qquad (8)$$

Again an addendum of (8) is different from 0 if and only if the associated table, which is constructed using $e, \sigma_2, \dots, \sigma_{2n}$ as columns, is a semi-normalized Latin rectangle. As rows are in S_{2n}, each number of I_n appears exactly n times in the Latin rectangle associated to the addendum. This proves the first part of the proposition.

The value of the addendum is 1, if the Latin rectangle is even, and -1, if it is odd. ∎

Now, we can prove Proposition 1.

Proof. From the Laplace rule (6),

$$\det_C(H_{2n}) = \sum_{J_2,\dots,J_n \in \boldsymbol{P}(n, I_{2n})} (-1)^{(\Sigma_{i_1 \in I_n} i_1 + \Sigma_{i_2 \in J_2} i_2 + \cdots + \Sigma_{i_n \in J_n} i_n)} \times \det_C\big(H_{2n}(I_n | J_2 | \cdots | J_{2n})\big) \times \det_C\big(H_{2n}(I_{2n} - I_n | I_{2n} - J_2 | \cdots | I_{2n} - J_n)\big). \tag{9}$$

From Proposition 4, for any non-zero addendum of (9),

$$(-1)^{(\Sigma_{i_1 \in I_n} i_1 + \Sigma_{i_2 \in J_2} i_2 + \cdots + \Sigma_{i_n \in J_n})} = (-1)^{n^2(2n+1)} = (-1)^n. \tag{10}$$

If $h_{i_1, i_2, \dots, i_{2n}}$ is an entry of $H_{2n}(I_n | J_2 | \cdots | J_{2n})$, $h_{2n+1-i_1,\, 2n+1-i_2, \dots, 2n+1-i_{2n}}$ is an entry of $H_{2n}(I_{2n} - I_n | I_{2n} - J_2 | \cdots | I_{2n} - J_n)$, and

$$h_{i_1, i_2, \dots, i_{2n}} = (-1)^n h_{2n+1-i_1, 2n+1-i_2, \dots, 2n+1-i_{2n}}. \tag{11}$$

Using (3), (4), (9), (10), and (11), we get that $\det_C(H_n)$ is a sum of square numbers. ∎

4. SPECIAL ADDENDA OF $\det_C(H_{2n})$

4.1. *First Addendum*

For Proposition 1, one can prove the Alon−Tarsi conjecture if one can state that one addendum of $\det_C(H_{2n})$ is different from zero. So it is enough to find $J_2, J_3, \dots, J_{2n}$ such that

$$\det_C\big(H_{2n}(I_n | J_2 | \cdots | J_{2n})\big) \neq 0.$$

In order to prove Proposition 2, we study the addendum obtained by choosing

$$J_1 = J_2 = \cdots = J_n = I_n, \qquad J_{n+1} = J_{n+2} = \cdots = J_{2n} = I_{2n} - I_n.$$

A semi-normalized Latin rectangle R, $n \times 2n$, associated to a non-zero addendum of $\det_C(H_{2n}(J_1 | J_2 | \cdots | J_{2n}))$, as defined in Proposition 4, belongs to

$$\mathrm{SNLR}\big(J_1 | J_2 | \cdots | J_{2n}\big)$$

and it is the union of two Latin squares of order n. The left Latin square, R_1, is constructed with the numbers of I_n and has its first column equal to

I_n; the right Latin square, R_r, is constructed with the numbers of $I_{2n} - I_n$. Subtracting n to each entry of R_r, we obtain a Latin square $R_r^{\#}$. The map

$$\mathrm{SNLR}(J_1|J_2|\cdots|J_{2n}) \to \mathrm{SNLS}(n) \times \mathrm{LS}(n)$$
$$R \mapsto \left(R_l, R_r^{\#}\right)$$

is bijective and

$$\mathrm{sgn}(R) = \mathrm{sgn}(R_l)\mathrm{sgn}\left(R_r^{\#}\right).$$

Then

$$\mathrm{snelr}(J_1|J_2|\cdots|J_{2n}) = \mathrm{snels}(n)\mathrm{els}(n) + \mathrm{snols}(n)\mathrm{ols}(n),$$
$$\mathrm{snolr}(J_1|J_2|\cdots|J_{2n}) = \mathrm{snels}(n)\mathrm{ols}(n) + \mathrm{snols}(n)\mathrm{els}(n),$$
$$\mathrm{snelr}(J_1|J_2|\cdots|J_{2n}) - \mathrm{snolr}(J_1|J_2|\cdots J_{2n}) = (\mathrm{snels}(n) - \mathrm{snols}(n))(\mathrm{els}(n) - \mathrm{ols}(n)).$$

Hence, for (2), it follows that

$$\det\nolimits_C\left(H_{2n}(J_1|J_2|\cdots|J_{2n})\right) = \begin{cases} n!((n-1)!)^2(\mathrm{AT}(n))^2 & \text{if } n \text{ is even} \\ 0 & \text{if } n \text{ is odd.} \end{cases}$$

Now Proposition 2 follows easily from Proposition 1.

4.2. *Second Addendum*

In the case in which n is odd, the previous result is not useful, so we choose a different addendum. In the following we assume that n is odd.

Let us study $\det_C(H_{2n}(J_1|J_2|\cdots|J_{2n}))$, where

$$\begin{aligned}
J_1 &= \{1,2,3,\ldots,n-2,n-1,n\} = I_n, & J_{n+1} &= I_{2n} - J_n,\\
J_2 &= \{1,2,3,\ldots,n-2,n-1,n+1\}, & J_{n+2} &= I_{2n} - J_{n-1},\\
J_3 &= \{1,2,3,\ldots,n-2,n,n+1\}, & J_{n+3} &= I_{2n} - J_{n-2},\\
&\vdots & &\vdots\\
J_{n-1} &= \{1,2,4,\ldots,n-1,n,n+1\}, & J_{2n-1} &= I_{2n} - J_2,\\
J_n &= \{1,3,4,\ldots,n-1,n,n+1\}, & J_{2n} &= I_{2n} - J_1. \quad (12)
\end{aligned}$$

Divide every semi-normalized Latin rectangle $R \in \mathrm{SNLR}(J_1|J_2|\cdots|J_{2n})$ into a left square table R_l and a right square table R_r.

In R_l we find numbers from 1 to $n+1$ and the number 1 stays in every row and in every column, while each number of $(2,3,\ldots,n,n+1)$ is not present exactly in one row and in one column. In R_r we find the elements of $I_{2n} - I_{n+1}$ in every row and in every column; the elements of $(2, 3, \dots, n, n+1)$ appear only once in R_r, and each number is found in a different row and column of R_r. The following example shows the situation:

$$R = \begin{pmatrix} 1 & 2 & 6 & 5 & 4 & 8 & 3 & 10 & 9 & 7 \\ 2 & 3 & 5 & 1 & 6 & 7 & 8 & 4 & 10 & 9 \\ 3 & 1 & 2 & 4 & 5 & 10 & 7 & 9 & 8 & 6 \\ 4 & 6 & 1 & 2 & 3 & 9 & 10 & 7 & 5 & 8 \\ 5 & 4 & 3 & 6 & 1 & 2 & 9 & 8 & 7 & 10 \end{pmatrix}$$

$$R_l = \begin{pmatrix} 1 & 2 & 6 & 5 & 4 \\ 2 & 3 & 5 & 1 & 6 \\ 3 & 1 & 2 & 4 & 5 \\ 4 & 6 & 1 & 2 & 3 \\ 5 & 4 & 3 & 6 & 1 \end{pmatrix}, \qquad R_r = \begin{pmatrix} 8 & \underline{3} & 10 & 9 & 7 \\ 7 & 8 & \underline{4} & 10 & 9 \\ 10 & 7 & 9 & 8 & \underline{6} \\ 9 & 10 & 7 & \underline{5} & 8 \\ \underline{2} & 9 & 8 & 7 & 10 \end{pmatrix}$$

There is only one way to extend R_l to a Latin square of order $n+1$: joining the row $(n+1, n, \dots, 2, 1)$ in the bottom and one column $(t_1, t_2, \dots, t_n, 1)$ at the right. The sign of the adjoint row is $(-1)^{(n+1)/2}$. Denote by σ the permutation $(t_1 - 1,\ t_2 - 1, \dots, t_n - 1)$; the sign of the adjoint column is $-\mathrm{sign}(\sigma)$; the sign of the product of the other rows and columns is equal to the sign of R_l.

From this Latin square we get a normalized Latin square of order $n+1$, R_l^*, operating a permutation of the columns. This permutation does not change the sign, since the order is even. Hence

$$\mathrm{sign}(R_l^*) = (-1)^{(n-1)/2}\mathrm{sign}(\sigma)\mathrm{sign}(R_l).$$

The previous example extends to

$$\begin{pmatrix} 1 & 2 & 6 & 5 & 4 & \mathbf{3} \\ 2 & 3 & 5 & 1 & 6 & \mathbf{4} \\ 3 & 1 & 2 & 4 & 5 & \mathbf{6} \\ 4 & 6 & 1 & 2 & 3 & \mathbf{5} \\ 5 & 4 & 3 & 6 & 1 & \mathbf{2} \\ \mathbf{6} & \mathbf{5} & \mathbf{4} & \mathbf{3} & \mathbf{2} & \mathbf{1} \end{pmatrix} \quad \text{and} \quad R_l^* = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 3 & 4 & 6 & 1 & 5 \\ 3 & 1 & 6 & 5 & 4 & 2 \\ 4 & 6 & 5 & 3 & 2 & 1 \\ 5 & 4 & 2 & 1 & 6 & 3 \\ 6 & 5 & 1 & 2 & 3 & 4 \end{pmatrix},$$

with $\sigma = (2, 3, 5, 4, 1)$.

The positions of the numbers $2, 3, \dots, n, n+1$ in R_r are determined uniquely by R_l (underlined numbers in the example). It is easy to check that, operating with the permutation σ on the rows of R_r, we obtain a square table where the numbers $2, 3, \ldots, n, n+1$ stay in the main diagonal. Replacing $2, 3, \ldots, n$ with $n+1$ in R_{r}, and subtracting n from each entry, we obtain a diagonal Latin square $R_{\mathrm{r}}^{\#}$ of order n. Following the example, the action of σ and the other operations lead respectively to

$$\begin{pmatrix} 2 & 9 & 8 & 7 & 10 \\ 8 & 3 & 10 & 9 & 7 \\ 7 & 8 & 4 & 10 & 9 \\ 9 & 10 & 7 & 5 & 8 \\ 10 & 7 & 9 & 8 & 6 \end{pmatrix}, \qquad R_{\mathrm{r}}^{\#} = \begin{pmatrix} 1 & 4 & 3 & 2 & 5 \\ 3 & 1 & 5 & 4 & 2 \\ 2 & 3 & 1 & 5 & 4 \\ 4 & 5 & 2 & 1 & 3 \\ 5 & 2 & 4 & 3 & 1 \end{pmatrix}.$$

The substitutions, that we have done do not change the sign of the table R_{r}; hence, as n is odd,

$$\operatorname{sign}\left(R_{\mathrm{r}}^{\#}\right) = \operatorname{sign}(\sigma)\operatorname{sign}(R_{\mathrm{r}}).$$

Now the sign of R can be related with the signs of R_{l}^{*} and $R_{\mathrm{r}}^{\#}$. Observing that

$$\begin{aligned} \operatorname{sign}(R) &= (-1)^{\sum_{k=2}^{n}(n+1-k)} \operatorname{sign}(R_{\mathrm{l}})\operatorname{sign}(R_{\mathrm{r}}) \\ &= (-1)^{(n-1)n/2} \operatorname{sign}(R_{\mathrm{l}})\operatorname{sign}(R_{\mathrm{r}}), \end{aligned}$$

we get

$$\operatorname{sign}(R) = \operatorname{sign}(R_{\mathrm{l}}^{*})\operatorname{sign}\left(R_{\mathrm{r}}^{\#}\right). \tag{13}$$

Now we can prove Proposition 3.

Proof. For n odd and $J_1, J_2, \ldots, J_{2n}$ as in (12), we have defined a map

$$\begin{aligned} \phi \colon \mathrm{SNLR}(J_1, J_2, \ldots, J_{2n}) &\to \mathrm{NLS}(n+1) \times \mathrm{DLS}(n) \\ R &\mapsto \left(R_{\mathrm{l}}^{*}, R_{\mathrm{r}}^{\#}\right). \end{aligned}$$

Looking at the given definitions, we can easily check that ϕ is bijective.

From (13) it follows that

$$\operatorname{snelr}(I_n, J_2, \ldots, J_{2n}) = \operatorname{nels}(n+1)\operatorname{dels}(n) + \operatorname{nols}(n+1)\operatorname{dols}(n),$$

$$\operatorname{snolr}(I_n, J_2, \ldots, J_{2n}) = \operatorname{nels}(n+1)\operatorname{dols}(n) + \operatorname{nols}(n+1)\operatorname{dels}(n),$$

$$\begin{aligned} &\operatorname{snelr}(I_n, J_2, \ldots, J_{2n}) - \operatorname{snolr}(I_n, J_2, \ldots, J_{2n}) \\ &\quad = \left(\operatorname{nels}(n+1) - \operatorname{nols}(n+1)\right)\left(\operatorname{dels}(n) - \operatorname{dols}(n)\right). \end{aligned}$$

As $\operatorname{els}(n+1) - \operatorname{ols}(n+1) = (n+1)!n!(\operatorname{nels}(n+1) - \operatorname{nols}(n+1))$, Proposition 3 follows from (2) and Propositions 1, 4. ∎

5. FINAL REMARKS

5.1. *Another Meaning of $AT(n)$*

We need some more definitions. Define a Latin square to be of type $(+,-)$ if the sign of the product of the rows is positive, while the sign of the product of the columns is negative. Similar definitions follow for types $(+,+)$, $(-,+)$ and $(-,-)$. Consequently the notations of sets of Latin squares must be modified to indicate sets of fixed type. For example, $\mathrm{ls}(+,+)(n)$ denotes the cardinality of the set of Latin squares of order n of type $(+,+)$.

The tilde operator, introduced by the author in [6], associates to a Latin square Q a Latin square $\tilde{Q}$, such that its columns are the inverse permutations of the rows of Q. It is a bijective map and has period 3. Also, its restriction to NLS(n) is bijective onto SNDLS(n),

$$\mathrm{NLS}(n) \overset{\sim}{\to} \mathrm{SNDLS}(n).$$

The tilde operator transforms Latin squares according to the following:

$$\text{if } n \equiv 0,1 \pmod 4, \qquad \text{then} \qquad \begin{cases} (+,+) \to (+,+) \\ (+,-) \to (-,+) \\ (-,+) \to (-,-) \\ (-,-) \to (+,-); \end{cases}$$

$$\text{if } n \equiv 2,3 \pmod 4, \qquad \text{then} \qquad \begin{cases} (+,+) \to (-,+) \\ (+,-) \to (+,+) \\ (-,+) \to (+,-) \\ (-,-) \to (-,-). \end{cases}$$

Transposing a Latin square does not change the sign; then

$$\mathrm{nls}(-,+)(n) = \mathrm{nls}(+,-)(n).$$

Using the properties of the tilde operator and (1), we get

$$\begin{aligned} \mathrm{AT}(n) &= \mathrm{sndls}(+,+)(n) + \mathrm{sndls}(-,-)(n) - \mathrm{sndls}(-,+)(n) \\ &\qquad - \mathrm{sndls}(+,-)(n) \\ &= \begin{cases} \text{if } n \equiv 0,1 \pmod 4, & \mathrm{nls}(+,+)(n) - \mathrm{nls}(-,-)(n) \\ \text{if } n \equiv 2,3 \pmod 4, & \mathrm{nls}(-,-)(n) - \mathrm{nls}(+,+)(n). \end{cases} \end{aligned}$$

A final remark: if n is even,

$$\mathrm{AT}(n) = \mathrm{nels}(n) - \mathrm{nols}(n)$$

but that is true only for n even (contra-example for $n = 7$).

5.2. CONCLUSIONS

Proposition 2 states that, if the Alon–Tarsi conjecture is true for Latin squares of order c, then it is true also for Latin squares of order $2^r c$, where r is a positive integer.

Drisko [2] proved that the Alon–Tarsi conjecture is true for Latin squares of order $p + 1$, where p is an odd prime. So the Alon–Tarsi conjecture is true for Latin squares of order $2^r(p + 1)$.

$|\mathrm{AT}(n)|$ increases very quickly with n. The value of $\mathrm{AT}(n)$ has been computed until order 8 [5]:[1]

$$\begin{array}{ll} \mathrm{AT}(2) = 1 & \mathrm{AT}(3) = -1 \\ \mathrm{AT}(4) = 4 & \mathrm{AT}(5) = -24 \\ \mathrm{AT}(6) = 2,304 & \mathrm{AT}(7) = 368,640 \\ \mathrm{AT}(8) = 6,210,846,720 & \mathrm{AT}(9) = ? \end{array}$$

Proposition 3 permits us to state that the Alon–Tarsi conjecture is true also for Latin squares of order 10, which cannot be obtained from Drisko's results and Proposition 2.

Other approaches to the study of $\det_{\mathrm{C}}(H_{2n})$ are possible. It seems interesting to study the Cayley determinant of subhypermatrices of H_{2n} of order 2. In fact, let $J_1, J_2, \dots, J_{2n}$ be sets of two elements, satisfying the conditions of Proposition 4; each associated Latin rectangle can be interpreted as a single permutation τ (the inverse of the first row composed the second row); all the permutations constructed from $\det_{\mathrm{C}}(H_{2n}(J_1|J_2|\cdots|J_{2n}))$ have the same decomposition in cycles and moreover

$$\det_{\mathrm{C}}\big(H_{2n}(J_1|J_2|\cdots|J_{2n})\big) = \begin{cases} 0 & \text{if } \tau \text{ has a cycle of order odd,} \\ 2^{\lambda-1} & \text{otherwise,} \end{cases}$$

where λ is the number of cycles of τ.

[1] In this paper the values of $\mathrm{AT}(n)$ are given only for n even; Pirillo kindly informed me that $\mathrm{nls}(+,+)(7) = 4,120,320$; $\mathrm{nls}(+,-)(7) = \mathrm{nls}(-,+)(7) = 4,166,400$; $\mathrm{nls}(-,-)(7) = 4,488,960$.

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