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Abstract Multiparameter Theory, II

PATRICK J. BROWNE*

Department of Mathematics, Statistics, and Computing Science, University of Calgary, Calgary, Alberta T2N 1N4, Canada

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In this paper we continue the study of multiparameter spectral theory commenced in our earlier paper of the same title. Here we relax the condition that the weight operator Δ_0 be strictly bounded away from zero. The case in which Δ_0 has 0 as a point of its continuous spectrum is discussed and a generalized eigenvector expansion and Parseval equality are obtained.

1. INTRODUCTION

In this paper we continue our study of multiparameter spectral theory. Our earlier works [2, 3] contain references to the recent literature on multiparameter problems both in the abstract setting and in the case of linked systems of ordinary differential equations.

We introduce some notation. Let $H_1, ..., H_k$ be separable Hilbert spaces and $H = \bigotimes_{r=1}^{k} H_r$ be their tensor product. In each space H_r assume we have operators T_r, V_{rs}, $s = 1,..., k$ such that

(i) $T_r, V_{rs}: H_r \to H_r$ are Hermitian (i.e., self-adjoint and continuous),

(ii) for any choice of $f_r \in H_r$, $f_r \neq 0$, $r = 1,..., k$ we have

$$\det\{(V_{rs}f_r, f_r)_r\} > 0, \tag{1}$$

where $(\cdot, \cdot)_r$ denotes the inner product in H_r.

Each of these operators induces an operator on the space H. The induced operators are denoted by $T_r^\dagger$, $V_{rs}^\dagger$. For a decomposable tensor $V_{rs}^\dagger f = f_1 \otimes \cdots \otimes f_{r-1} \otimes V_{rs}f_r \otimes f_{r+1} \otimes \cdots \otimes f_k$. $V_{rs}^\dagger$ is then extended to the whole space H by continuity and linearity. $T_r^\dagger$ is defined similarly.

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On H we define operators $\Delta_0, \Delta_1, \ldots, \Delta_k$ by means of the formal determinantal expansion

$$\sum_{s=0}^{k} \alpha_s \Delta_s = \begin{vmatrix} \alpha_0 & \alpha_1 & \cdots & \alpha_k \\ T_1^\dagger & V_{11}^\dagger & \cdots & V_{1k}^\dagger \\ \cdots & \cdots & \cdots & \cdots \\ T_k^\dagger & V_k^\dagger & \cdots & V_{kk}^\dagger \end{vmatrix},$$

where $\alpha_0, \ldots, \alpha_k$ are arbitrary complex numbers. This method of defining operators on a tensor product space is by now a standard technique in multiparameter theory (see, e.g., [2, 3]). The operators $\Delta_0, \ldots, \Delta_k$ are Hermitian on H and in view of our definiteness condition (1) we have $(\Delta_0 f, f) \geqslant 0$ for all $f \in H$. In [2, 3] we assumed $(\Delta_0 f, f) \geqslant \mu(f, f)$ for all $f \in H$ where $\mu > 0$. It is our intent here to investigate the case in which Δ_0 has 0 as a point of its continuous spectrum. We achieve this by proposing the following:

Assumption. *There exist spectral measures $P_r(M_r)$ defined for $M_r \in B$ (the bounded Borel sets in $\mathbb{R}$) and having values in the set of orthogonal projections on H_r such that*

(i) $V_{rs} P_r(M_r) = P_r(M_r) V_{rs}$, $T_r P_r(M_r) H_r \subseteq P_r(M_r) H_r$, $r, s = 1, \ldots, k$,

(ii) *if $M = M_1 \times \cdots \times M_k$, where each M_r is a bounded Borel subset of $\mathbb{R}$ and if $P(M) = P_1^\dagger(M_1) \cdots P_k^\dagger(M_k)$, then*

$$(\Delta_0 P(M) f, P(M) f) \geqslant \mu(M) (P(M) f, P(M) f), \qquad \text{for all } f \in H,$$

where $\mu(M) > 0$ and $\mu(M) \to 0$ as $M \to \mathbb{R}^k$.

As an example of this situation let $H_r = L^2(0, \infty)$ (Lebesgue measure) and suppose we have real-valued bounded continuous functions $a_{rs}(x_r)$, $0 \leqslant x_r < \infty$, $r, s = 1, \ldots, k$ with $|A|(x) = \det\{a_{rs}(x_r)\} > 0$ for $x \in [0, \infty) \times \cdots \times [0, \infty)$ (k factors). Suppose also that $|A|(x) \to 0$ as $\|x\| \to \infty$. Then if we set $(V_{rs} f_r)(x_r) = a_{rs}(x_r) f_r(x_r)$ our initial assumptions are satisfied. In this case $(\Delta_0 f)(x) = |A|(x) f(x)$ and the operator Δ_0 has 0 as a point of its continuous spectrum. For the spectral measures P_r we take $P_r(M_r) f_r = \chi_{M_r} f_r$. The operators T_r may then be chosen arbitrarily subject to condition (i) of the assumption.

The bilinear form $[f, g] = (\Delta_0 f, g)$ is an inner product on H under which H is not complete. $H^\wedge$ denotes the completion of H with respect to $[\cdot, \cdot]$. The customary notation T^* denotes the adjoint with respect to $(\cdot, \cdot)$ of an operator T densely defined in H. $T^\#$ denotes the $[\cdot, \cdot]$ adjoint of an operator T densely defined in $H^\wedge$. The norm in $H^\wedge$ is written $|||\cdot|||$.

2. Commuting Self-Adjoint Operators

Lemma 1. *Δ_0: $H \to H$ has a unique $[\cdot, \cdot]$-Hermitian extension Δ_0: $H^\wedge \to H^\wedge$.*

Proof. Let $f \in H$. Then

$$[\Delta_0 f, \Delta_0 f] = (\Delta_0^2 f, \Delta_0 f) = (\Delta_0^{3/2} f, \Delta_0^{3/2} f)$$
$$\leqslant \| \Delta_0 \|^2 (\Delta_0^{1/2} f, \Delta_0^{1/2} f) = \| \Delta_0 \|^2 [f, f].$$

This establishes that Δ_0 has a $[\cdot, \cdot]$ continuous extension to $H^\wedge$. The Hermiticity of the extension is trivial. Note that the spectrum of Δ_0 is a subset of the non-negative real axis so that the fractional powers of Δ_0 used above are well defined.

Lemma 2. *$P(M)$: $H \to H$ has a unique $[\cdot, \cdot]$-Hermitian extension $P(M)$: $H^\wedge \to H^\wedge$.*

Proof. Notice that $P(M)$ commutes with Δ_0 on H and so commutes with functions of Δ_0. Let $f \in H$. Then

$$[P(M)f, P(M)f] = (\Delta_0 P(M)f, P(M)f) = (\Delta_0^{1/2} P(M)f, \Delta_0^{1/2} P(M)f)$$
$$= (P(M)\Delta_0^{1/2} f, P(M)\Delta_0^{1/2} f)$$
$$\leqslant \| P(M) \|^2 [f, f],$$

showing that $P(M)$ has an extension to $H^\wedge$—in fact, of the same norm. Hermiticity of the extension is trivial.

Lemma 3. *$\lim_{M \to \mathbb{R}^k} P(M) = I$ in the strong $[\cdot, \cdot]$-operator topology.*

Proof. Let $f \in H^\wedge$ and $\epsilon > 0$. Select $g \in H$ such that $|||f - g||| < \epsilon$. Then

$$|||P(M)f - f||| \leqslant |||P(M)f - P(M)g||| + |||P(M)g - g||| + |||g - f|||$$
$$\leqslant 2\epsilon + \| \Delta_0 \| \, \| P(M)g - g \|$$
$$\to 2\epsilon \quad \text{as} \quad M \to \mathbb{R}^k.$$

Lemma 4. (i) *$P(M) H^\wedge \subset H$;* (ii) *$\Delta_0 H^\wedge \subset H$.*

Proof. (i) Let $f \in H^\wedge$ and select a sequence $f_n \in H$ such that $|||f - f_n||| \to 0$. Then $P(M)f_n \to P(M)f$ in $H^\wedge$. Now

$$|||P(M)f_n - P(M)f_m|||^2 = (\Delta_0 P(M)(f_n - f_m), P(M)(f_n - f_m))$$
$$\geqslant \mu(M)(P(M)(f_n - f_m), P(M)(f_n - f_m)).$$

This shows that $P(M)f_n$ is an H-Cauchy sequence and so has an H-limit. It now follows that this limit must coincide with $P(M)f$, and thus, $P(M)f \in H$.

(ii) Let $f \in H^\wedge$. Then $H - \lim_{M \to \mathbb{R}^k} \Delta_0 P(M) f = \Delta_0 f$. Also

$$\| \Delta_0 (P(M) - P(N) f \|^2 = (\Delta_0 (P(M) - P(N)) f, \Delta_0 (P(M) - P(N)) f) \leqslant \| \Delta_0 \| [(P(M) - P(N)) f, (P(M) - P(N)) f].$$

Thus $\Delta_0 P(M) f$ is an H-Cauchy sequence and so $\Delta_0 f \in H$.

Note that $\Delta_0^{-1}: \mathscr{D}(\Delta_0^{-1}) \subset H \to H^\wedge$ exists as a densely defined operator since 0 is a point of the continuous spectrum of Δ_0. Further, by our assumption we see that for $f \in H^\wedge$, $\Delta_s P(M) f \in P(M) H \subset \mathscr{D}(\Delta_0^{-1})$. This leads us to define operators $\Gamma_1, ..., \Gamma_k$ as follows:

(i) $\mathscr{D}(\Gamma_s) = \{ f \in H^\wedge \mid \Delta_0^{-1} \Delta_s P(M) f$ has a $[\cdot, \cdot]$ limit as $M \to \mathbb{R}^k \}$;

(ii) for $f \in \mathscr{D}(\Gamma_s)$, $\Gamma_s f = [\cdot, \cdot] \lim_{M \to \mathbb{R}^k} \Delta_0^{-1} \Delta_s P(M) f$.

As in our earlier works [2, 3], it is the operators Γ_s which form the basis of the multiparameter spectral theory.

THEOREM 1. *The operators* Γ_s, $s = 1, ..., k$ *are* $[\cdot, \cdot]$*-self-adjoint, i.e.,* $\Gamma_s^{\#} = \Gamma_s$.

Proof. Let $f, g \in \mathscr{D}(\Gamma_s)$. Then

$$\begin{aligned}[\Gamma_s f, g] &= \lim_{M \to \mathbb{R}^k} (\Delta_s P(M) f, P(M) g) \\ &= \lim_{M \to \mathbb{R}^k} (P(M) f, \Delta_s P(M) g) \\ &= \lim_{M \to \mathbb{R}^k} (\Delta_0 P(M) f, \Delta_0^{-1} \Delta_s P(M) g) \\ &= [f, \Gamma_s g].\end{aligned}$$

This shows that Γ_s is $[\cdot, \cdot]$-symmetric. Now let $g \in \mathscr{D}(\Gamma_s^{\#})$. Then for any $f \in H$, $P(M) f \in \mathscr{D}(\Gamma_s)$, so that

$$[\Gamma_s P(M) f, g] = [P(M) f, \Gamma_s^{\#} g],$$
$$(\Delta_s P(M) f, P(M) g) = (\Delta_0 P(M) f, P(M) \Gamma_s^{\#} g),$$
$$(f, \Delta_s P(M) g) = (f, \Delta_0 P(M) \Gamma_s^{\#} g).$$

This holds for all $f \in H$ and all M. Thus we have

$$\Delta_s P(M) g = \Delta_0 P(M) \Gamma_s^{\#} g,$$
$$\Delta_0^{-1} \Delta_s P(M) g = P(M) \Gamma_s^{\#} g.$$

Hence $[\cdot, \cdot]$-$\lim_{M \to \mathbb{R}^k} \Delta_0^{-1} \Delta_s P(M) g = \Gamma_s^{\#} g$ showing that $g \in \mathscr{D}(\Gamma_s)$ and so the result is established.

Commutativity of unbounded self-adjoint operators is defined [5, p. 261] via commutativity of their spectral measures. Using the fact that Γ_s is the strong limit of the operators $\Delta_0^{-1}\Delta_s P(M)$ and an argument using Rellich's theorem [6, p. 369] much as in the proof of [3, Theorem 2], we need only establish the commutativity of the operators $\Delta_0^{-1}\Delta_s P(M)$. To this end we may concentrate on the subspace $P(M)H$, for the operators in question vanish on its ortho-complement. Consider the multiparameter array of operators acting on $P(M)H = \otimes_{r=1}^{k} P_r(M_r) H_r$,

$$\begin{vmatrix} T_1^{\dagger} & V_{11}^{\dagger} & \cdots & V_{1k}^{\dagger} \\ \cdots & \cdots & \cdots & \cdots \\ T_k^{\dagger} & V_{k1}^{\dagger} & \cdots & V_{kk}^{\dagger} \end{vmatrix}.$$

Note that our assumption guarantees that these operators carry $P(M)H$ into $P(M)H$. On $P(M)H$, Δ_0 is strictly positive definite, so we may appeal to the theory of [2, Theorem 2] which shows that $\Delta_0^{-1}\Delta_s P(M)$ are pairwise commutative. We should point out that the need for the solvability condition used in [2, p. 251, Section preceding Theorem 2] has recently been removed by Källström and Sleeman [4]. Accordingly, we may now claim

THEOREM 2. *The operators Γ_s, $s = 1, 2, \dots, k$ are pairwise commutative.*

3. MULTIPARAMETER SPECTRAL THEORY

In previous works on multiparameter theory, an eigenvalue $\lambda = (\lambda_1, \dots, \lambda_k)$ and eigenvector $f = f_1 \otimes \cdots \otimes f_k \in H$ were defined as a k-tuple of (necessarily) real numbers and a decomposable vector $0 \neq f \in H$ such that

$$T_r f_r + \sum_{s=1}^{k} \lambda_s V_{rs} f_r = 0, \qquad r = 1, \dots, k.$$

We adopt the same terminology and now claim

THEOREM 3. *Let $\lambda \in \mathbb{R}^k$ and $f = f_1 \otimes \cdots \otimes f_k$ be an eigenvalue and corresponding eigenvector. Then $f \in \mathscr{D}(\Gamma_s)$, $s = 1, 2, \dots, k$ and $\Gamma_s f = \lambda_s f$.*

Proof. Appealing again to our earlier work [2], we claim

$$T_r P(M) + \sum_{s=1}^{k} V_{rs}^{\dagger}\Delta_0^{-1}\Delta_s P(M) = 0.$$

Then we have for the eigenvalue λ and eigenvector f,

$$T_r^{\dagger} P(M) f - T_r^{\dagger} f + \sum_{s=1}^{k} V_{rs}^{\dagger}(\Delta_0^{-1}\Delta_s P(M) - \lambda_s I) f = 0, \qquad r = 1, 2, \dots, k.$$

We note that $T_r^\dagger P(M)f - T_r^\dagger f \to 0$ as $M \to \mathbb{R}^k$ in H (and also in $H^\wedge$). Thus

$$\lim_{M\to\mathbb{R}^k} \sum_{s=1}^{k} V_{rs}^\dagger(\Delta_0^{-1}\Delta_s P(M) - \lambda_s I)f = 0$$

in H. Now it follows that $\Delta_s P(M)f \to \lambda_s \Delta_0 f$ in H and so $\Delta_s f = \lambda_s \Delta_0 f$, $\Delta_0^{-1}\Delta_s f = \lambda_s f$ and the result is established. We have used here the fact that if Δ_{0rs} denotes the cofactor of $V_{rs}^\dagger$ in the expansion of Δ_0 then $\sum_{r=1}^{k} \Delta_{0rt} V_{rs}^\dagger = \Delta_0$ if $t = s$, 0 if $t \neq s$; see [1, Theorem 6.4.1, p. 106].

The upshot of the theorem is that an eigenvalue and decomposable eigenvector correspond to a simultaneous eigenvector for the operators $\Gamma_1, \ldots, \Gamma_k$. The possibility remains for $\Gamma_1, \ldots, \Gamma_k$ to have a simultaneous eigenvector f not of the form $f = f_1 \otimes \cdots \otimes f_k \in H$.

The generalized eigenvector expansion and Parseval equality for vectors $f \in H^\wedge$ is now a ready consequence of the theory of several commuting self-adjoint operators (see [5, pp. 270–285]) used in our previous discussion of multiparameter spectral theory [3, Sect. 4]. The arguments are similar and we leave the reader to supply details. Accordingly, we conclude this section with

THEOREM 4. *There is a spectral measure $E(M)$ defined on the Borel subsets of $\mathbb{R}^k$, vanishing outside the Cartesian product of the spectra of the operators $\Gamma_1, \ldots, \Gamma_k$, taking values in the set of orthogonal projections on $H^\wedge$ and such that for $f \in H^\wedge$*

(i) $[f, f] = \int_{\mathbb{R}^k} [E(d\lambda), f, f]$,

(ii) $f = \int_{\mathbb{R}^k} E(d\lambda) f$,

this integral converging in the norm of $H^\wedge$.

REFERENCES

1. F. V. ATKINSON, "Multi-parameter Eigenvalue Problems, Vol. I: Matrices and Compact Operators," Academic Press, New York, 1972.
2. P. J. BROWNE, Multi-parameter spectral theory, *Indiana Univ. J. Math.* **24** (1974), 249–257.
3. P. J. BROWNE, Abstract multiparameter theory, I, *J. Math. Anal. Appl.* **60** (1977), 259–273.
4. A. KÄLLSTRÖM AND B. D. SLEEMAN, Solvability of a linear operator system, *J. Math. Anal. Appl.* **55** (1976), 785–793.
5. E. PRUGOVEČKI, "Quantum Mechanics in Hilbert Space," Academic Press, New York, 1971.
6. F. RIESZ AND B. SZ.-NAGY, "Functional Analysis" (L. F. Baron, Trans.), Frederick Ungar, New York, 1971.