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J. Math. Anal. Appl. 279 (2003) 168–179

Journal of MATHEMATICAL ANALYSIS AND APPLICATIONS

www.elsevier.com/locate/jmaa

Nonclassical symmetries as special solutions of heir-equations

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Received 25 September 2002

Submitted by William F. Ames

Abstract

In Phys. D 78 (1994) 124, we have found that iterations of the nonclassical symmetries method give rise to new nonlinear equations, which inherit the Lie point symmetry algebra of the given equation. In the present paper, we show that special solutions of the right-order heir-equation correspond to classical and nonclassical symmetries of the original equations. An infinite number of nonlinear equations which possess nonclassical symmetries are derived.

1. Introduction

The most famous and established method for finding exact solutions of differential equations is the classical symmetries method (CSM), also called group analysis, which originated in 1881 from the pioneering work of Lie [19]. Many good books have been dedicated to this subject and its generalizations [4,7,8,16,31,34,35,38].

The nonclassical symmetries method (NSM) was introduced in 1969 by Bluman and Cole [6] in order to obtain new exact solutions of the linear heat equation, i.e., solutions not deducible from the CSM. The NSM consists of adding the invariant surface condition to the given equation, and then applying the CSM. The main difficulty of this approach is that the determining equations are no longer linear. On the other hand, the NSM may yield more solutions than the CSM. The NSM has been successfully applied to various equations [10,11,14,18,30],[1] for the purpose of finding new exact solutions.

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[1] Just to cite some of numerous papers on this subject.

doi:10.1016/S0022-247X(02)00706-0

Galaktionov [13] and King [17] have found exact solutions of certain evolution equations which apparently do not seem to be derived by either the CSM or NSM. In [28], we have shown how these solutions can be obtained by iterating the NSM. A special case of the NSM generates a new nonlinear equation (the so-called G-equation [27]), which inherits the prolonged symmetry algebra of the original equation. Another special case of the NSM is then applied to this heir-equation to generate another heir-equation, and so on. Invariant solutions of these heir-equations are exactly the solutions derived in [13] and [17].

In this paper, we show that the difficulty of finding nonclassical symmetries can be overcome by determining the right-order heir-equation, and looking for a particular solution which has an a priori known form. Both classical and nonclassical symmetries can be found in this way. Therefore, our method may give an answer to the question "How can one establish a priori if a given equation admits nonclassical symmetries?" We limit our analysis to single evolution equations in two independent variables. In the present paper, we will not deal with systems, although heir-equations for systems were introduced in [3].

In Section 2, first we recall what heir-equations are and then we present our method. In Section 3, some examples are given. In Section 4, we make some final comments.

The use of a symbolic manipulator becomes imperative, because the heir-equations can be quite long: one more independent variable is added at each iteration. We employ our own interactive REDUCE programs [25,26] to generate the heir-equations.

2. Heir-equations and outline of the method

Let us consider an evolution equation in two independent variables and one dependent variable:

$$u_t = H(t, x, u, u_x, u_{xx}, u_{xxx}, \ldots). \tag{1}$$

The invariant surface condition is given by

$$V_1(t, x, u)u_t + V_2(t, x, u)u_x = F(t, x, u). \tag{2}$$

Let us take the case with $V_1 = 0$ and $V_2 = 1$, so that (2) becomes[2]

$$u_x = G(t, x, u). \tag{3}$$

Then, an equation for G is easily obtained. We call this equation G-equation [27]. Its invariant surface condition is given by

$$\xi_1(t, x, u, G)G_t + \xi_2(t, x, u, G)G_x + \xi_3(t, x, u, G)G_u = \eta(t, x, u, G). \tag{4}$$

Let us consider the case $\xi_1 = 0$, $\xi_2 = 1$, and $\xi_3 = G$, so that (4) becomes

$$G_x + GG_u = \eta(t, x, u, G). \tag{5}$$

Then, an equation for η is derived. We call this equation η-equation. Clearly

$$G_x + GG_u \equiv u_{xx} \equiv \eta. \tag{6}$$

[2] We have replaced $F(t, x, u)$ with $G(t, x, u)$ in order to avoid any ambiguity in the following discussion.

We could keep iterating to obtain the Ω-equation, which corresponds to

$$\eta_x + G\eta_u + \eta\eta_G \equiv u_{xxx} \equiv \Omega(t, x, u, G, \eta), \tag{7}$$

the ρ-equation, which corresponds to

$$\Omega_x + G\Omega_u + \eta\Omega_G + \Omega\Omega_\eta \equiv u_{xxxx} \equiv \rho(t, x, u, G, \eta, \Omega), \tag{8}$$

and so on. Each of these equations inherits the symmetry algebra of the original equation, with the right prolongation: first prolongation for the G-equation, second prolongation for the η-equation, and so on. Therefore, these equations are named heir-equations.

This iterating method yields both partial symmetries as given by Vorobev in [40], and differential constraints as given by Olver [32]. Also, it should be noticed that the $u_{\underbrace{xx\ldots}_{n}}$-equation of (1) is just one of many possible n-extended equations as defined by Guthrie in [15]. More details can be found in [28].

Now, we describe the method that allows one to find nonclassical symmetries of (1) by using a suitable heir-equation. For the sake of simplicity, let us assume that the highest order x-derivative appearing in (1) is two, i.e.,

$$u_t = H(t, x, u, u_x, u_{xx}). \tag{9}$$

First, we use (9) to replace u_t into (2), with the condition $V_1 = 1$, i.e.,

$$H(t, x, u, u_x, u_{xx}) + V_2(t, x, u)u_x = F(t, x, u). \tag{10}$$

Then, we generate the η-equation with $\eta = \eta(x, t, u, G)$ and replace $u_x = G$, $u_{xx} = \eta$ in (10), i.e.,

$$H(t, x, u, G, \eta) = F(t, x, u) - V_2(t, x, u)G. \tag{11}$$

For Dini's theorem, we can isolate η in (11), e.g.,

$$\eta = \big[h_1(t, x, u, G) + F(t, x, u) - V_2(t, x, u)G\big]h_2(t, x, u, G), \tag{12}$$

where $h_i(t, x, u, G)$ $(i = 1, 2)$ are known functions. Thus, we have obtained a particular solution of η which must yield an identity if replaced into the η-equation. The only unknowns are $V_2 = V_2(t, x, u)$ and $F = F(t, x, u)$. Let us recall to the reader that there are two sorts of nonclassical symmetries, those where in (2) the infinitesimal V_1 is nonzero, and those where it is zero [10]. In the first case, we can assume without loss of generality that $V_1 = 1$, while in the second case we can assume $V_2 = 1$, which corresponds to generate the G-equation. If there exists a nonclassical symmetry,[3] our method will recover it. Otherwise, only the classical symmetries will be found. If we are interested in finding only nonclassical symmetries, then we should impose F and V_2 to be functions only of the dependent variable u. Moreover, any such solution should be singular, i.e., should not form a group.

If we are dealing with a third order equation, then we need to construct the heir-equation of order three, i.e., the Ω-equation. Then, a similar procedure will yield a particular solution of the Ω-equation given by a formula similar to

[3] Of course, we mean one such that $V_1 \neq 0$, i.e., $V_1 = 1$.

$$\Omega = \big[h_1(t,x,u,G,\eta) + F(t,x,u) - V_2(t,x,u)G\big]h_2(t,x,u,G,\eta), \tag{13}$$

where $h_i(t,x,u,G,\eta)$ $(i = 1,2)$ are known functions.

In the case of a fourth order equation, we need to construct the heir-equation of order four, i.e., the ρ-equation. Then, a similar procedure will yield a particular solution of the ρ-equation given by a formula similar to

$$\rho = \big[h_1(t,x,u,G,\eta,\Omega) + F(t,x,u) - V_2(t,x,u)G\big]h_2(t,x,u,G,\eta,\Omega), \tag{14}$$

where $h_i(t,x,u,G,\eta,\Omega)$ $(i = 1,2)$ are known functions.

And so on.

3. Some examples

We present some examples of how the method works. We consider some families of evolution equations of second and third order. For each of them, we derive the corresponding heir-equations up to the appropriate order. Then, we look for the particular solution which yields nonclassical symmetries. We would like to underline how easy this method is in comparison with the existing one. The only difficulty consists is deriving the heir-equations, which become longer and longer. However, they can be automatically determined by using any computer algebra system.

3.1. *Example 1: $u_t = u_{xx} + R(u,u_x)$*

Let us consider the following family of second order evolution equations:

$$u_t = u_{xx} + R(u,u_x) \tag{15}$$

with $R(u,u_x)$ a known function of u and u_x. Famous equations known to possess nonclassical symmetries belong to (15): Burgers' [4], Fisher's [9], real Newell–Whitehead's [24], Fitzhugh–Nagumo's [30], and Huxley's equation [5,9].

The G-equation of (15) is

$$R_G(GG_u + G_x) + GR_u + 2G_{xu}G + G_{uu}G^2 - G_uR - G_t + G_{xx} = 0. \tag{16}$$

The η-equation of (15) is

$$\begin{aligned}&2R_{uG}\eta G + R_{GG}\eta^2 + R_G\eta_x + GR_G\eta_u + R_{uu}G^2 - GR_u\eta_G + R_u\eta + 2\eta_{xG}\eta\\&\quad + 2\eta_{uG}\eta G + \eta_{GG}\eta^2 - \eta_t + 2\eta_{xu}G + \eta_{xx} + \eta_{uu}G^2 - R\eta_u = 0.\end{aligned} \tag{17}$$

The particular solution (12) that we are looking for is

$$\eta = -R(u,G) + F(t,x,u) - V_2(t,x,u)G, \tag{18}$$

which replaced into (17) yields an overdetermined system in the unknowns F and V_2 if $R(u,G)$ has a given expression. Otherwise, after solving a first order linear partial differential equation in $R(u,G)$, we obtain that Eq. (15) may possess a nonclassical symmetry (2) with $V_1 = 1$, $V_2 = v(u)$, $F = f(u)$ if $R(u,u_x)$ has the following form:

$$R(u,u_x)=\frac{u_x}{f^2}\left(\left(-\frac{df}{du}fu_x+\frac{dv}{du}\right)fu_x^2+\Psi(\xi)u_x^2+2f^2v-3fu_xv^2+u_x^2v^3\right), \tag{19}$$

with f, v arbitrary functions of u, and Ψ arbitrary function of

$$\xi=\frac{f(u)}{u_x}-v(u). \tag{20}$$

This means that infinitely many cases can be found.

Here we present just three examples of (19) which are new, as far as we know. Equation (15) with $R(u,u_x)$ given by

$$R(u,u_x)=(2u_x+u^4)\frac{u_x}{u} \tag{21}$$

admits a nonclassical symmetry with $v=u^3/2$ and $f=-u^7/12$. It is interesting to notice that the corresponding reduction leads to the solution of the following ordinary differential equation in u and x,

$$u_{xx}=-2u_x^2/u-3u^3u_x/2-u^7/12,$$

which is linearizable. In fact, it admits a Lie symmetry algebra of dimension eight [20].

A second example is given by Eq. (15) with

$$\begin{aligned}R(u,u_x)=u_x\Big(&16\log\big(\big(-a_1^2u^3-3a_1a_2u^2-2a_1a_6u^2-2a_1u_xu+4a_3a_7u+4a_4a_7\\&-4a_5a_7u_x\big)/(4a_7u_x)\big)a_7^2u_x^2+a_1^5u^7+7a_1^4a_2u^6+4a_1^4a_6u^6+15a_1^3a_2^2u^5\\&+16a_1^3a_2a_6u^5-8a_1^3a_3a_7u^5-8a_1^3a_4a_7u^4+4a_1^3a_6^2u^5+9a_1^2a_2^3u^4\\&+12a_1^2a_2^2a_6u^4-32a_1^2a_2a_3a_7u^4-32a_1^2a_2a_4a_7u^3+4a_1^2a_2a_6^2u^4\\&-16a_1^2a_3a_6a_7u^4-16a_1^2a_4a_6a_7u^3-24a_1a_2^2a_3a_7u^3-24a_1a_2^2a_4a_7u^2\\&-16a_1a_2a_3a_6a_7u^3-16a_1a_2a_4a_6a_7u^2+16a_1a_3^2a_7^2u^3+32a_1a_3a_4a_7^2u^2\\&+16a_1a_4^2a_7^2u+16a_2a_3^2a_7^2u^2+32a_2a_3a_4a_7^2u+16a_2a_4^2a_7^2\Big)\\&\Big/\big(a_1^4u^6+6a_1^3a_2u^5+4a_1^3a_6u^5+9a_1^2a_2^2u^4+12a_1^2a_2a_6u^4-8a_1^2a_3a_7u^4\\&-8a_1^2a_4a_7u^3+4a_1^2a_6^2u^4-24a_1a_2a_3a_7u^3-24a_1a_2a_4a_7u^2\\&-16a_1a_3a_6a_7u^3-16a_1a_4a_6a_7u^2+16a_3^2a_7^2u^2+32a_3a_4a_7^2u\\&+16a_4^2a_7^2\big),\end{aligned} \tag{22}$$

where a_j $(j=1,\dots,7)$ are arbitrary constants. It admits a nonclassical symmetry with

$$v=\frac{a_1u+a_2+2a_6}{2}$$

and

$$f=\frac{-a_1^2u^3-3a_1a_2u^2-2a_1a_6u^2+4a_3a_7u+4a_4a_7}{4}.$$

A third example is given by Eq. (15) with

$$R(u,u_x)=\frac{u_x}{\sin(u)^2}\big(\cos(u)^3u_x^2-3\cos(u)^2\sin(u)u_x+2\cos(u)\sin(u)^2 \\ -\cos(u)\sin(u)u_x-\sin(u)^2u_x^2+\Psi(\xi)u_x^2\big), \tag{23}$$

where Ψ is an arbitrary function of $\xi=(-\cos(u)u_x+\sin(u))/u_x$. It admits a nonclassical symmetry with $v=\cos(u)$ and $f=\sin(u)$.

The subclass of Eq. (15) with $R=r(u)$ was considered in [10], where classical and nonclassical symmetries were retrieved. Just to show that our method is a lot simpler than the existing one, we give the details of the calculations in the case

$$r(u)=-u^3-bu^2-cu-d,$$

which admits a nonclassical symmetry [10, Ansatz 4.2.1]. We replace (18) into (17), which becomes a third degree polynomial in G. The corresponding coefficients (let us call them $mm3, mm2, mm1, mm0$, respectively) must all become equal to zero. From $mm3=0$, we obtain

$$V_2=A_1(t,x)u+A_2(t,x),$$

while $mm2=0$ yields

$$F=A_3(t,x)u+A_4(t,x)+\frac{\partial A_1}{\partial x}u^2-\frac{A_1^2u^3}{3}-A_1A_2u^2$$

with $A_k(t,x)$ $(k=1,\dots,4)$ arbitrary functions. Now, none of the remaining arbitrary functions depends on u. Since $mm1$ is a third degree polynomial in u, then its coefficients (let us call them $mm1k3, mm1k2, mm1k1, mm1k0$, respectively) must all become equal to zero. Equaling $mm1k3$ to zero yields two cases: either $A_1=0$, or $A_1^2=9/2$. If we assume $A_1=3/\sqrt{2}$, then $mm1k2=0$, $mm1k1=0$, $mm1k0=0$ lead to $A_2=b/\sqrt{2}$, $A_4=-3c/2$, $A_3=-3d/2$, respectively. These values yield $mm0=0$. Thus, the nonclassical symmetry found in [10] is recovered, i.e.,

$$V_2=\frac{b+3u}{\sqrt{2}},\qquad F=\frac{3}{2}(-u^3-bu^2-cu-d). \tag{24}$$

A similar result holds if we assume $A_1=-3/\sqrt{2}$, then F is the same, and $V_2=-(b+3u)/\sqrt{2}$. The case $A_1=0$ leads to either $V_2=1$, $F=0$ (trivial classical symmetry), or $d=-b(2b^2-9c)/27$ with

$$V_2=A_2,\qquad F=-\frac{1}{3}(b+3u)\frac{\partial A_2}{\partial x},$$

where $A_2(t,x)$ must satisfy

$$\frac{\partial A_2}{\partial t}-3\frac{\partial^2A_2}{\partial x^2}+2A_2\frac{\partial A_2}{\partial x}=0, \\ 3\frac{\partial^3A_2}{\partial x^3}-3A_2\frac{\partial^2A_2}{\partial x^2}+(b^2-3c)\frac{\partial A_2}{\partial x}=0. \tag{25}$$

Solving (25) results into two more cases which can be found in [10, Table 2], fourth and fifth row, respectively.

3.2. Example 2: $u_t = u^{-2}u_{xx} + R(u, u_x)$

Let us consider another family of second order evolution equations:

$$u_t = u^{-2}u_{xx} + R(u, u_x). \tag{26}$$

The G-equation of (26) is

$$\begin{aligned} &R_G(GG_u + G_x)u^3 + u^3 G R_u + 2uGG_{xu} + uG^2G_x - u^3 R G_u \\ &\quad - 2G^2G_u - u^3G_t + uG_{xx} - 2GG_x = 0. \end{aligned} \tag{27}$$

The η-equation of (26) is

$$\begin{aligned} &2R_{uG}\eta G u^4 + R_{GG}\eta^2u^4 + R_G\eta_x u^4 + R_G\eta_u G u^4 + R_{uu}G^2u^4 - R_u\eta_G G u^4 \\ &\quad + R_u\eta u^4 + 2\eta_{xG}\eta u^2 + 2\eta_{uG}\eta G u^2 + \eta_{GG}\eta^2u^2 - 2\eta_G\eta G u - \eta_t u^4 + 2\eta_{xu}Gu^2 \\ &\quad + \eta_{xx}u^2 - 4\eta_x G u + \eta_{uu}G^2u^2 - \eta_u R u^4 - 4\eta_u G^2 u - 2\eta^2 u + 6\eta G^2 = 0. \end{aligned} \tag{28}$$

The particular solution (12) that we are looking for is

$$\eta = \big[-R(u, G) + F(t, x, u) - V_2(t, x, u)G\big]u^2, \tag{29}$$

which replaced into (28) and imposing $V_2 = v(u)$, $F = f(u)$ yields a first order linear partial differential equation in $R(u, u_x)$. Then, Eq. (26) may possess a nonclassical symmetry (2) with $V_1 = 1$, $V_2 = v(u)$, $F = f(u)$, if $R(u, u_x)$ has the following form:

$$R(u, u_x) = -vu_x + f - \frac{df}{du}\frac{u_x^2}{uf} + \left(\frac{dv}{du}f + \Psi(\xi)\right)\frac{u_x^3}{f^2u^2}, \tag{30}$$

with f, v arbitrary functions of u and Ψ an arbitrary function of the same ξ as given in (20). This means that infinitely many cases can be found.

Here we present just two examples of (30) which are new, as far as we know. In both examples, Ψ is an arbitrary function of ξ as shown.

Equation (26) with $R(u, u_x)$ given by

$$R(u, u_x) = 2\frac{u_x^3}{u^6} + \Psi\left(\frac{u^5}{u_x} - u^2\right)\frac{u_x^3}{u^{12}} - 5\frac{u_x^2}{u^3} - (1 + u^2)u_x + u^5 \tag{31}$$

admits a nonclassical symmetry with $v = u^2 + 1$ and $f = u^5$.

Equation (26) with $R(u, u_x)$ given by

$$R(u, u_x) = \Psi\left(\frac{1}{uu_x} - u\right)u_x^3 + \frac{u_x^3}{u} + \frac{u_x^2}{u^3} - uu_x + \frac{1}{u} \tag{32}$$

admits a nonclassical symmetry with $v = u$ and $f = 1/u$.

Now we would like to show how our method works with an equation which does not admit nonclassical symmetries. Let us consider Eq. (26) with $R = -2u^{-3}u_x^2 + 1$ [32, p. 519], i.e.,

$$u_t = (u^{-2}u_x)_x + 1. \tag{33}$$

Its η-equation admits a solution of the type (29) only if

$$V_2 = \frac{c_1 + x}{c_3 - 2t}, \qquad F = \frac{-2u}{c_3 - 2t},$$

where c_j $(j = 1, 3)$ are arbitrary constants. It corresponds to the three-dimensional Lie point symmetry algebra admitted by (33). Nonclassical symmetries do not exist. However, it is known that Eq. (33) admits higher order symmetries [22], which may be retrieved searching for particular solutions of its higher order heir-equations, as we conjecture in the final comments.

3.3. Example 3: $u_t = u_{xxx} + R(u, u_x, u_{xx})$

Let us consider the following family of third order evolution equations:

$$u_t = u_{xxx} + R(u, u_x, u_{xx}), \tag{34}$$

with $R(u, u_x, u_{xx})$ a known function of u, u_x and u_{xx}. We derive the Ω-equation[4] of (34) and look for the particular solution (13), i.e.,

$$\Omega = -R(u, G, \eta) + F(t, x, u) - V_2(t, x, u)G, \tag{35}$$

which replaced into the Ω-equation and assuming $V_2 = v(u)$, $F = f(u)$ yields the following first order linear partial differential equation in $R(u, u_x, u_{xx}) \equiv R(u, G, \eta)$:

$$\begin{aligned}
&R(u, G, \eta)\frac{d}{du}f(u) + \frac{\partial}{\partial\eta}R(u, G, \eta)G^3\frac{d^2}{du^2}v(u) - \frac{\partial}{\partial\eta}R(u, G, \eta)G^2\frac{d^2}{du^2}f(u)\\
&\quad - 3G\eta\frac{d^2}{du^2}f(u) - \frac{\partial}{\partial\eta}R(u, G, \eta)\eta\frac{d}{du}f(u) - \frac{\partial}{\partial G}R(u, G, \eta)G\frac{d}{du}f(u)\\
&\quad + 6G^2\eta\frac{d^2}{du^2}v(u) - 3\frac{d}{du}v(u)v(u)G^2 + 3G\frac{d}{du}v(u)f(u) - 4R(u, G, \eta)\frac{d}{du}v(u)G\\
&\quad + \frac{\partial}{\partial G}R(u, G, \eta)G^2\frac{d}{du}v(u) + 3\frac{\partial}{\partial\eta}R(u, G, \eta)G\eta\frac{d}{du}v(u) - \frac{\partial}{\partial u}R(u, G, \eta)f(u)\\
&\quad + 3\eta^2\frac{d}{du}v(u) - G^3\frac{d^3}{du^3}f(u) + G^4\frac{d^3}{du^3}v(u) = 0.
\end{aligned} \tag{36}$$

Thus, Eq. (34) may possess a nonclassical symmetry (2) with $V_1 = 1$, $V_2 = v(u)$, $F = f(u)$ if $R(u, u_x, u_{xx}) \equiv R(u, G, \eta)$ satisfies (36). Note that the complete integral of (36) involves an arbitrary function $\Phi = \Phi(\xi_1, \xi_2)$ of $\xi_1 \equiv \xi$ as given in (20) and

$$\xi_2 = \frac{f(u)}{u_x^3}\left(u_{xx}\,f(u) - u_x^2\frac{d}{du}f(u) + u_x^3\frac{d}{du}v(u)\right). \tag{37}$$

This means that infinitely many cases can be found.

Here we present two classes of solutions of (36) which have never been described, as far as we know.

Equation (34) with $R(u, u_x, u_{xx})$ given by

$$-3\frac{u^6}{u_x} - 15u^4 - 6\frac{u_{xx}u^3}{u_x} - 12u_xu^2 - (3u_x + 12u_{xx})u + 12u_x^2$$

[4] Here we do not present anyone of the heir-equations due to their long expression.

$$
\begin{aligned}
&-3\frac{u_{xx}^2}{u_x}+\frac{-6u_x^2+3u_{xx}u_x}{u}+3\frac{u_x^2}{u^2}-4\frac{u_x^3}{u^3}+3\frac{u_x^3}{u^4}-\frac{u_x^4}{u^5}+\frac{u_x^4}{u^6}\\
&-\frac{u_x^4}{u^9}\Phi\left(\frac{u(u_x+u^2)}{u_x},-\frac{u^3(u_{xx}u^3-u_x^3-3u_x^2u^2)}{u_x^3}\right)
\end{aligned}
\tag{38}
$$

admits a nonclassical symmetry with $v=1-u$ and $f=u^3$.

Equation (34) with $R(u,u_x,u_{xx})$ given by

$$
\begin{aligned}
&\frac{1}{4u^3u_x(-2\sqrt{u}\,u_x+u)}\Bigg(12u^3u_{xx}u_x^2+12u^{9/2}u_x^2+6u^{3/2}u_x^5-3u_x^4u^2\\
&-48u_x^3u^4+24u^{7/2}u_xu_{xx}^2-24u^{5/2}u_x^3u_{xx}\\
&+4(u-2\sqrt{u}\,u_x)u_x^5\Phi\left(-\frac{\sqrt{u}\,u_x-u}{u_x},u\left(u_{xx}u-u_x^2+\frac{u_x^3}{2\sqrt{u}}\right)u_x^{-3}\right)\\
&-36u^3{u_x}^5+64u^{7/2}u_x^4+8u^{5/2}u_x^6-12u^4u_{xx}^2\Bigg)
\end{aligned}
\tag{39}
$$

admits a nonclassical symmetry with $v=\sqrt{u}$ and $f=u$.

Finally, we would like to show how our method works with a third order evolution equation which does not admit nonclassical symmetries. Let us consider the modified Korteweg–de Vries equation (mKdV):

$$
u_t=u_{xxx}-6u^2u_x.
\tag{40}
$$

In [27], the G-equation of (40) was derived:

$$
\begin{aligned}
&3GG_{xxu}+3GG_uG_{xu}+3G_xG_{xu}+G^3G_{uuu}+3G^2G_{xuu}+3G^2G_uG_{uu}\\
&+3GG_xG_{uu}-G_t+G_{xxx}-6u^2G_x-12uG^2=0.
\end{aligned}
\tag{41}
$$

The η-equation of (40) is

$$
\begin{aligned}
&\eta^3\eta_{GGG}+3\eta^2\eta_{xGG}+3\eta^2\eta_{uG}+3\eta^2\eta_{uGG}G+3\eta^2\eta_G\eta_{GG}+3\eta\eta_x\eta_{GG}+3\eta\eta_{xxG}\\
&+3\eta\eta_{xu}+6\eta\eta_{xuG}G+3\eta\eta_{xG}\eta_G+3\eta\eta_u\eta_{GG}G+3\eta\eta_{uu}G+3\eta\eta_{uuG}G^2\\
&+3\eta\eta_{uG}\eta_GG-36\eta uG-\eta_t+3\eta_x\eta_{xG}+3\eta_x\eta_{uG}G-6\eta_xu^2+\eta_{xxx}+3\eta_{xxu}G\\
&+3\eta_{xuu}G^2+3\eta_{xG}\eta_uG+3\eta_u\eta_{uG}G^2+\eta_{uuu}G^3+12\eta_GuG^2-12G^3=0.
\end{aligned}
\tag{42}
$$

The Ω-equations of (40) is

$$
\begin{aligned}
&6\Omega_{xu\eta}\Omega G+6\Omega_{xG\eta}\Omega\eta+3\Omega_{xx\eta}\Omega+3\Omega_{x\eta}\Omega_\eta\Omega+3\Omega_{x\eta}\Omega_x+3\Omega_{x\eta}\Omega_uG\\
&+3\Omega_{x\eta}\Omega_G\eta+6\Omega_{uG\eta}\Omega\eta G+3\Omega_{uu\eta}\Omega G^2+3\Omega_{u\eta}\Omega_\eta\Omega G+3\Omega_{u\eta}\Omega_xG\\
&+3\Omega_{u\eta}\Omega_uG^2+3\Omega_{u\eta}\Omega_G\eta G+3\Omega_{u\eta}\Omega\eta+3\Omega_{GG\eta}\Omega\eta^2+3\Omega_{G\eta}\Omega_\eta\Omega\eta\\
&+3\Omega_{G\eta}\Omega_x\eta+3\Omega_{G\eta}\Omega_u\eta G+3\Omega_{G\eta}\Omega_G\eta^2+3\Omega_{G\eta}\Omega^2+\Omega_{\eta\eta\eta}\Omega^3+3\Omega_{x\eta\eta}\Omega^2\\
&+3\Omega_{u\eta\eta}\Omega^2G+3\Omega_{G\eta\eta}\Omega^2\eta+3\Omega_{\eta\eta}\Omega_\eta\Omega^2+3\Omega_{\eta\eta}\Omega_x\Omega+3\Omega_{\eta\eta}\Omega_u\Omega G\\
&+3\Omega_{\eta\eta}\Omega_G\Omega\eta+36\Omega_\eta\eta uG+12\Omega_\eta G^3-\Omega_t+6\Omega_{xuG}\eta G+3\Omega_{xuu}G^2
\end{aligned}
$$

$$
\begin{aligned}
&+3\Omega_{xu}\eta+3\Omega_{xGG}\eta^2+3\Omega_{xG}\Omega+\Omega_{xxx}+3\Omega_{xxu}G+3\Omega_{xxG}\eta-6\Omega_x u^2\\
&+3\Omega_{uGG}\eta^2G+3\Omega_{uG}\Omega G+3\Omega_{uG}\eta^2+\Omega_{uuu}G^3+3\Omega_{uuG}\eta G^2+3\Omega_{uu}\eta G\\
&+\Omega_{GGG}\eta^3+3\Omega_{GG}\Omega\eta+12\Omega_G uG^2-48\Omega uG-36\eta^2u-72\eta G^2=0. \qquad (43)
\end{aligned}
$$

The particular solution (13) that we are looking for is

$$\Omega=6u^2G+F(t,x,u)-V_2(t,x,u)G, \qquad (44)$$

which replaced into (43) yields an overdetermined system in the unknowns F and V_2. It is very easy to prove that nonclassical symmetries do not exist, a well-known result. Instead, we obtain the classical symmetries admitted by (40), i.e.,

$$V_2=\frac{c_2+x}{c_1+3t}, \qquad F=-\frac{u}{c_1+3t}, \qquad (45)$$

with c_i $(i=1,2)$ arbitrary constants.

4. Final comments

We have determined an algorithm which is easier to implement than the usual method to find nonclassical symmetries admitted by an evolution equation in two independent variables. Moreover, one can retrieve both classical and nonclassical symmetries with the same method. Last but not least, we have shown that our method is able to retrieve an infinite number of equations admitting nonclassical symmetries.

Using the heir-equations raises many intriguing questions which we hope to address in future work:

- Could an a priori knowledge of the existence of nonclassical symmetries apart from classical be achieved by looking at the properties of the right-order heir-equation? We have shown that our method leads to both classical and nonclassical symmetries. Nonclassical symmetries could exist if we impose F and V_2 to be functions only of the dependent variable u in either (12), or (13), or (14), or .... Of course, any such solution of F and V_2 does not yield a nonclassical symmetry, unless it is isolated, i.e., does not form a group.
- What is integrability? The existence of infinitely many higher order symmetries is one of the criteria [22,33]. In [29], we have shown that invariant solutions of the heir-equations yield Zhdanov's conditional Lie–Bäcklund symmetries [43]. Higher order symmetries may be interpreted as special solutions of heir-equations (up to which order? see [33,37]). Another criterion for integrability consists of looking for Bäcklund transformations [2,36]. In [27], we have found that a nonclassical symmetry of the G-equation (41) for the mKdV equation (40) gives the known Bäcklund transformation between (40) and the KdV equation [23]. Another integrability test is the Painlevé property [41] which when satisfied leads to Lax pairs (hence, inverse scattering transform) [2], Bäcklund transformations, and Hirota bilinear formalism [39]. In [12], the singularity manifold of the mkdV Eq. (40) was found to be connected to an equation which is exactly the G-equation (41). Could heir-equations be the common link among all the integrability methods?

- In order to reduce a partial differential equation to ordinary differential equations, one of the first things to do is find the admitted Lie point symmetry algebra. In most instances, it is very small, and therefore not many reductions can be obtained. However, if heir-equations are considered, then many more ordinary differential equations can be derived using the same Lie algebra [3,21,28]. Of course, the classification of all dimension subalgebras [42] becomes imperative [21]. In the case of known integrable equations such as (40), it would be interesting to investigate which ordinary differential equations result from using the admitted Lie point symmetry algebra and the corresponding heir-equations. Do all these ordinary differential equations possess the Painlevé property (see the Painlevé conjecture as stated in [1])?
- In recent years, researchers often find solutions of partial differential equations which apparently do not come from any symmetry reduction. Are the heir-equations the ultimate method which keeps Lie symmetries at center stage?

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