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Finite index supergroups and subgroups of torsionfree abelian groups of rank two

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Abstract

Every torsionfree abelian group A of rank two is a subgroup of $\mathbb{Q} \oplus \mathbb{Q}$ and is expressed by a direct limit of free abelian groups of rank two with lower diagonal integer-valued 2×2-matrices as the bonding maps. Using these direct systems we classify all subgroups of $\mathbb{Q} \oplus \mathbb{Q}$ which are finite index supergroups of A or finite index subgroups of A. Using this classification we prove that for each prime p there exists a torsionfree abelian group A satisfying the following, where $A \leqslant \mathbb{Q} \oplus \mathbb{Q}$ and all supergroups are subgroups of $\mathbb{Q} \oplus \mathbb{Q}$:

(1) for each natural number s there are $\sum_{q \mid s,\, \gcd(p,q)=1} q$ s-index supergroups and also $\sum_{q \mid s,\, \gcd(p,q)=1} q$ s-index subgroups;
(2) each pair of distinct s-index supergroups are non-isomorphic and each pair of distinct s-index subgroups are non-isomorphic.

Keywords: Torsionfree abelian group; Rank two; Finite index; Subgroup; Supergroup

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1. Introduction and main results

This investigation originally started from a classification of finite-sheeted covering maps on connected compact abelian groups. When groups are 1-dimensional, a classification is fairly easy [2], which is reduced in principle to Baer's classification of torsionfree abelian groups of rank one. As a next step we have investigated the 2-dimensional case, which will appear in another paper [4]. In that paper we proved the following for a connected compact group Y:

(a) Every finite-sheeted covering map from a connected space over Y is equivalent to a covering homomorphism from a compact, connected group. Moreover, if Y is abelian, then the domain of the homomorphism is abelian.
(b) Let $f : X \to Y$ and $f' : X' \to Y$ be finite-sheeted covering homomorphisms over Y. Then f and f' are equivalent as covering maps if and only if the two homomorphisms are equivalent as topological homomorphisms.

Accordingly we can reduce all things to the category of compact abelian groups, and then, by the Pontrjagin duality, it reduces further to an investigation of the equivalence class of finite index supergroups of torsionfree abelian groups of rank two. Here, two supergroups B and C of a group A are equivalent, if there exists an isomorphism between B and C which fixes every element of A. When B and C are finite index supergroups of A, the embedding of A to the direct sum of two copies of the rational group $\mathbb{Q} \oplus \mathbb{Q}$ induces embeddings of B and C to $\mathbb{Q} \oplus \mathbb{Q}$ and then equivalent supergroups B and C are mapped onto the same subgroup of $\mathbb{Q} \oplus \mathbb{Q}$. From now on, when we consider a supergroup of a torsionfree abelian group A of rank two, we assume that A is embedded into $\mathbb{Q} \oplus \mathbb{Q}$ and the supergroup is a subgroup of $\mathbb{Q} \oplus \mathbb{Q}$.

Every torsionfree abelian group A of rank two is presented by $A = \varinjlim(A_n, g_n\colon n < \omega)$ where A_n's are copies of $\mathbb{Z} \oplus \mathbb{Z}$ and $g_n = \left[\begin{smallmatrix} p_n & 0 \\ \alpha_n & t_n \end{smallmatrix}\right] \in M_2(\mathbb{Z})$ such that $p_n, t_n > 0$ and $0 \leqslant \alpha_n < p_n$. For a natural number s let F_s be the set of all positive integers q satisfying $\gcd(p_n, q) = 1$ for almost all n and that there exists q_1 such that $qq_1r = s$ and

(a) $\gcd(p_n, q_1) = \gcd(t_n, r) = 1$ for almost all n;
(b) if $q_1 > 1$, the $\gcd(t_n, q_1) \neq 1$ for infinitely many n's.

Under the above presentation of A we prove the following:

(1) For a natural number s, the number of distinct s-index supergroups of A is $\sum_{q \in F_s} q$ and the number of s-index subgroups of A is also $\sum_{q \in F_s} q$.
(2) Let $(\alpha_n\colon n < \omega)$ be semi-periodic and p a positive integer. If $p_n = p$, $t_n = 1$ for almost all n or if $p_n = t_n = p$ for almost all n, then finite index supergroups of A are isomorphic to A, and all finite index subgroups of A are also isomorphic to A (Corollary 6.7).
(3) Let p be a prime and $p_n = p$, $t_n = 1$ for every n and q be a natural number with $q > 1$ and $\gcd(p, q) = 1$. Let a p-adic integer $\sum_{n=0}^{\infty} \alpha_n p^n$ is not quadratic over $\mathbb{Q}$. Then, for each natural number s, distinct s-index supergroups of A are non-isomorphic. Moreover, distinct s-index subgroups of A are non-isomorphic (Theorem 5.2).

A restricted form of (3) was asserted in our former paper [3].

In the second section of the present paper we explain how to express a rank 2 torsionfree abelian group, its finite index supergroups, and its finite index subgroups by a sequence of integer-valued matrices. In Section 3 we define super-admissible sequences and sub-admissible sequences and prove a classification theorem of the finite index supergroups and the subgroups of A. In Section 4 we concentrate to a certain kind of rank 2 torsionfree abelian groups and prove Theorem 5.2. In Section 5 we investigate about other groups of this kind.

Let ω denote the set of all non-negative integers and also denote the least infinite ordinal. Hence $n \in \omega$ and $n < \omega$ have the same meaning. Let $\mathbb{N}$ be the set of positive integers, i.e. $\mathbb{N} \cup \{0\} = \omega$. When we use the word "integer" without the adjective "p-adic," it always means rational integer.

The major work of rank 2 torsionfree abelian groups is the work of Beaumont and Pierce [1]. They did not present groups in a particular manner, but they introduced invariants for subgroups of $\mathbb{Q} \oplus \mathbb{Q}$ and using them they showed a complete system of the invariants for the quasi-isomorphism classes of rank 2 torsionfree abelian groups. Two abelian groups are quasi-isomorphic, if they have finite index isomorphic subgroups. Therefore, our approach is related to the isomorphism problem of determining whether two given quasi-isomorphic groups are isomorphic. On the other hand, recent works by logicians indicate that there is no complete answer to this problem [7,8,10], because the equivalence introduced in [1] has the same complexity level as that of the isomorphism types of the rank 1 torsionfree abelian groups, but the complexity level of the isomorphism types of the rank 2 torsionfree abelian groups is strictly harder than that of the rank 1 ones. However, it is still necessary to clarify relationship between our approach using direct systems and the invariants for the quasi-isomorphism classes. We will mention this slightly in Section 6.

2. Putting into limit systems

Definition 2.1. Let $M_n(\mathbb{Z})$ be the set of integer-valued $n \times n$-matrices and $CM_n(\mathbb{Z})$ be the set of $n \times n$-matrices $c = [c_{ij}] \in M_n(\mathbb{Z})$ such that $c_{ii} > 0$, $c_{ij} = 0$ for $j < i$ and $0 \leqslant c_{ij} < c_{ii}$ for $j > i$. Similarly let $CM_n^*(\mathbb{Z})$ be the set of $c = [c_{ij}] \in M_n(\mathbb{Z})$ such that $c_{ii} > 0$, $c_{ij} = 0$ for $j < i$ and $0 \leqslant c_{ij} < c_{jj}$ for $i < j$. For $f \in M_n(\mathbb{Z})$ let ${}^t f$ denote the transposed matrix of f. An element of a free abelian group of rank n is denoted by a column vector and so matrices act from the left.

The notation "CM_n" comes from covering homomorphisms [4].

Lemma 2.2. *For each regular matrix $h \in M_n(\mathbb{Z})$ there exist unique $c \in CM_n(\mathbb{Z})$ and $f \in GL_n(\mathbb{Z})$ such that $h = cf$. Consequently there exist unique $c \in CM_n(\mathbb{Z})$ and $f \in GL_n(\mathbb{Z})$ such that $h = f\,{}^t c$.*

Similarly, there exist unique $c \in CM_n^*(\mathbb{Z})$ and $f \in GL_n(\mathbb{Z})$ such that $h = fc$.

Proof. Starting from h we get such a c by successive use of elementary column operations. Hence, the existence of c and f is clear. Considering h and c to be homomorphisms mapping column vectors, we have $\mathrm{Im}(c) = \mathrm{Im}(h)$. On the other hand, if $\mathrm{Im}(c) = \mathrm{Im}(c')$ for $c, c' \in CM_n(\mathbb{Z})$, then $c = c'$. Therefore the uniqueness of c is clear and consequently f is also unique. Apply this for ${}^t h$, then we have the second statement.

Similarly, considering successive use of elementary row operations as the preceding argument, we see the existence of $c \in CM_n^*(\mathbb{Z})$ and $f \in GL_n(\mathbb{Z})$ such that $h = fc$. □

Now the next lemma is clear.

Lemma 2.3. *Let B be a free abelian group of rank n and let $h : \mathbb{Z}^n \to B$ be an injective homomorphism. Then there exists a unique base for B such that h is expressed as a matrix c such that ${}^t c$ is in $CM_n(\mathbb{Z})$. Moreover the matrix c is unique.*

Similarly let C be a free abelian group of rank n and let $h : C \to \mathbb{Z}^n$ be an injective homomorphism. Then there exists a unique base for C such that h is expressed as a matrix c in $CM^*_n(\mathbb{Z})$. Moreover the matrix c is unique.

Lemma 2.4. *Let A, X and Y be subgroups of a group B. Then $X + Y = \{x + y\colon x \in X, y \in Y\}$ is a subgroup of B. If $A \cap X \subseteq Y \subseteq A$, then $(X + Y) \cap A = Y$. If $Y \subseteq A \subseteq X + Y$, then $(X \cap A) + Y = A$.*

Proof. The first statement is clear. To prove the second one it suffices to show $(X + Y) \cap A \subseteq Y$. Let $x \in X$, $y \in Y$ and $x + y = a \in A$. Then $x = a - y \in A$ and hence $x \in A \cap X \subseteq Y$, which implies $x + y \in Y$. To show the third one, let $a \in A$. We have $x \in X$ and $y \in Y$ such that $a = x + y$. Since $x = a - y \in A$, $a \in (X \cap A) + Y$. □

For an abelian group A let $A^* = \mathrm{Hom}(A, \mathbb{Z})$. Let A be a subgroup of B such that B/A is finite. Then the correspondence $h \mapsto h \mid A$ for $h \in B^*$ is a injection. Hence we identify B^* with a subgroup of A^*.

Lemma 2.5. *Let B be a finitely generated free abelian group and A its finite index subgroup. Then B^* is a finite index subgroup of A^*.*

Proof. We have an exact sequence

$$0 \to (B/A)^* \to B^* \to A^* \to \mathrm{Ext}(B/A, \mathbb{Z}) \to \mathrm{Ext}(B, \mathbb{Z}).$$

Since $(B/A)^* = 0$, $\mathrm{Ext}(B/A, \mathbb{Z}) \cong B/A$ and $\mathrm{Ext}(B, \mathbb{Z}) = 0$, we have the conclusion. □

Lemma 2.6. *Let B_2 be a finitely generated free abelian group and B_1, A_1, A_2 be subgroups such that $A_2 + B_1 = B_2$ and $A_2 \cap B_1 = A_1$ and B_2/A_1 is finite. Then $A^*_2 + B^*_1 = A^*_1$ and $A^*_2 \cap B^*_1 = B^*_2$ under the above identification.*

Proof. We first show $A^*_2 \cap B^*_1 = B^*_2$. Since $B^*_2 \leqslant A^*_2 \cap B^*_1$, it suffices to show $A^*_2 \cap B^*_1 \leqslant B^*_2$. If $h \in A^*_1$ belongs to $A^*_2 \cap B^*_1$, then we have $h_1 \in A^*_2$ and $h_2 \in B^*_1$ such that $h_1 \mid A_1 = h_2 \mid A_1 = h$. Define $\bar{h}(a + b) = h_1(a) + h_2(b)$ for $a \in A_2$ and $b \in B_1$. Then $\bar{h}$ is well defined and $\bar{h} \in B^*_2$.

To show $A^*_2 + B^*_1 = A^*_1$ by contradiction, suppose the negation. Since $A^*_2 + B^*_1 \leqslant A^*_1$ and A^*_1 is free and $A^*_2 + B^*_1$ is a finite index subgroup of A^*_1 by Lemma 2.5, we have $h : A^*_2 + B^*_1 \to \mathbb{Z}$ which does not extend on A^*_1. We apply the preceding result to $A^*_2 + B^*_1$. Then we have $(A^*_2 + B^*_1)^* = A^{**}_2 \cap B^{**}_1$. Now h belongs to $A^{**}_2 \cap B^{**}_1$, which is naturally isomorphic to $A_2 \cap B_1 = A_1$. Hence h can extend on A^*_1, which is a contradiction. □

Lemma 2.7. *Let B be a torsionfree abelian group of rank m and A be a subgroup of finite index. Then the following hold.*

(1) *For each $n \in \omega$, let A_n be free abelian groups of rank m such that $A_n \subseteq A_{n+1}$ and $A = \bigcup_{n<\omega} A_n$. Then there exist $n_0 < \omega$ and free abelian groups B_n of rank m such that $A_n \subseteq B_n \subseteq B$ and $B_n/A_n \simeq B/A$ for $n \geqslant n_0$ and $B = \bigcup_{n \geqslant n_0} B_n$.*

(2) *For each $n \in \omega$, let B_n be free abelian groups of rank m such that $B_n \subseteq B_{n+1}$ and $B = \bigcup_{n<\omega} B_n$. Then there exist $n_0 < \omega$ such that $B_n/(B_n \cap A) \simeq B/A$ for $n \geqslant n_0$. Consequently each $B_n \cap A$ is a free abelian group of rank m and $A = \bigcup_{n \geqslant n_0}(B_n \cap A)$.*

Proof. (1) Choose a finite subset F of B so that $\langle A \cup F\rangle = B$. We have n_0 such that $\langle F\rangle \cap A \subseteq A_{n_0}$. Let $B_n = \langle F\rangle + A_n$. Then $B_n \cap A = A_n$ for $n \geqslant n_0$ by Lemma 2.4. Hence $B_n/A_n \simeq B/A$.

(2) Choose a finite subset F of B so that $\langle A \cup F\rangle = B$. We have n_0 such that $\langle F\rangle \subseteq B_{n_0}$. Then $B_n/(B_n \cap A) \simeq B/A$ for $n \geqslant n_0$. The other statements are clear. $\square$

Definition 2.8. When A is a subgroup of a group B, we call B as a *supergroup* of A. When A is a finite index subgroup of a group B, we say B a finite index supergroup of A.

When we express a homomorphism by a matrix, we assume that an element of the domain is expressed by a column vector and the matrix acts from the left. We frequently identify matrices and homomorphisms between free abelian groups of finite rank. By Lemma 2.2 and an easy induction we have

Lemma 2.9. *Let A be a torsionfree abelian group of rank 2. Then there exist lower diagonal matrices $f_n = \left[\begin{smallmatrix} p_n & 0 \\ \alpha_n & t_n \end{smallmatrix}\right] \in M_2(\mathbb{Z})$ such that $p_n, t_n > 0$ and $0 \leqslant \alpha_n < p_n$, i.e. ${}^t f_n \in CM_2(\mathbb{Z})$, and the direct limit $\varinjlim(A_n, f_n\colon n < \omega)$ is isomorphic to A where each A_n is a copy of $\mathbb{Z} \oplus \mathbb{Z}$.*

Lemma 2.10. *Let A be the direct limit $\varinjlim(A_n, f_n\colon n < \omega)$ given in Lemma 2.9. If B is a torsionfree abelian group which contains A as a finite index subgroup, then there exist $n_0 < \omega$ and lower diagonal matrices $g_n = \left[\begin{smallmatrix} p'_n & 0 \\ \beta_n & t'_n \end{smallmatrix}\right] \in M_2(\mathbb{Z})$ and $h_n = \left[\begin{smallmatrix} q_n & 0 \\ c_n & r_n \end{smallmatrix}\right]$ for $n \geqslant n_0$ which satisfy the following*:

(1) *$p'_n, t'_n, q_n, r_n > 0$ and $0 \leqslant c_n < q_n$;*
(2) *the direct limit $\varinjlim(B_n, g_n\colon n < \omega)$ is isomorphic to B where B_n's are copies of $\mathbb{Z} \oplus \mathbb{Z}$;*
(3) *the diagram*

$$\begin{tikzcd} B_n \arrow[r, "g_n"] & B_{n+1} \\ A_n \arrow[u, "h_n"] \arrow[r, "f_n"'] & A_{n+1} \arrow[u, "h_{n+1}"'] \end{tikzcd}$$

commutes and $\mathrm{Im}(g_n) + \mathrm{Im}(h_{n+1}) = B_{n+1}$ *and* $\mathrm{Im}(g_n) \cap \mathrm{Im}(h_{n+1}) = \mathrm{Im}(h_{n+1} \circ f_n) = \mathrm{Im}(g_n \circ h_n)$;
(4) *$B/A \simeq B_n/\mathrm{Im}(h_n)$ for each n.*

Proof. Since each f_n is injective, we may assume that A_n is a subgroup of A. Then we have $n_0 < \omega$ and subgroups B'_n of B such that $B'_n/A_n \simeq B/A$ for $n \geqslant n_0$, by Lemma 2.7(1). Since each B'_n is isomorphic to $\mathbb{Z} \oplus \mathbb{Z}$, we fix a basis for each B'_n and express the inclusion map from A_n to B'_n by a matrix $h'_n \in M_2(\mathbb{Z})$. For each h'_n there exist an invertible matrix $i_n \in M_2(\mathbb{Z})$ and $q_n, r_n > 0$ and $0 \leqslant c_n < q_n$ such that for $n \geqslant n_0$, $i_n h'_n = \left[\begin{smallmatrix} q_n & 0 \\ c_n & r_n \end{smallmatrix}\right]$. We denote the inclusion map from B'_n to B'_{n+1} by g'_n. For each n, let B_n denote the range of i_n, then $B_n \simeq \mathbb{Z} \oplus \mathbb{Z}$. Further, let $h_n = i_n h'_n$ and $g_n = i_{n+1} g'_n i_n^{-1}$. Since each i_n is invertible in $M_2(\mathbb{Z})$, $g_n \in M_2(\mathbb{Z})$ and $\varinjlim(B_n, g_n\colon n < \omega)$ is isomorphic to $\varinjlim(B'_n, g'_n\colon n < \omega) = B$. $\square$

By Lemma 2.6 we have an inverse system which is dual to the direct system in Lemma 2.10.

$$\begin{tikzcd} B_n^* \arrow[d, "h_n^*"'] & B_{n+1}^* \arrow[l, "g_n^*"'] \arrow[d, "h_{n+1}^*"] \\ A_n^* & A_{n+1}^* \arrow[l, "f_n^*"] \end{tikzcd}$$

commutes and $\mathrm{Im}(f_n^*) + \mathrm{Im}(h_n^*) = A_n^*$ and $\mathrm{Im}(f_n^*) \cap \mathrm{Im}(h_n^*) = \mathrm{Im}(f_n^* \cdot h_{n+1}^*) = \mathrm{Im}(h_n^* \cdot g_n^*)$.

Since the dual matrix is given by transposed matrix, we have

$$f_n^* = \begin{bmatrix} p_n & \alpha_n \\ 0 & t_n \end{bmatrix}, \quad g_n^* = \begin{bmatrix} p'_n & \beta_n \\ 0 & t'_n \end{bmatrix} \quad \text{and} \quad h_n^* = \begin{bmatrix} q_n & c_n \\ 0 & r_n \end{bmatrix} \quad \text{for } n < \omega.$$

Lemma 2.11. *For $n < \omega$, let $f_n, g_n, h_n \in M_2(\mathbb{Z})$ be given by $f_n = \left[\begin{smallmatrix} p_n & \alpha_n \\ 0 & t_n \end{smallmatrix}\right]$, $h_n = \left[\begin{smallmatrix} q_n & c_n \\ 0 & r_n \end{smallmatrix}\right]$, $h_{n+1} = \left[\begin{smallmatrix} q_{n+1} & c_{n+1} \\ 0 & r_{n+1} \end{smallmatrix}\right]$ such that*

- $p_n, t_n, q_n, r_n, q_{n+1}, r_{n+1} > 0$;
- $0 \leqslant \alpha_n$ *and* $h_n, h_{n+1} \in CM_2(\mathbb{Z})$;
- $\mathrm{Im}(h_n) + \mathrm{Im}(f_n) = \mathbb{Z} \oplus \mathbb{Z}$ *and* $\mathrm{Im}(h_n) \cap \mathrm{Im}(f_n) = \mathrm{Im}(f_n h_{n+1}) = \mathrm{Im}(h_n g_n)$.

Let $d = \gcd(p_n, q_n)$ and $p_n = p^* d$ and $q_n = q^* d$.

Then $\gcd(t_n, r_n) = \gcd(p_n, q_n, c_n t_n - r_n \alpha_n) = 1$, $q_n = q_{n+1} d$, $r_n d = r_{n+1}$ and $q^* = q_{n+1}$. Consequently there exist q and r such that for sufficiently large n, $q_n = q$, $r_n = r$, $\gcd(p_n, q) = \gcd(t_n, r) = 1$ and $g_n = \left[\begin{smallmatrix} p_n & (p_n c_{n+1} + \alpha_n r - c_n t_n) q^{-1} \\ 0 & t_n \end{smallmatrix}\right]$.

Proof. By the assumption $\mathrm{Im}(h_n) + \mathrm{Im}(f_n) = \mathbb{Z} \oplus \mathbb{Z}$, we have $\gcd(t_n, r_n) = \gcd(p_n, q_n, c_n t_n - r_n \alpha_n) = 1$. We remark that $\gcd(d, c_n t_n - r_n \alpha_n) = 1$. By the other assumption $\mathrm{Im}(h_n) \cap \mathrm{Im}(f_n) = \mathrm{Im}(f_n h_{n+1}) = \mathrm{Im}(h_n g_n)$, we have $\mathbb{Z} \oplus \mathbb{Z}/\mathrm{Im}(h_n) \simeq \mathrm{Im}(f_n)/\mathrm{Im}(f_n h_{n+1}) \simeq \mathbb{Z} \oplus \mathbb{Z}/\mathrm{Im}(h_{n+1})$ and hence $q_n r_n = q_{n+1} r_{n+1}$. Since

$$\begin{aligned} g_n &= h_n^{-1} f_n h_{n+1} \\ &= \begin{bmatrix} q_n^{-1} & -c_n q_n^{-1} r_n^{-1} \\ 0 & r_n^{-1} \end{bmatrix} \begin{bmatrix} p_n q_{n+1} & p_n c_{n+1} + \alpha_n r_{n+1} \\ 0 & t_n r_{n+1} \end{bmatrix} \\ &= \begin{bmatrix} p_n q_n^{-1} q_{n+1} & (p_n c_n + \alpha_n r_{n+1} - c_n r_n^{-1} r_{n+1} t_n) q_n^{-1} \\ 0 & t_n r_n^{-1} r_{n+1} \end{bmatrix} \in M_2(\mathbb{Z}), \end{aligned}$$

we have $r_n \mid r_{n+1} t_n$, $q_n \mid q_{n+1} p_n$ and $(p_n c_n + \alpha_n r_{n+1} - c_n r_n^{-1} r_{n+1} t_n) q_n^{-1} \in \mathbb{Z}$. Since $\gcd(r_n, t_n) = 1$ and $\gcd(p^*, q^*) = 1$, there are positive integers k, k' such that $r_n k = r_{n+1}$ and $q^* k' = q_{n+1}$. We have $q^* d r_n = q_n r_n = q_{n+1} r_{n+1} = q^* k' k r_n$ and consequently $d = kk'$. We have

$$\begin{aligned}\bigl(p_n c_n + \alpha_n r_{n+1} - c_n r_n^{-1} r_{n+1} t_n\bigr) q_n^{-1} &= (p_n c_n + \alpha_n r_n k - c_n t_n k) q_n^{-1} \\ &= \bigl(p_n c_n + (\alpha_n r_n - c_n t_n) k\bigr) q_n^{-1}.\end{aligned}$$

Since $d \mid q_n$ and $d \mid p_n$, $d \mid (\alpha_n r_n - c_n t_n)k$. Hence $\gcd(d, c_n t_n - r_n \alpha_n) = 1$. This implies $d \mid k$, which implies $d = k$ and $k' = 1$. Now we have $q_n = q_{n+1} d$, $r_n d = r_{n+1}$ and $q^* = q_{n+1}$.

Since $q_n \geqslant q_{n+1} > 0$, there exists n_0 such that $d = \gcd(p_n, q_n) = 1$ for each $n \geqslant n_0$. That is $q_n = q_{n+1} = q$ and $r_n = r_{n+1} = r$. This completes the proof. □

In Lemma 2.10 we are interested in finite index supergroups. We use Lemma 2.7 to analyze finite index subgroups.

Lemma 2.12. *Let A be the direct limit $\varinjlim(A_n, f_n\colon n < \omega)$ given in Lemma 2.9. If C is a finite index subgroup of A, then there exist $n_0 < \omega$ and lower diagonal matrices $e_n = \left[\begin{smallmatrix} p'_n & 0 \\ \beta_n & t'_n \end{smallmatrix}\right] \in M_2(\mathbb{Z})$ and $h_n = \left[\begin{smallmatrix} q_n & 0 \\ c_n & r_n \end{smallmatrix}\right]$ for $n \geqslant n_0$ which satisfy the following*:

(1) *$p'_n, t'_n, q_n, r_n > 0$ and $0 \leqslant c_n < r_n$;*
(2) *the direct limit $\varinjlim(C_n, e_n\colon n < \omega)$ is isomorphic to C where C_n's are copies of $\mathbb{Z} \oplus \mathbb{Z}$;*
(3) *the diagram*

$$\begin{array}{ccc} A_n & \xrightarrow{f_n} & A_{n+1} \\ {\scriptstyle h_n}\uparrow & & \uparrow{\scriptstyle h_{n+1}} \\ C_n & \xrightarrow[e_n]{} & C_{n+1} \end{array}$$

commutes and $\mathrm{Im}(f_n) + \mathrm{Im}(h_{n+1}) = A_{n+1}$ *and* $\mathrm{Im}(f_n) \cap \mathrm{Im}(h_{n+1}) = \mathrm{Im}(h_{n+1} \cdot e_n) = \mathrm{Im}(f_n \cdot h_n)$;
(4) *$A/C \simeq A_n / \mathrm{Im}(h_n)$ for every $n \leqslant n_0$.*

Proof. Since f_n's are injective, we may assume that A_n's are subgroups of A. Then we have $n_0 < \omega$ and subgroups C'_n of C such that $A_n / C'_n \simeq A/C$ for $n \geqslant n_0$, by Lemma 2.7(2). Since C'_n's are isomorphic to $\mathbb{Z} \oplus \mathbb{Z}$, we fix a base for each C'_n and express the inclusion map from C'_n to A_n by a matrix $h'_n \in M_2(\mathbb{Z})$. For each h'_n there exist an invertible matrix $i_n \in M_2(\mathbb{Z})$, integers $q_n, r_n > 0$, and $0 \leqslant c_n < r_n$ such that for $n \geqslant n_0$, $h'_n i_n = \left[\begin{smallmatrix} q_n & 0 \\ c_n & r_n \end{smallmatrix}\right]$. We denote the inclusion map from C'_n to C'_{n+1} by e'_n. Let C_n be a copy of $\mathbb{Z} \oplus \mathbb{Z}$ which is the domain of i_n and $h_n = h'_n i_n$ and $e_n = i_{n+1} e'_n i_n^{-1}$. Since i_n's are invertible in $M_2(\mathbb{Z})$, $e_n \in M_2(\mathbb{Z})$ and $\varinjlim(C_n, e_n\colon n < \omega)$ is isomorphic to $\varinjlim(C'_n, e'_n\colon n < \omega) = C$. □

Lemma 2.13. *Let $f_n, e_n, h_n, h_{n+1} \in M_2(\mathbb{Z})$ be given as $f_n = \left[\begin{smallmatrix} p_n & 0 \\ \alpha_n & t_n \end{smallmatrix}\right]$, $h_n = \left[\begin{smallmatrix} q_n & 0 \\ c_n & r_n \end{smallmatrix}\right]$ and $h_{n+1} = \left[\begin{smallmatrix} q_{n+1} & 0 \\ c_{n+1} & r_{n+1} \end{smallmatrix}\right]$ such that*

- $p_n, t_n, q_n, r_n, q_{n+1}, r_{n+1} > 0$;
- $0 \leqslant \alpha_n$ *and* $h_n, h_{n+1} \in CM_2^*(\mathbb{Z})$;
- $\mathrm{Im}(h_{n+1}) + \mathrm{Im}(f_n) = \mathbb{Z} \oplus \mathbb{Z}$ *and* $\mathrm{Im}(h_{n+1}) \cap \mathrm{Im}(f_n) = \mathrm{Im}(f_n h_n) = \mathrm{Im}(h_{n+1} e_n)$.

Let $d = \gcd(t_n, r_{n+1})$ and $t_n = t^* d$ and $r_{n+1} = r^* d$.

Then $\gcd(p_n, q_{n+1}) = \gcd(t_n, r_{n+1}, c_{n+1}p_n - q_{n+1}\alpha_n) = 1$, $q_{n+1}d = q_n$, $r_{n+1} = r_n d$ and $r^* = r_n$. Consequently, there exist q and r such that for sufficiently large n we have $q_n = q$, $r_n = r$, $\gcd(p_n, q) = \gcd(t_n, r) = 1$ and $e_n = \left[\begin{smallmatrix} p_n & 0 \\ (-p_n c_{n+1} + \alpha_n q + c_n t_n) r^{-1} & t_n \end{smallmatrix}\right]$.

Proof. Since the proof is similar to that of Lemma 2.11, we omit the reasoning and only indicate the changes for e_n, k, and k'. By the fact

$$\begin{aligned} e_n &= h_{n+1}^{-1} f_n h_n \\ &= \begin{bmatrix} q_{n+1}^{-1} & 0 \\ -c_{n+1} q_{n+1}^{-1} r_{n+1}^{-1} & r_{n+1}^{-1} \end{bmatrix} \begin{bmatrix} p_n q_n & 0 \\ q_n \alpha_n + c_n t_n & t_n r_n \end{bmatrix} \\ &= \begin{bmatrix} p_n q_n q_{n+1}^{-1} & 0 \\ (-p_n c_{n+1} q_n q_{n+1} + q_n \alpha_n + c_n t_n) r_{n+1}^{-1} & t_n r_n r_{n+1}^{-1} \end{bmatrix} \in M_2(\mathbb{Z}), \end{aligned}$$

we have $q_{n+1} \mid q_n$ and $r^* \mid r_n$. Let $q_{n+1}k = q_n$ and $r^* k' = r_n$. The equation $q_n r_n = q_{n+1} r_{n+1}$ implies $d = kk'$ as before. The fact $r_{n+1} \mid -p_n c_{n+1} q_n q_{n+1} + q_n \alpha_n + c_n t_n$ implies $d \mid k$ and $k' = 1$ also as before. $\square$

3. Classification of finite index supergroups and subgroups

Using results in the previous section we classify finite index supergroups and subgroups of a torsionfree abelian group of rank two.

Definition 3.1. Let A be a torsionfree abelian group of rank two which is expressed as in Lemma 2.9, that is, $A = \varinjlim(A_n, f_n\colon n < \omega)$, A_n's are copies of $\mathbb{Z} \oplus \mathbb{Z}$ and $f_n = \left[\begin{smallmatrix} p_n & 0 \\ \alpha_n & t_n \end{smallmatrix}\right] \in M_2(\mathbb{Z})$ such that $p_n, t_n > 0$ and $0 \leqslant \alpha_n < p_n$.

Assume $\gcd(p_n, q) = \gcd(t_n, r) = 1$ for sufficiently large n. A sequence $\mathbf{c}_{qr}$ is super-admissible, if

- $\mathbf{c}_{qr} : [n_0, \omega) \to \{0, 1, \ldots, q-1\}$ for some $n_0 < \omega$;
- $p_n \mathbf{c}_{qr}(n+1) \equiv t_n \mathbf{c}_{qr}(n) - r\alpha_n \bmod q$.

A sequence $\mathbf{c}_{qr}$ is sub-admissible, if

- $\mathbf{c}_{qr} : [n_0, \omega) \to \{0, 1, \ldots, r-1\}$ for some $n_0 < \omega$;
- $p_n \mathbf{c}_{qr}(n+1) \equiv t_n \mathbf{c}_{qr}(n) + q\alpha_n \bmod r$.

Two super-admissible sequences $\mathbf{c}_{qr}$ and $\mathbf{c}'_{q'r'}$ are equivalent, if $q = q'$, $r = r'$ and $\mathbf{c}_{qr}(n) = \mathbf{c}'_{q'r'}(n)$ for sufficiently large n. For sub-admissible sequences the equivalence is defined in the same way.

When we try to construct a super-admissible sequence, we can start from arbitrary $\mathbf{c}_{qr}(0) \in \{0, 1, \ldots, q-1\}$ and can inductively define $\mathbf{c}_{qr}(n+1)$ from $\mathbf{c}_{qr}(n)$, because the given equation concerns mod q and $\gcd(p_n, q) = 1$. However in the case of sub-admissible sequences to define

$\mathbf{c}_{qr}(n)$'s for all n the choice of $\mathbf{c}_{qr}(0) \in \{0, 1, \ldots, r-1\}$ may be restricted. In spite of this, many things still go parallel with super-admissible and sub-admissible sequences. We will state definitions and statements in pairs.

Definition 3.2. For a super-admissible sequence $\mathbf{c}_{qr}$ defined, define a sequence $\mathbf{g}_{qr} : [n_0, \omega) \to \mathbb{Z}$ by $\mathbf{g}_{qr}(n) = (p_n \mathbf{c}_{qr}(n+1) - t_n \mathbf{c}_{qr}(n) + r\alpha_n)/q$. For a sub-admissible sequence $\mathbf{c}_{qr}$ defined, define a sequence $\mathbf{e}_{qr} : [n_0, \omega) \to \mathbb{Z}$ by $\mathbf{e}_{qr}(n) = (-p_n \mathbf{c}_{qr}(n+1) + t_n \mathbf{c}_{qr}(n) + q\alpha_n)/r$.

In the sequel an abelian group A always denotes a torsionfree abelian group of rank two embedded into $\mathbb{Q} \oplus \mathbb{Q}$ and expressed as $\varinjlim(A_n, f_n\colon n < \omega)$ as in Lemma 2.9. Since the proof of the next lemma is straightforward we omit the proof.

Lemma 3.3. (1) *Let $\mathbf{c}_{qr}$ be a super-admissible sequence and $\mathbf{g}_{qr}$ be a sequence defined in Definition* 3.2. *For each $n \geqslant n_0$ let $g_n = \left[\begin{smallmatrix} p_n & 0 \\ \mathbf{g}_{qr}(n) & t_n \end{smallmatrix}\right]$ and, for each n, B_n be a copy of $\mathbb{Z} \oplus \mathbb{Z}$. Let $h_n = \left[\begin{smallmatrix} q & 0 \\ \mathbf{c}_{qr}(n) & r \end{smallmatrix}\right]$.*

Then $g_n h_n = h_{n+1} f_n$ for $n \geqslant n_0$. The homomorphic image of $\varinjlim(A_n, f_n\colon n < \omega)$ by $\varinjlim(h_n\colon n < \omega)$ is a finite index subgroup of $\varinjlim(B_n, g_n\colon n < \omega)$.

(2) *Let $\mathbf{c}_{qr}$ be a sub-admissible sequence and $\mathbf{e}_{qr}$ be a sequence defined in Definition* 3.2. *For each $n \geqslant n_0$ let $e_n = \left[\begin{smallmatrix} p_n & 0 \\ \mathbf{e}_{qr}(n) & t_n \end{smallmatrix}\right]$ and let C_n be a copy of $\mathbb{Z} \oplus \mathbb{Z}$. Let $h_n = \left[\begin{smallmatrix} q & 0 \\ \mathbf{c}_{qr}(n) & r \end{smallmatrix}\right]$. Then $f_n h_n = h_{n+1} e_n$ for $n \geqslant n_0$. The homomorphic image of $\varinjlim(C_n, e_n\colon n < \omega)$ by $\varinjlim(h_n\colon n < \omega)$ is a finite index subgroup of $\varinjlim(A_n, f_n\colon n < \omega)$.*

Lemma 3.4. *Let B be a subgroup of $\mathbb{Q} \oplus \mathbb{Q}$ which is a finite index supergroup of A. If $\mathbf{c}_{qr}$ and $\mathbf{c}'_{q'r'}$ be super-admissible sequences constructed from B, then $\mathbf{c}_{qr}$ and $\mathbf{c}'_{q'r'}$ are equivalent.*

Proof. We have subgroups B_n of B and subgroups B'_n of B' where each B_n and B'_n are free abelian groups and $B_n/A_n \simeq B/A$ and $B'_n/A_n \simeq B'/A$ for sufficiently large n. Choose $b_1, \ldots, b_s \in B$ and $b'_1, \ldots, b'_s \in B$ so that $\bigcup_{k=1}^s (b_k + A) = B$ and $\bigcup_{k=1}^s (b'_k + A) = B$. Take n_0 so large that $B_n/A_n \simeq B/A$ and $B'_n/A_n \simeq B'/A$ for every $n \geqslant n_0$ and $b_1, \ldots, b_s \in B_{n_0}$ and $b'_1, \ldots, b'_s \in B'_{n_0}$. We have $n_1 \geqslant n_0$ such that $b_1, \ldots, b_s \in B'_{n_1}$ and $b'_1, \ldots, b'_s \in B_{n_1}$. Since $\bigcup_{k=1}^s b_k + A_n = B_n$ and $\bigcup_{k=1}^s b'_k + A_n = B'_n$ for every $n \geqslant n_0$, $B_n = B'_n$ for $n \geqslant n_1$. By Lemma 2.3 $\mathbf{c}_{qr}(n) = \mathbf{c}'_{q'r'}(n)$ for $n \geqslant n_1$ and $q = q'$ and $r = r'$. $\square$

For a subgroup B of $\mathbb{Q} \oplus \mathbb{Q}$ which is a finite index supergroup of A, we have a super-admissible sequence $\mathbf{c}_{qr}$ by Lemmas 2.10 and 2.11. Let $[\mathbf{c}_{qr}]$ denote the equivalence class containing the one containing $\mathbf{c}_{qr}$ respectively. By $\Phi(B)$, we denote the equivalence class $[\mathbf{c}_{qr}]$.

Theorem 3.5. *Let $\varinjlim(A_n, f_n\colon n < \omega)$ be a subgroup of $\mathbb{Q} \oplus \mathbb{Q}$ expressed as in Lemma* 2.9. *Then Φ defines a one to one correspondence between the class of subgroups of $\mathbb{Q} \oplus \mathbb{Q}$ which are finite index supergroups of A and the equivalence classes of super-admissible sequences.*

Proof. Lemma 3.4 implies the well-definedness of Φ as a map from the class of subgroups of $\mathbb{Q} \oplus \mathbb{Q}$ which are finite index supergroups of A. To see that Φ is injective, suppose that B and B' induce $\mathbf{c}_{qr}$ and $\mathbf{c}'_{q'r'}$, respectively, and $[\mathbf{c}_{qr}] = [\mathbf{c}'_{q'r'}]$. Then $q = q'$ and $r = r'$ and there exists n_0 such that $\mathbf{c}_{qr}(n) = \mathbf{c}'_{q'r'}(n)$ for all $n \geqslant n_0$, which implies B and B' are equivalent as supergroups of A and hence $B = B'$. For a given super-admissible sequence $\mathbf{c}_{qr}$ which is defined on $[n_1, \omega)$, let $h_n = \left[\begin{smallmatrix} q & 0 \\ \mathbf{c}_{qr}(n) & r \end{smallmatrix}\right]$ and $g_n = \left[\begin{smallmatrix} p_n & 0 \\ \mathbf{g}_{qr}(n) & t_n \end{smallmatrix}\right]$ for each $n \geqslant n_1$. Since $g_n h_n = f_n h_{n+1}$ for each $n \geqslant n_1$, we have a supergroup B such that $\Phi([B]) = [\mathbf{c}_{qr}]$, that is, Φ is surjective. □

We have the similar statement for finite index subgroups. For a finite index subgroup C of A, we have a sub-admissible sequence $\mathbf{c}_{qr}$ by Lemmas 2.12 and 2.13. By $\Psi(C)$, we denote the sub-admissible sequence $[\mathbf{c}_{qr}]$. Since the proof is similar to that of Theorem 3.5, we omit it.

Theorem 3.6. *Let $\varinjlim(A_n, f_n\colon n < \omega)$ be a subgroup of $\mathbb{Q} \oplus \mathbb{Q}$ expressed as in Lemma 2.9. Then Ψ defines a one to one correspondence between the class of finite index subgroups of A and the equivalence classes of sub-admissible sequences.*

As we mentioned before, there is a difference between super-admissible sequences and sub-admissible ones. The following four results show both the similarities and also the differences between them.

Lemma 3.7. *Let $r = ur_0$ such that $u \mid \prod_{k=0}^{n-1} p_k$ and $\gcd(r_0, p_k) = 1$ and $\gcd(r, t_k) = 1$ for every $0 \leqslant k \leqslant n-1$ and $(\alpha_k\colon 0 \leqslant k < n)$ be a finite sequence of integers.*

Then there exist exactly r_0 integers $0 \leqslant a < r$ such that $a = c_0$ for some sequence $(c_k\colon 0 \leqslant k < n)$ such that $p_k c_{k+1} \equiv c_k t_k + q\alpha_k \bmod r$.

Proof. First we show the number of such integers a is at most r_0. Let $p_k c_{k+1} \equiv c_k t_k + q\alpha_k \bmod r$ and $p_k c'_{k+1} \equiv c'_k t_k + q\alpha_k \bmod r$.

Multiplying $\prod_{k=0}^{i-1} p_k \prod_{k=i+1}^{n} t_k$ to the equation $p_i c_{i+1} \equiv c_i t_i + q\alpha_i \bmod r$ we have

$$\prod_{k=0}^{i} p_k \prod_{k=i+1}^{n} t_k c_{i+1} \equiv \prod_{k=0}^{i-1} p_k \prod_{k=i}^{n-1} t_k c_i + \prod_{k=0}^{i-1} p_k \prod_{k=i+1}^{n-1} t_k q\alpha_k \mod r$$

for $0 \leqslant i \leqslant n-1$.

Adding the left-hand terms and the right ones respectively, we have

$$(1) \quad \prod_{k=0}^{n-1} p_k c_n \equiv \prod_{k=0}^{n-1} t_k c_0 + \sum_{i=0}^{n-1} \prod_{k=0}^{i-1} p_k \prod_{k=i+1}^{n-1} t_k q\alpha_i \mod r$$

and similarly

$$(2) \quad \prod_{k=0}^{n-1} p_k c'_n \equiv \prod_{k=0}^{n-1} t_k c'_0 + \sum_{i=0}^{n-1} \prod_{k=0}^{i-1} p_k \prod_{k=i+1}^{n-1} t_k q\alpha_i \mod r.$$

Hence we have

$$\prod_{k=0}^{n-1} t_k (c_0 - c'_0) \equiv 0 \mod u,$$

which implies $c_0 - c'_0 \equiv 0 \bmod u$, because $\gcd(t_k, r) = 1$ and so $\gcd(t_k, u) = 1$ for every $0 \leqslant k \leqslant n-1$.

To see that there exist r_0 such a's, take $0 \leqslant b, b' < r$ so that $b \not\equiv b' \bmod r_0$. Then we have sequences $(c_k\colon\ 0 \leqslant k \leqslant n)$ and $(c'_k\colon\ 0 \leqslant k \leqslant n)$ such that $c_n = b$, $c'_n = b'$, $p_k c_{k+1} \equiv c_k t_k + q\alpha_k \pmod r$ and $p_k c'_{k+1} \equiv c'_k t_k + q\alpha_k \pmod r$ and $0 \leqslant c_k, c'_k < r$ for every $0 \leqslant k \leqslant n$. Since $\gcd(p_k, r_0) = 1$ for $0 \leqslant k \leqslant n-1$, $c_k \not\equiv c'_k \bmod r_0$ and particularly $c_0 \not\equiv c'_0 \bmod r_0$. These imply the conclusion. $\square$

Lemma 3.8. *We assume the setting of Definition* 3.1.

(1) *Let $r = s_0^{n_0} \cdots s_k^{n_k} r_0$ such that each s_i is a prime and $\{n : s_i \mid p_n\}$ is infinite for each s_i and* $\gcd(r_0, p_n) = 1$ *for almost all n. Then there exist exactly r_0 equivalence classes of sub-admissible sequences $\mathbf{c}_{qr}$.*

(2) *Similarly, let $q = s_0^{n_0} \cdots s_k^{n_k} q_0$ such that each s_i is a prime and $\{n : s_i \mid t_n\}$ is infinite for each s_i and* $\gcd(q_0, t_n) = 1$ *for almost all n. Then there exist exactly q_0 equivalence classes of super-admissible sequences $\mathbf{c}_{qr}$.*

Proof. (1) There exists a positive integer m_0 so that $\gcd(r_0, p_n) = 1$ for all $n \geqslant m_0$. It suffices to show that there exist exactly r_0 sub-admissible sequences $\mathbf{c}_{qr}$ whose domain is $[m_0, \omega)$.

Let $m_1 \geqslant m_0$ be such that $s_0^{n_0} \cdots s_k^{n_k} \mid \prod_{k=m_0}^{m_1} p_k$. Then we apply Lemma 3.7 to $u = s_0^{n_0} \cdots s_k^{n_k}$ and a sequence $(\alpha_k\colon\ m_0 \leqslant i \leqslant n-1)$ for $n > m_1$. Let $n' > n > m_1$. Since each sequence obtained by Lemma 3.7 for n has a unique extension for n', we can see that there exist exactly r_0 sub-admissible sequences $\mathbf{c}_{qr}$ whose domain is $[m_0, \omega)$.

(2) Let $u = s_0^{n_0} \cdots s_k^{n_k}$. We have m_0 and $m_1 \geqslant m_0$ so that $\gcd(q_0, t_n) = 1$ for all $n \geqslant m_0$ and $u \mid \prod_{k=m_0}^{m_1-1} t_k$. Suppose that $p_k c_{k+1} \equiv c_k t_k - r\alpha_k \bmod q$ and $p_k c'_{k+1} \equiv c'_k t_k - r\alpha_k \bmod q$.

By a similar argument as in the proof of Lemma 3.7 we have

$$\prod_{k=m_0}^{m_1-1} p_k c_{m_1} \equiv \prod_{k=m_0}^{m_1-1} t_k c_0 - \sum_{i=m_0}^{m_1-1} \prod_{k=m_0}^{i-1} p_k \prod_{k=i+1}^{m_1-1} t_k r \alpha_i \mod q$$

and

$$\prod_{k=m_0}^{m_1-1} p_k c'_{m_1} \equiv \prod_{k=m_0}^{m_1-1} t_k c'_0 - \sum_{i=m_0}^{m_1-1} \prod_{k=m_0}^{i-1} p_k \prod_{k=i+1}^{m_1-1} t_k r \alpha_i \mod q.$$

We have

$$\prod_{k=m_0}^{m_1-1} p_k \bigl(c_{m_1} - c'_{m_1}\bigr) \equiv 0 \mod u$$

and hence $c_{m_1} - c'_{m_1} \equiv 0 \bmod u$. This implies that there exist at most q_0 equivalence classes of super-admissible sequences $\mathbf{c}_{qr}$. On the other hand, for given $c_k \not\equiv c'_k \bmod q_0$, we have $t_k c_k \not\equiv t_k c'_k \bmod q_0$, since $\gcd(t_k, q_0) = 1$. Since $\gcd(p_k, q) = 1$, there exist unique c_{k+1} and c'_{k+1} such that $p_k c_{k+1} \equiv c_k t_k - r\alpha_k \bmod q$ and $p_k c'_{k+1} \equiv c'_k t_k - r\alpha_k \bmod q$. Since $t_k c_k \not\equiv t_k c'_k \bmod q_0$, we have $p_k c_{k+1} \not\equiv p_k c'_{k+1} \bmod q_0$ and hence $c_{k+1} \not\equiv c'_{k+1} \bmod q_0$.

These imply the conclusion. $\square$

Now we can count the number of equivalence classes of index s supergroups of A and also the number of index s subgroups of A by Theorems 3.5 and 3.6 and Lemma 3.8.

Corollary 3.9. *Let a torsionfree abelian group A of rank 2 be presented as in Definition 3.1. For each natural number s, the number of s-index subgroups of A is equal to the number of equivalence classes of s-index supergroups of A, which is equal to $\sum_{q\in F_s} q$ in Section 1.*

Proof. Let $s = s_0^{n_0} \cdots s_k^{n_k}$ such that each s_i is a prime and let $I = \{i\colon \gcd(s_i, p_n) = 1$ for almost all $n\}$, $J = \{i\colon \gcd(s_i, t_n) = 1$ for almost all $n\}$ and $K = I \cap J$. There exists a super-admissible sequence $\mathbf{c}_{qr}$ with $s = qr$ if and only if there exists a sub-admissible sequence $\mathbf{c}_{qr}$ with $s = qr$ if and only if $I \cup J = \{0, \ldots, k\}$. Let $q_* = \prod_{i\in I} s_i^{n_i}$, $r_* = \prod_{i\in J} s_i^{n_i}$ and $s_* = \prod_{i\in I\cap J} s_i^{n_i}$. Let $qr = s$. There exists a super-admissible sequence $\mathbf{c}_{qr}$, if and only if $q \mid q_*$ and $r \mid r_*$. This condition is also equivalent to the existence of a sub-admissible sequence $\mathbf{c}_{qr}$. Let $q_0 r_0 = q_1 r_1 = s$ so that

(1) $q_0 \mid q_*$, $r_0 \mid r_*$, $q_1 \mid q_*$ and $r_1 \mid r_*$;
(2) $q_0 = q_1 u$ and $r_1 = r_0 u$ for some u with $u \mid s_*$, $\gcd(q_1, s_*) = \gcd(r_0, s_*) = 1$.

The numbers of equivalence classes of super-admissible sequences $\mathbf{c}_{q_0 r_0}$ and those of sub-admissible sequences $\mathbf{c}_{q_1 r_1}$ are the same as the numbers of u with $u \mid s_*$. Now we have the conclusion. $\square$

The next corollary follows from Corollary 3.9 straightforwardly.

Corollary 3.10. *Let p be a natural number and A be a torsionfree abelian group of rank 2 presented as the canonical form in Lemma 2.9, where $p_n = p$ and $t_n = 1$ for every n. For a natural number s the number of s-index subgroups of A and the number of equivalence classes of s-index supergroups of A are equal to $\sum_{q\mid s,\, \gcd(p,q)=1} q$.*

Remark 3.11. (1) Wickless has informed to the first author that Corollary 3.9 holds for every torsionfree abelian group A of finite rank. We thank him for this information with its simple proof. Here, we outline his short proof of this fact. Let A be a subgroup of $\mathbb{Q}^m$. For a prime p, let r_p be a non-negative integer such that $A/pA \simeq (\mathbb{Z}/p\mathbb{Z})^{r_p}$. We have $A/p^i A \simeq (\mathbb{Z}/p^i\mathbb{Z})^{r_p}$ for every positive integer i. For a positive integer s, let $s = \prod_{i=0}^k s_i^{n_i}$ where s_i is a prime. Then $A/sA \simeq \bigoplus_{i=0}^k (\mathbb{Z}/s_i^{n_i}\mathbb{Z})^{r_{s_i}}$.

We denote the cardinality of a set X by $|X|$. Every s-index subgroup of A contains sA and hence $|\{S \leqslant A\colon |A/S| = s\}| = |\{S \leqslant A\colon sA \leqslant S,\ |S| = s\}|$. On the other hand, every s-index supergroup of A is a subgroup of $(1/s)A$. Since $(1/s)A$ is isomorphic to A and this isomorphism induces the isomorphism from A to sA, $|\{S \leqslant \mathbb{Q}^m\colon A \leqslant S,\ |S/A| = s\}| = |\{S \leqslant A\colon sA \leqslant S,\ |S| = s\}| = |\{S \leqslant A\colon |A/S| = s\}|$ and this cardinality is equal to $|\{S \leqslant \bigoplus_{i=0}^k (\mathbb{Z}/s_i^{n_i}\mathbb{Z})^{r_{s_i}}\colon |S| = s\}|$. When A is presented as in Definition 3.1, the last cardinality is equal to $\sum_{q\in F_s} q$.

(2) We remark that Lemma 2.11 and Theorem 3.5 were firstly proved for covers on toroidal spaces, the correspondence of them will appear in [4].

4. When are groups A_α not isomorphic?

In this section we introduce groups A_α for p-adic integer α and state some basic facts about these groups. First we recall how the p-adic integers relate to the direct product of countably many copies of the integers $\mathbb{Z}$, i.e., $\mathbb{Z}^\omega$. A p-adic integer is presented as a formal sum $\alpha = \sum_{n=0}^{\infty} \alpha(n) p^n$ [5]. When $0 \leqslant \alpha(n) < p$, we call $\sum_{n=0}^{\infty} \alpha(n) p^n$ a canonical presentation. We regard α as a p-adic integer and call $\alpha(n)$ its nth digit.

Let p be a prime. For $x \in \mathbb{Z}^\omega$ let $f(x)_n : \mathbb{Z} \oplus \mathbb{Z} \to \mathbb{Z} \oplus \mathbb{Z}$ be a homomorphism represented by a matrix $\left[\begin{smallmatrix} p & 0 \\ x(n) & 1 \end{smallmatrix}\right]$ for each $n \in \omega$, where $x(n) \in \mathbb{Z}$.

The direct limit $\varinjlim(\mathbb{Z} \oplus \mathbb{Z}, f(x)_n\colon n < \omega)$ is isomorphic to a subgroup of $\mathbb{Q} \oplus \mathbb{Q}$, i.e.

$$A_x = \left\{ up^{-n} \begin{bmatrix} 1 \\ -\sum_{i=0}^{n-1} x(i) p^i \end{bmatrix} + v \begin{bmatrix} 0 \\ 1 \end{bmatrix} \colon u, v \in \mathbb{Z},\ n < \omega \right\}.$$

By the next lemma we see that in the investigation of A_α we may use any presentation of a p-adic integer α.

Lemma 4.1. *If $\sum_{n=0}^{\infty} x(n) p^n = \sum_{n=0}^{\infty} y(n) p^n$ for $x, y \in \mathbb{Z}^\omega$, then groups A_x and A_y are equal.*

Proof. Since $\sum_{i=0}^{n-1} x(i) p^i \equiv \sum_{i=0}^{n-1} y(i) p^i \mod p^n$, we define a_n's by $a_0 = 0$ and $p^n a_n = \sum_{i=0}^{n-1} x(i) p^i - \sum_{i=0}^{n-1} y(i) p^i$ for $n \geqslant 1$. Then $x(n) + a_n = y(n) + p a_{n+1}$ and hence

$$\begin{bmatrix} p & 0 \\ x(n) & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ a_n & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ a_{n+1} & 1 \end{bmatrix} \begin{bmatrix} p & 0 \\ y(n) & 1 \end{bmatrix}.$$

Since $\left[\begin{smallmatrix} 1 & 0 \\ a_n & 1 \end{smallmatrix}\right] \in GL_2(\mathbb{Z})$, the direct limits are isomorphic. Since groups A_x and A_y are defined from these direct systems by embedding the first copy of $\mathbb{Z} \oplus \mathbb{Z}$ into $\mathbb{Q} \oplus \mathbb{Q}$ canonically, A_x and A_y are equal. $\square$

This lemma is generalized as follows. Since the proof is similar, we omit it.

Lemma 4.2. *Let $0 \neq u_n \in \mathbb{Z}$. If $\sum_{i=0}^{n-1} x(i)(\prod_{j=0}^{i} u_j) \equiv \sum_{i=0}^{n-1} y(i)(\prod_{j=0}^{i} u_j) \mod \prod_{i=0}^{n-1} u_i$, then the direct limit groups obtained by using $\left[\begin{smallmatrix} u_i & 0 \\ x(i) & 1 \end{smallmatrix}\right]$ and $\left[\begin{smallmatrix} u_i & 0 \\ y(i) & 1 \end{smallmatrix}\right]$ as bonding maps are isomorphic.*

Lemma 4.3. *For $x \in \mathbb{Z}^\omega$, let $-x$ be the inverse element in the abelian group $\mathbb{Z}^\omega$, i.e. $(-x)(n) = -x(n)$. Then, A_{-x} is isomorphic to A_x.*

Proof. We have

$$\begin{aligned} A_{-x} &= \left\{ up^{-n} \begin{bmatrix} 1 \\ \sum_{i=0}^{n-1} x(i) p^i \end{bmatrix} + v \begin{bmatrix} 0 \\ 1 \end{bmatrix} \colon u, v \in \mathbb{Z},\ n < \omega \right\} \\ &= \left\{ up^{-n} \begin{bmatrix} 1 \\ \sum_{i=0}^{n-1} x(i) p^i \end{bmatrix} + v \begin{bmatrix} 0 \\ -1 \end{bmatrix} \colon u, v \in \mathbb{Z},\ n < \omega \right\}. \end{aligned}$$

Hence an isomorphism between A_x and A_{-x} is given by $\left[\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right]$. $\square$

Here we show trivial cases of limits.

Proposition 4.4. *Let $x, y \in \mathbb{Z}^\omega$. The direct limit group obtained by $\left[\begin{smallmatrix} 1 & 0 \\ y(n) & x(n) \end{smallmatrix}\right]$ is isomorphic to the group obtained by $\left[\begin{smallmatrix} 1 & 0 \\ 0 & x(n) \end{smallmatrix}\right]$ which is isomorphic to $\mathbb{Q}(x) \oplus \mathbb{Z}$, where $\mathbb{Q}(x) = \{a/\prod_{i=n_0}^n x(i)\colon a \in \mathbb{Z},\ n \geqslant n_0\}$ if $x(n) \neq 0$ for almost all n, where $x(n) \neq 0$ for $n \geqslant n_0$, and $\mathbb{Q}(x) = \{0\}$ otherwise.*

Proof. Define $z(n)$'s inductively as $z(0) = 0$ and $z(n+1) = x(n)z(n) + y(n)$. Then we have

$$\begin{bmatrix} 1 & 0 \\ y(n) & x(n) \end{bmatrix} \begin{bmatrix} 1 & 0 \\ z(n) & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ z(n+1) & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & x(n) \end{bmatrix}$$

and the conclusion holds. □

Suppose that A_α and A_β are isomorphic. Then there exists a rational matrix $\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$ which defines the isomorphism. It is easy to see that a, b, c, d belong to $\mathbb{Q}(p^\infty) = \{ap^n\colon a, n \in \mathbb{Z}\} = \mathbb{Z}[1/p]$. (When we are interested in the structure of abelian groups, we use $\mathbb{Q}(p^\infty)$ and otherwise $\mathbb{Z}[1/p]$.) Since the inverse matrix is also of such a form, the determinant $\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$ is of the form $\pm p^n$ for $n \in \mathbb{Z}$. By the same argument as in the Goodearl and Rushing paper [6] we have $a\alpha - b\alpha\beta + c - d\beta = 0$. Since this fact is crucial in the sequel we review a line of its proof here. Let $g = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$, $\mathbf{e}_0 = \left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right]$ and $\mathbf{e}_1 = \left[\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right]$. Let

$$A_{x,n} = \left\{ up^{-n} \begin{bmatrix} 1 \\ -\sum_{i=0}^{n-1} x(i)p^i \end{bmatrix} + v \begin{bmatrix} 0 \\ 1 \end{bmatrix} \colon u, v \in \mathbb{Z} \right\},$$

$$\mathbf{z}_n = p^{-n} \begin{bmatrix} 1 \\ -\sum_{i=0}^{n-1} \alpha(i)p^i \end{bmatrix} \quad \text{and} \quad \mathbf{w}_n = p^{-n} \begin{bmatrix} 1 \\ -\sum_{i=0}^{n-1} \beta(i)p^i \end{bmatrix}.$$

Then $A_{\alpha,n} = \langle \mathbf{z}_n \rangle + \langle \mathbf{e}_1 \rangle$ and $A_{\beta,n} = \langle \mathbf{w}_n \rangle + \langle \mathbf{e}_1 \rangle$. We have m_0 and the minimal $k(n)$ so that $g(\mathbf{e}_1) \in A_{\alpha,m_0}$ and $g(\mathbf{w}_n) \in A_{\alpha,k(n)}$. Then it is easy to see that $\lim_{n\to\infty} k(n) = \infty$. We have integers a_n and b_n such that $g(w_n) = a_n \mathbf{z}_{k(n)} + b_n \mathbf{e}_1$. Since $p\mathbf{z}_{k(n)} \in A_{\alpha,k(n)-1}$, a_n and p are relatively prime by the minimality of $k(n)$, if $k(n) > m_0$. Hence we have

$$p^{-n}\left(a - b\sum_{i=0}^{n-1} \beta(i)p^i\right) = p^{-k(n)} a_n,$$

$$p^{-n}\left(c - d\sum_{i=0}^{n-1} \beta(i)p^i\right) = p^{-k(n)}\left(-a_n \sum_{i=0}^{k(n)-1} \alpha(i)p^i + b_n p^{k(n)}\right).$$

By cross-multiplication we have

$$a_n\left(c - d\sum_{i=0}^{n-1} \beta(i)p^i\right) = \left(a - b\sum_{i=0}^{n-1} \beta(i)p^i\right)\left(-a_n \sum_{i=0}^{k(n)-1} \alpha(i)p^i + b_n p^{k(n)}\right).$$

Since $a, b, c, d \in \mathbb{Z}[1/p]$ and a_n and p are relatively prime for sufficiently large n, b_n is divided by a_n in the ring of p-adic integers and we have

$$c - d\sum_{i=0}^{n-1} \beta(i)p^i = \left(a - b\sum_{i=0}^{n-1} \beta(i)p^i\right)\left(-\sum_{i=0}^{k(n)-1} \alpha(i)p^i + (b_n/a_n)p^{k(n)}\right).$$

Taking the limit gives the conclusion.

5. Finite index supergroups and subgroups of A_α for a non-quadratic α

Let α_0 be a p-adic integer which is not quadratic, that is, α_0 does not satisfy equations of form $\alpha_0^2 + b\alpha_0 + c = 0$ for any rational numbers b, c. We recall Definition 3.2. For a super-admissible sequence $\mathbf{c}_{qr} : [n_0, \omega) \to \{0, \dots, q-1\}$ such that $p\mathbf{c}_{qr}(n+1) \equiv \mathbf{c}_{qr}(n) - r\alpha_0(n) \bmod q$, define $\mathbf{g}_{qr} : [n_0, \omega) \to \mathbb{Z}$ by

$$\mathbf{g}_{qr}(n) = \frac{p\mathbf{c}_{qr}(n+1) - \mathbf{c}_{qr}(n) + r\alpha_0(n)}{q}.$$

Similarly for a sub-admissible sequence $\mathbf{c}_{qr} : [n_0, \omega) \to \{0, \dots, r-1\}$ such that $p\mathbf{c}_{qr}(n+1) \equiv \mathbf{c}_{qr}(n) + q\alpha_0(n) \bmod r$, define $\mathbf{e}_{qr} : [n_0, \omega) \to \mathbb{Z}$ by

$$\mathbf{e}_{qr}(n) = \frac{-p\mathbf{c}_{qr}(n+1) + \mathbf{c}_{qr}(n) + q\alpha_0(n)}{r}.$$

Lemma 5.1. *Let $q_0r_0 = q_1r_1$ and* $\gcd(q_0, p) = \gcd(q_1, p) = 1$. *Suppose that $a\alpha + c - d\beta = 0$ for some $a, c, d \in \mathbb{Z}[1/p]$ and α and β in the following* (1) *and* (2)*, respectively, and $ad = \pm p^n$ for some $n \in \mathbb{Z}$.*

(1) *If $\mathbf{c}_{q_0r_0}$ and $\mathbf{c}_{q_1r_1}$ are super-admissible sequences such that $\alpha = \sum_{n=n_0}^{\infty} \mathbf{g}_{q_0r_0}(n)p^n$ and $\beta = \sum_{n=n_0}^{\infty} \mathbf{g}_{q_1r_1}(n)p^n$, then $(q_0, \mathbf{c}_{q_0r_0}(n_0)) = (q_1, \mathbf{c}_{q_1r_1}(n_0))$ holds.*

(2) *If $\mathbf{c}_{q_0r_0}$ and $\mathbf{c}_{q_1r_1}$ are sub-admissible sequences such that $\alpha = \sum_{n=n_0}^{\infty} \mathbf{e}_{q_0r_0}(n)p^n$ and $\beta = \sum_{n=n_0}^{\infty} \mathbf{e}_{q_1r_1}(n)p^n$, then $(r_0, \mathbf{c}_{q_0r_0}(n_0)) = (r_1, \mathbf{c}_{q_1r_1}(n_0))$ holds.*

Proof. For simplicity of notation, we abuse a map $x : [n_0, \omega) \to \mathbb{Z}$ with the p-adic integer $\sum_{n=n_0}^{\infty} x(n)p^n$. By multiplying a sufficiently large p^m we may assume $a, c, d \in \mathbb{Z}$ without any loss of generality.

Suppose that $ad = \pm p^m$ in case (1). Let $w_0 = \sum_{n=0}^{n_0} \alpha_0(n)$. Multiplying p^n's to the equations

$$q\mathbf{g}_{qr}(n) = p\mathbf{c}_{qr}(n+1) - \mathbf{c}_{qr}(n) + r\alpha_0(n)$$

and taking the sum, we have

$$\begin{aligned} q\mathbf{g}_{qr} &= p\sum_{n=n_0}^{\infty} p^n\mathbf{c}_{qr}(n+1) - \sum_{n=n_0}^{\infty} p^n\mathbf{c}_{qr}(n) + r(\alpha_0 - w_0) \\ &= -p^{n_0}\mathbf{c}_{qr}(n_0) + r(\alpha_0 - w_0). \end{aligned}$$

Let $c_0 = \mathbf{c}_{q_0 r_0}(n_0)$ and $c_1 = \mathbf{c}_{q_1 r_1}(n_0)$. Then we have $q_0 \mathbf{g}_{q_0 r_0} = -p^{n_0} c_0 + r_0(\alpha_0 - w_0)$ and $q_1 \mathbf{g}_{q_1 r_1} = -p^{n_0} c_1 + r_1(\alpha_0 - w_0)$. Since $a q_0 q_1 \alpha + c q_0 q_1 - d q_0 q_1 \beta = 0$, we have

$$\begin{aligned}(a q_1 r_0 - d q_0 r_1)\alpha_0 + c q_0 q_1 - a q_1\bigl(p^{n_0} c_0 + r_0 w_0\bigr) + d q_0\bigl(p^{n_0} c_1 + r_1 w_0\bigr)\\ = a q_1\bigl(-p^{n_0} c_0 + r_0\alpha_0 - r_0 w_0\bigr) + c q_0 q_1 - d q_0\bigl(-p^{n_0} c_1 + r_1 \alpha_0 - r_1 w_0\bigr) = 0.\end{aligned}$$

Since α_0 is not rational, we have $a q_1 r_0 - d q_0 r_1 = 0$. Since $q_0 r_0 = q_1 r_1 > 0$, we have $a q_1^2 = d q_0^2$. Hence we have $a = d$ and $q_0 = q_1$ and also $r_0 = r_1$. Then we have $c q_0 = a p^{n_0}(c_0 - c_1)$. Since $\gcd(p, q_0) = 1$, $\gcd(a, q_0) = 1$, and $0 \leqslant c_0, c_1 < q_0$, we have $c_0 = c_1$.

Next, suppose that $ad = \pm p^m$ in case (2). The argument goes similarly. Let $c_0 = \mathbf{c}_{q_0 r_0}(n_0)$ and $c_1 = \mathbf{c}_{q_1 r_1}(n_0)$. Then, we have $r_0 \mathbf{e}_{q_0 r_0} = p^{n_0} c_0 + q_0(\alpha_0 - w_0)$ and $r_1 \mathbf{e}_{q_1 r_1} = p^{n_0} c_1 + q_1(\alpha_0 - w_0)$ and consequently

$$(a r_1 q_0 - d r_0 q_1)\alpha_0 + r_0 r_1 c + a r_1\bigl(p^{n_0} c_0 - q_0 w_0\bigr) - d r_0\bigl(p^{n_0} c_1 - q_1 w_0\bigr) y = 0.$$

By a similar argument as above, we have $a = d$, $q_0 = q_1$ and $r_0 = r_1$. Also we have $r_0 c + a p^{n_0}(c_0 - c_1) = 0$. Let $r_0 = s_0 p^m$ with $\gcd(p, s_0) = 1$. Then $c_0 \equiv c_1 \bmod s_0$. We apply the proof of Lemma 3.7 to the case that r_0 for r, p^m for u, $p_k = p$, $t_k = 1$ and $(c_k\colon 0 \leqslant k < n) = (\alpha_0(k)\colon n_0 \leqslant k < n_0 + n)$ for a sufficiently large n, i.e. $n \geqslant m$. Then we have $c_0 \equiv c_1 \bmod p^m$. Since $0 \leqslant c_0, c_1 < r_0 = p^m s_0$, we conclude $c_0 = c_1$. □

The next theorem strengthens [3, Corollary 1] extensively.

Theorem 5.2. *Let p be a prime and α_0 be a p-adic integer which is not quadratic over a rational field. Let $\gcd(p, q_0) = \gcd(p, q_1) = 1$ and $q_0 r_0 = q_1 r_1$. Suppose one of the following holds*:

(1) $\mathbf{c}_{q_0 r_0}$ *and* $\mathbf{c}_{q_1 r_1}$ *are super-admissible sequences such that* $(q_0, \mathbf{c}_{q_0 r_0}(n_0)) \neq (q_1, \mathbf{c}_{q_1 r_1}(n_0))$ *and* $\alpha = \sum_{n=n_0}^{\infty} \mathbf{g}_{q_0 r_0}(n) p^n$ *and* $\beta = \sum_{n=n_0}^{\infty} \mathbf{g}_{q_1 r_1}(n) p^n$; *and*

(2) $\mathbf{c}_{q_0 r_0}$ *and* $\mathbf{c}_{q_1 r_1}$ *are sub-admissible sequences such that* $(r_0, \mathbf{c}_{q_0 r_0}(n_0)) \neq (r_1, \mathbf{c}_{q_1 r_1}(n_0))$ *and* $\alpha = \sum_{n=n_0}^{\infty} \mathbf{e}_{q_0 r_0}(n) p^n$ *and* $\beta = \sum_{n=n_0}^{\infty} \mathbf{e}_{q_1 r_1}(n) p^n$.

Then A_α and A_β are not isomorphic. Consequently, for each natural number s distinct s-index supergroups of A_{α_0} are non-isomorphic and also distinct s-index subgroups of A_{α_0} are non-isomorphic.

Proof. Suppose the negation of the conclusion. Then there exist $a, b, c, d \in \mathbb{Z}[1/p]$ such that $a\alpha - b + c\alpha\beta - d\beta = 0$ and $ad - bc = \pm p^n$ for some $n \in \mathbb{Z}$. When $c = 0$ we have a contradiction by Lemma 5.1. Otherwise, using equations

$$q \mathbf{g}_{qr} = -\mathbf{c}_{qr}(n_0) + r(\alpha_0 - w_0)$$

and

$$r \mathbf{e}_{qr} = \mathbf{c}_{qr}(n_0) + q(\alpha_0 - w_0),$$

where $w_0 = \sum_{n=0}^{n_0 - 1} p^n \alpha_0(n)$, we deduce a contradiction to the assumption that α_0 is not quadratic over a rational field in each case. □

6. More on groups A_α and on some other groups

Theorem 6.1. *Let α be a p-adic integer which is not quadratic over $\mathbb{Q}$. Then for integers m and n, $A_{m\alpha} \simeq A_{n\alpha}$ if and only if $m = \pm np^i$ for some integer i.*

Proof. If $m = \pm np^i$ for some integer i, we may assume i is non-negative. First we deal with the case $m = np^i$. Let $x(k) = 0$ for $k < i$ and $x(k) = n\alpha(k-i)$ for $k \geqslant i$. Then $A_x \simeq A_{m\alpha}$ by Lemma 4.1. On the other hand $A_x \simeq A_{n\alpha}$ by the property of the direct limit. In case $m = -np^i$, we have $A_{m\alpha} \simeq A_{-n\alpha} \simeq A_{n\alpha}$ by Lemma 4.3.

To show the other direction of the statement, suppose that $A_{m\alpha} \simeq A_{n\alpha}$. Then we have $a, b, c, d \in \mathbb{Z}[1/p]$ such that $ad - bc = \pm p^i$ for some integer i and $am\alpha - b + cmn\alpha^2 - dn\alpha = 0$. If $m = 0$, then $dn = b = 0$ since α is not rational. Then $ad \neq 0$ and hence $n = 0$. A similar statement for n holds and so we may assume $mn \neq 0$. Since α is not quadratic over $\mathbb{Q}$, we have $c = 0$ and also $am - dn = 0$ and $b = 0$. Then $ad = \pm p^i$, which implies $a = \pm p^{i_0}$ and $d = \pm p^{i_1}$ for some integers i_0, i_1. Hence we have $m = \pm np^{i_1 - i_0}$. $\square$

In [3, Theorem 1] the statement was restricted to the case $p > 2$. Here we prove a more general statement.

Theorem 6.2.

(1) *Let $G_n = \left[\begin{smallmatrix} s & 0 \\ t & 1 \end{smallmatrix}\right]$ for every n. Then, the direct limit group A whose bonding maps are G_n's is isomorphic to $\mathbb{Q}(s^\infty) \oplus \mathbb{Z}$.*

(2) *Let $G_n = \left[\begin{smallmatrix} s & 0 \\ t & s-1 \end{smallmatrix}\right]$ for every n. Then, the direct limit group A whose bonding maps are G_n's is isomorphic to $\mathbb{Q}(s^\infty) \oplus \mathbb{Q}((s-1)^\infty)$.*

Proof. (1) Let $d = \gcd(s-1, t)$ and $s - 1 = ds'$ and $t = dt'$. There exist integers x_0, y_0 such that $s'y_0 - t'x_0 = 1$. Since

$$\begin{bmatrix} s & 0 \\ t & 1 \end{bmatrix}^n = \begin{bmatrix} s^n & 0 \\ t\sum_{i=0}^{n-1} s^i & 1 \end{bmatrix},$$

A is isomorphic to

$$\left\{us^{-n}\begin{bmatrix} 1 \\ t\sum_{i=0}^{n-1} s^i \end{bmatrix} + v\begin{bmatrix} 0 \\ 1 \end{bmatrix}\colon u, v \in \mathbb{Z},\ n \in \mathbb{N}\right\}.$$

We remark $t\sum_{i=0}^{n-1} s^i = (s^n - 1)t/(s-1) = (s^n - 1)t'/s'$.

We show $A = B_1 \oplus B_2$ where $B_1 = \{us^{-n}\left[\begin{smallmatrix} s' \\ t' \end{smallmatrix}\right]\colon u \in \mathbb{Z},\ n \in \mathbb{N}\}$ and $B_2 = \{v\left[\begin{smallmatrix} x_0 \\ y_0 \end{smallmatrix}\right]\colon v \in \mathbb{Z}\}$. Since $s'y_0 - t'x_0 = 1$, $B_1 \cap B_2 = \{0\}$. Since B_2 is obviously a subgroup of A, we show $s^{-n}\left[\begin{smallmatrix} s' \\ t' \end{smallmatrix}\right] \in A$. Since

$$s^{-n}t' + s's^{-n}t\sum_{i=0}^{n-1} s^i = s^{-n}\left(t' + s't\sum_{i=0}^{n-1} s^i\right)$$

$$= s^{-n}\left(t' + s't\frac{s^n - 1}{s - 1}\right)$$
$$= s^{-n}t'\bigl(1 + s^n - 1\bigr) = t',$$

$$s^{-n}\begin{bmatrix} s' \\ t' \end{bmatrix} = s's^{-n}\begin{bmatrix} 1 \\ -t\sum_{i=0}^{n-1} s^i \end{bmatrix} + t'\begin{bmatrix} 0 \\ 1 \end{bmatrix} \in A.$$

Hence we have $B_1 \oplus B_2 \leqslant A$. To see $A \leqslant B_1 \oplus B_2$, it suffices to show $s^{-n}\left[\begin{smallmatrix} 1 \\ -t\sum_{i=0}^{n-1} s^i \end{smallmatrix}\right]$, $\left[\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right] \in B_1 \oplus B_2$,

$$\begin{aligned}
\left(y_0 + x_0 t\sum_{i=0}^{n-1} s^i\right)s^{-n}\begin{bmatrix} s' \\ t' \end{bmatrix} - t'\begin{bmatrix} x_0 \\ y_0 \end{bmatrix} &= \bigl(s'y_0 - t'x_0 + x_0t's^n\bigr)s^{-n}/s'\begin{bmatrix} s' \\ t' \end{bmatrix} - t'\begin{bmatrix} x_0 \\ y_0 \end{bmatrix} \\
&= \begin{bmatrix} s^{-n} + x_0t' - x_0t' \\ t'(1 + x_0t's^n)s^{-n}/s' - t'y_0 \end{bmatrix} \\
&= \begin{bmatrix} s^{-n} \\ t'(s^{-n} + t'x_0 - s'y_0)/s' \end{bmatrix} \\
&= \begin{bmatrix} s^{-n} \\ -s^{-n}t'(s^n - 1)/s' \end{bmatrix} \\
&= s^{-n}\begin{bmatrix} 1 \\ -\sum_{i=0}^{n-1} s^i \end{bmatrix},
\end{aligned}$$

$$\bigl(-x_0s^n\bigr)s^{-n}\begin{bmatrix} s' \\ t' \end{bmatrix} + s'\begin{bmatrix} x_0 \\ y_0 \end{bmatrix} = \begin{bmatrix} -x_0s' + s'x_0 \\ -x_0t' + s'y_0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Now we have shown $A = B_1 \oplus B_2$, which is isomorphic to $\mathbb{Q}(s^\infty) \oplus \mathbb{Z}$.

(2) Since $\left[\begin{smallmatrix} s & 0 \\ t & s-1 \end{smallmatrix}\right]^n = \left[\begin{smallmatrix} s^n & 0 \\ t\sum_{i=0}^{n-1} s^i(s-1)^{n-1-i} & (s-1)^n \end{smallmatrix}\right]$ and $t\sum_{i=0}^{n-1} s^i(1-s)^{n-1-i} = t(s^n - (s-1)^n)$, A is isomorphic to

$$\begin{aligned}
&\left\{u\begin{bmatrix} s^{-n} \\ -ts^{-n}(s-1)^{-n}(s^n - (s-1)^n) \end{bmatrix} + v\begin{bmatrix} 0 \\ (s-1)^{-n} \end{bmatrix}\colon u, v \in \mathbb{Z},\ n \in \mathbb{N}\right\} \\
&\quad = \left\{us^{-n}\begin{bmatrix} 1 \\ t \end{bmatrix} + (-tu + v)(s-1)^{-n}\begin{bmatrix} 0 \\ 1 \end{bmatrix}\colon u, v \in \mathbb{Z},\ n \in \mathbb{N}\right\} \\
&\quad \simeq \mathbb{Q}\bigl(s^\infty\bigr) \oplus \mathbb{Q}\bigl((s-1)^\infty\bigr). \qquad \square
\end{aligned}$$

A sequence $(x_n\colon n < \omega)$ is said to be *semi-periodic*, if there exist integers $m \geqslant 0$ and $k > 0$ such that $x_n = x_{n+k}$ for every $n \geqslant m$, and to be *periodic*, if there exists an integer $k > 0$ such that $x_n = x_{n+k}$ for every n.

Corollary 6.3. *Let α be a p-adic integer. Then A_α is isomorphic to $\mathbb{Q}(p^\infty) \oplus \mathbb{Z}$, if and only if α is rational, i.e. semi-periodic.*

Proof. The sufficiency follows from Theorem 6.2(1). Suppose that A_α is isomorphic to $\mathbb{Q}(p^\infty) \oplus \mathbb{Z}$. Since A_0 is isomorphic to $\mathbb{Q}(p^\infty) \oplus \mathbb{Z}$, we have $a, b, c, d \in \mathbb{Z}[1/p]$ such that $a\alpha + c = 0$ and $ad - bc = p^n$ for some $n \in \mathbb{Z}$. Then a is non-zero and hence α is rational. □

Now we have

Corollary 6.4. *Let α and β be p-adic integers. Suppose that α is rational. Then, A_β is isomorphic to A_α if and only if β is rational, i.e. semi-periodic.*

The following are complementary to Theorem 6.1.

Corollary 6.5. *Let α be a p-adic integer. Then $A_{\alpha+m}$ and $A_{-\alpha+m}$ are isomorphic to A_α for every $m \in \mathbb{Z}$.*

Proof. Let $\alpha = \sum_{n=0}^{\infty} \alpha(n) p^n$ is a canonical presentation. If α is rational, the conclusion follows from Corollary 6.3. So, we suppose that α is not rational. Then, for a non-negative $m \in \mathbb{Z}$ we have n_0 such that $\sum_{n=0}^{n_0} \alpha(n) p^n \geqslant m$. Hence the direct systems related to α and $\sum_{n=0}^{n_0} \alpha(n) p^n - m$ are the same eventually and we have $A_\alpha \simeq A_{\alpha-m}$. We apply this fact to $-\alpha$, then by Lemma 4.3 we have $A_\alpha \simeq A_{-\alpha} \simeq A_{-\alpha-m} \simeq A_{\alpha+m}$. We have the other cases similarly. □

To investigate solenoids Keesling and Mardesic [9] investigated a certain A_α. Now their following result is clear from Corollary 6.3.

> Let $(i_n\colon n < \omega)$ be a strictly increasing sequence of natural numbers such that $\lim_{n\to\infty} i_{n+1} - i_n = \infty$ and let α be the p-adic integer $\sum_{n=0}^{\infty} p^{i_n}$. Then A_α is not isomorphic to $\mathbb{Q}(p^\infty) \oplus \mathbb{Z}$.

Finally we prove results about finite index supergroups and subgroups of A_α's for rational α and related results. The results contrast with Theorem 5.2.

Lemma 6.6. *Let r be a positive integer, let s an integer, let $f : \{0, 1, \dots, r-1\} \to \{0, 1, \dots, r-1\}$ be a function, and let $x_n \in \{0, 1, \dots, r-1\}$ for $n < \omega$.*

(1) *If $x_{n+1} = f(x_n)$ for $n < \omega$, then the sequence $(x_n\colon n < \omega)$ is semi-periodic.*
(2) *If $x_n = f(x_{n+1})$ for $n < \omega$, then the sequence $(x_n\colon n < \omega)$ is periodic.*

Proof. (1) Since $x_n \in \{0, 1, \dots, r-1\}$, there are $0 \leqslant i_0 < j_0 \leqslant r$ such that $x_{i_0} = x_{j_0}$. Then $(x_n\colon n < \omega)$ is $(j_0 - i_0)$-periodic on $[i_0, \omega)$.

(2) Fix n. There are $0 \leqslant i < j \leqslant r$ such that $x_{n+i} = x_{n+j}$. Then $x_n = x_{n+j-i}$. Let $k_n > 0$ be the minimal $k > 0$ such that $x_n = x_{n+k_n}$. We have $x_n = f(x_{n+1}) = f(x_{n+k_{n+1}+1}) = x_{n+k_{n+1}}$ and so $k_n \leqslant k_{n+1}$. We claim $k_n = k_{n+1}$ for every $n < \omega$. To show this by contradiction, suppose that $k_n < k_{n+1}$. Let $m = n + k_n$. Since $x_m = x_{n+k_n} = x_n = x_{n+k_{n+1}}$, $k_m \leqslant k_{n+1} - k_n < k_{n+1}$, which contradicts $m \geqslant n + 1$. □

Corollary 6.7. *For each $n < \omega$, let B_n be a copy of $\mathbb{Z} \oplus \mathbb{Z}$ and $f_n = \begin{bmatrix} s & 0 \\ \alpha_n & 1 \end{bmatrix}$ for all n. If $(f_n\colon n < \omega)$ is semi-periodic, all finite index supergroups and subgroups are isomorphic to*

$\varinjlim(B_n, f_n\colon n < \omega)$ itself. Consequently, for a rational p-adic integer α all finite index supergroups and subgroups of A_α are isomorphic to A_α itself.

Proof. Let $(f_n\colon n < \omega)$ be k-periodic on $[m, \infty)$. Then we have $\left[\begin{smallmatrix} s^k & 0 \\ t & 1 \end{smallmatrix}\right] = f_{m+k-1} \cdots f_{m+1} f_m$ for some t and let $g_n = \left[\begin{smallmatrix} s^k & 0 \\ t & 1 \end{smallmatrix}\right]$ for every $n < \omega$. Then $\varinjlim(B_n, g_n\colon n < \omega) \simeq \varinjlim(B_n, f_n\colon n < \omega)$. Super-admissible and sub-admissible sequences for $(g_n\colon n < \omega)$ are periodic by Lemma 6.6. Fix a periodic admissible sequence and let k be the period. The direct limit is isomorphic to the direct limit given by iterated use of a matrix of form $\left[\begin{smallmatrix} s^k & 0 \\ u & 1 \end{smallmatrix}\right]$ for some u. Hence the direct limit is isomorphic to $\mathbb{Q}((s^k)^\infty) \oplus \mathbb{Z} \simeq \mathbb{Q}(s^\infty) \oplus \mathbb{Z}$ by Theorem 6.2. □

Proposition 6.8. *Let $g_n = \left[\begin{smallmatrix} s & 0 \\ \alpha_n & s \end{smallmatrix}\right]$ for every n. Then, the direct limit group A whose bonding maps are g_n's is isomorphic to $\mathbb{Q}(s^\infty) \oplus \mathbb{Q}(s^\infty)$ and every finite index supergroup or subgroup is isomorphic to A itself.*

Proof. Since $g_{n-1} \cdots g_0 = \left[\begin{smallmatrix} s^n & 0 \\ \sum_{i=0}^{n-1} \alpha_i & s^n \end{smallmatrix}\right]$, A is isomorphic to

$$\begin{aligned} &\left\{ us^{-n} \begin{bmatrix} 1 \\ -s^{-1} \sum_{i=0}^{n-1} \alpha_i \end{bmatrix} + vs^{-n} \begin{bmatrix} 0 \\ 1 \end{bmatrix}\colon u, v \in \mathbb{Z},\ n < \omega \right\} \\ &\quad = \left\{ us^{-n} \begin{bmatrix} 1 \\ 0 \end{bmatrix} + vs^{-n-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix}\colon u, v \in \mathbb{Z},\ n < \omega \right\} \\ &\quad \simeq \mathbb{Q}\big(s^\infty\big) \oplus \mathbb{Q}\big(s^\infty\big). \end{aligned}$$

Since α_i's are arbitrary, every finite index supergroup or subgroup is isomorphic to A itself. □

Remark 6.9. (1) There is a possibility of extending our result to the case of finite rank, but we have not done so.

(2) In Section 3 we introduced a necessary condition for the isomorphicness of A_α's. We do not know whether the condition is sufficient.

(3) The existence of a torsionfree abelian group of rank two having a finite index supergroup which is not isomorphic to the original group had been proved in [1, Theorem 9.6(2)] before [3] in a different method.

References

[1] R.A. Beaumont, R.S. Pierce, Torsionfree groups of rank two, Mem. Amer. Math. Soc. 38 (1961) 1–41.

[2] V. Matijević, Classifying finite-sheeted coverings of paracompact spaces, Rev. Mat. Complut. 16 (2003) 311–327.

[3] K. Eda, J. Mandić, V. Matijević, Torus-like continua which are not self-covering spaces, Topology Appl. (2005) 359–369.

[4] K. Eda, V. Matijević, Finite sheeted covering maps over 2-dimensional compact abelian groups, Topology Appl. 153 (2006) 1033–1045.

[5] L. Fuchs, Infinite Abelian Groups, vols. 1, 2, Academic Press, 1970, 1973.

[6] K.R. Goodearl, T.B. Rushing, Direct limit groups and the Keesling–Mardešić shape fibration, Pacific J. Math. 86 (1980) 471–476.

[7] G. Hjorth, Around nonclassifiability for countable torsion free abelian groups, in: Abelian Groups and Modules, in: Trends Math., Birkhäuser, 1999, pp. 269–292.
[8] A.S. Kechris, On the classification problem for rank 2 torsion-free abelian groups, J. London Math. Soc. 62 (2000) 437–450.
[9] J. Keesling, S. Mardešić, A shape fibration with fibers of different shape, Pacific J. Math. 84 (1979) 319–331.
[10] S. Thomas, The classification problem for torsion-free abelian groups of finite rank, J. Amer. Math. Soc. 16 (2003) 233–258.