ACADEMIC PRESS

Available online at www.sciencedirect.com

SCIENCE DIRECT®

Journal of Combinatorial Theory, Series A 103 (2003) 223–235

Journal of Combinatorial Theory Series A

http://www.elsevier.com/locate/jcta

Mitosis recursion for coefficients of Schubert polynomials

Ezra Miller[1]

Department of Mathematics, MIT, 77 Massachusetts Ave., Cambridge, MA 02139, USA

Received 31 July 2002

Abstract

Mitosis is a rule introduced by Knutson and Miller for manipulating subsets of the $n \times n$ grid. It provides an algorithm that lists the reduced pipe dreams (also known as rc-graphs) of Fomin and Kirillov for a permutation $w \in S_n$ by downward induction on weak Bruhat order, thereby generating the coefficients of Schubert polynomials of Lascoux and Schützenberger inductively. This note provides a short and purely combinatorial proof of these properties of mitosis.

1. Introduction

It has been a goal for some years, ever since Kohnert made his conjecture in [Koh91], to find inductive combinatorial rules on diagrams in the $n \times n$ grid that yield the coefficients of Schubert polynomials [LS82], when counted properly. The mitosis rule was offered in [KM03a] as a solution to this problem, but the proof was long, and involved some notions that strayed rather far from the elementary combinatorics of permutations. The purpose of this note is to bring mitosis entirely into the realm of combinatorics, by giving a short combinatorial proof of the fact (Theorem 15) that mitosis lists reduced pipe dreams (also known as rc-graphs) [FK96,BB93] recursively by induction on weak order in S_n, starting from the unique reduced pipe dream for the long permutation w_0.

More precisely, the proof here of Theorem 15, and the resulting diagrammatic recursion for the coefficients of Schubert polynomials in Corollary 16, rests only on

E-mail address: emiller@msri.org.

[1] Supported by the Sloan Foundation and NSF.

0097-3165/03/$ - see front matter © 2003 Published by Elsevier Inc.
doi:10.1016/S0097-3165(03)00020-7

the formula of Billey, Jockusch, and Stanley (Theorem 4), the characterization of Schubert polynomials by divided differences (Definition 3), and elementary combinatorial properties of reduced pipe dreams (Lemmas 5, 9 and 12 plus Proposition 13).

Mitosis was originally conceived in [KM03a] as a residual operation derived from more complicated combinatorial isobaric divided differences (Demazure operators) on standard monomials for certain determinantal ideals defined in the context of Schubert varieties in flag manifolds. As such, it served as a geometrically motivated improvement on Kohnert's rule [Koh91,Mac91,Win99,Win02], its advantages being the short combinatorial proof here and consistency with double Schubert polynomials as in [KM03a]. Mitosis is closely related to the construction of Schubert polynomials in terms of chains in Bruhat order in [LS02]. Other combinatorial algorithms producing Schubert polynomials include the chute and ladder moves on reduced pipe dreams [BB93], a different combinatorial divided difference operator on reduced pipe dreams [Len02], and an earlier construction of Bergeron [Ber92].

The plan of the paper is as follows. In the next two sections, we review the definition of the set $\mathcal{RP}(w)$ of reduced pipe dreams for a permutation $w \in S_n$, the BJS formula, and the mitosis algorithm on pipe dreams (subsets of the $n \times n$ grid). Section 4 provides an involution on $\mathcal{RP}(w)$ that is crucial for the proof of the main theorem and corollary in Section 5. The final section, which concerns the mitosis poset and is logically independent of the other sections, presents two definitions and a conjecture concerning a poset structure on the set $\mathcal{RP}_n$ of all reduced pipe dreams for permutations in S_n.

2. Pipe dreams

Consider a square grid $\mathbb{Z}_{>0} \times \mathbb{Z}_{>0}$ extending infinitely south and east, with the box in row i and column j labeled (i,j), as in an $\infty \times \infty$ matrix. If each box in the grid is covered with a square tile containing either ┼ or ╯╭, then one can think of the tiled grid as a network of pipes.

Definition 1. A *pipe dream* is a finite subset of $\mathbb{Z}_{>0} \times \mathbb{Z}_{>0}$, identified as the set of crosses in a tiling by *crosses* ┼ and *elbow joints* ╯╭. A pipe dream is *reduced* if each pair of pipes crosses at most once. The set $\mathcal{RP}(w)$ of reduced pipe dreams for the permutation $w \in S_n$ is the set of reduced pipe dreams D such that the pipe entering row i exits from column $w(i)$.

Although we always draw crossing tiles as some sort of cross (either '+' or '┼', the former with the square tile boundary and the latter without), we often leave the elbow tiles blank or denote them by dots, to make the diagrams less cluttered. Viewing n as fixed, we shall be interested in pipe dreams contained in the pipe dream

D_0 that has crosses in the triangular region strictly above the main antidiagonal (in spots (i,j) with $i+j \leqslant n$) and elbow joints elsewhere in the square grid $[n] \times [n]$ of size n. Note that D_0 is the unique reduced pipe dream for the *long permutation* $w_0 = n \ldots 321$ in S_n.

Example 2. The pipe dream D in Fig. 1 for $n = 8$ is a reduced pipe dream for the permutation $w = 13865742 \in S_8$. For clarity, we omit the square tile boundaries as well as the wavy "sea" of elbows ╯╭ below the main antidiagonal in the right pipe dream.

Since we need a statement of the BJS formula, we recall here the definition of Schubert polynomials of Lascoux and Schützenberger via divided differences. For notation, $s_i \in S_n$ denotes the transposition switching i and $i+1$, and length(w) denotes the number of inversions in a permutation w.

Definition 3 (Lascoux and Schützenberger [LS82]). The *i*th *divided difference operator* ∂_i takes each polynomial $f \in \mathbb{Z}[x_1, \ldots, x_n]$ to

$$\partial_i f(x_1, \ldots, x_n) = \frac{f(x_1, \ldots, x_n) - f(x_1, \ldots, x_{i-1}, x_{i+1}, x_i, x_{i+2}, \ldots, x_n)}{x_i - x_{i+1}}.$$

The *Schubert polynomial* for $w \in S_n$ is defined by the recursion

$$\mathfrak{S}_{ws_i}(x_1, \ldots, x_n) = \partial_i \mathfrak{S}_w(x_1, \ldots, x_n)$$

whenever length$(ws_i) <$ length(w), and the initial condition $\mathfrak{S}_{w_0}(x_1, \ldots, x_n) = \prod_{i=1}^{n} x_i^{n-i}$.

Theorem 4 (Billey et al. [BJS93], Fomin and Stanley [FS94]). $\mathfrak{S}_w(x_1, \ldots, x_n) = \sum_{D \in \mathcal{RP}(w)} \mathbf{x}^D$, *where* $\mathbf{x}^D = \prod_{(i,j) \in D} x_i$.

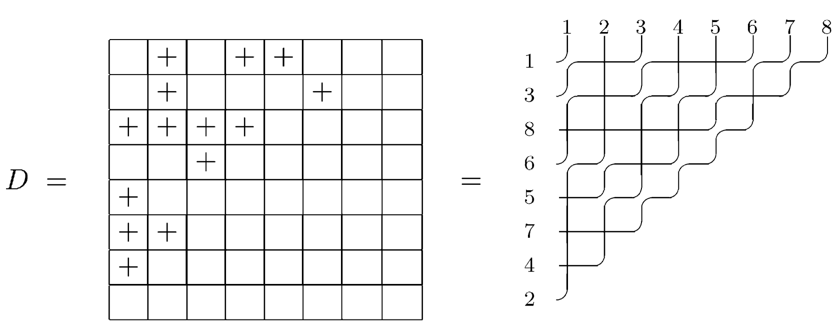
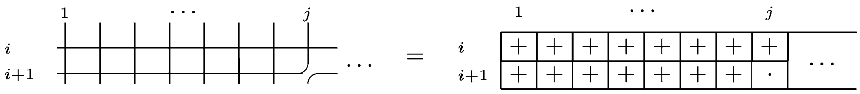


Fig. 1. A reduced pipe dream for $w = 13865742 \in S_8$.

The next lemma, which will be applied in Section 5, gives a criterion for when removing a '+' from a pipe dream $D \in \mathcal{RP}(w)$ leaves a pipe dream in $\mathcal{RP}(ws_i)$. Specifically, it concerns the removal of a cross at (i,j) from configurations that look like



at the left end of rows i and $i+1$ in D.

Lemma 5. *Let $D \in \mathcal{RP}(w)$ and j be a fixed column index with $(i+1,j) \notin D$, but $(i,p) \in D$ for all $p \leqslant j$, and $(i+1,p) \in D$ for all $p < j$. Then* $\text{length}(ws_i) < \text{length}(w)$, *and if $D' = D \backslash (i,j)$ then $D' \in \mathcal{RP}(ws_i)$.*

Proof. Removing (i,j) only switches the exit points of the two pipes starting in rows i and $i+1$, so the pipe starting in row k of D' exits out of column $ws_i(k)$ for each k. No pair of pipes can cross twice in D' because there are $\text{length}(ws_i)$ crossings. □

3. Mitosis algorithm

Given a pipe dream in $[n] \times [n]$, define

$$\begin{aligned}\text{start}_i(D) &= \text{column index of leftmost elbow in row } i \\ &= \min(\{j \mid (i,j) \notin D\} \cup \{n+1\}), \end{aligned} \tag{1}$$

so the ith row of D is filled solidly with crosses in the region to the left of $\text{start}_i(D)$. Let

$$\mathcal{J}_i(D) = \{\text{columns } j \text{ strictly to the left of } \text{start}_i(D) \mid (i+1,j) \text{ has no cross in } D\}.$$

For $p \in \mathcal{J}_i(D)$, construct the *offspring* D_p as follows. First delete the cross at (i,p) from D. Then take all crosses in row i of $\mathcal{J}_i(D)$ that are to the left of column p, and move each one down to the empty box below it in row $i+1$.

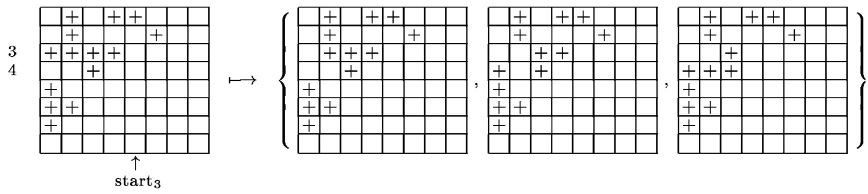
Definition 6. The ith *mitosis* operator sends a pipe dream D to

$$\text{mitosis}_i(D) = \{D_p \mid p \in \mathcal{J}_i(D)\}.$$

Write $\text{mitosis}_i(\mathcal{P}) = \bigcup_{D \in \mathcal{P}} \text{mitosis}_i(D)$ whenever $\mathcal{P}$ is a set of pipe dreams.

Observe that all of the action takes place in rows i and $i+1$, and $\text{mitosis}_i(D)$ is an empty set whenever $\mathcal{J}_i(D)$ is empty.

Example 7. The pipe dream D at left is the reduced pipe dream for $w = 13865742$ from Example 2.

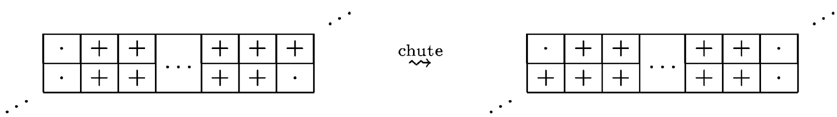


The set of three pipe dreams on the right is obtained by applying mitosis_3, since $\mathcal{J}_3(D)$ consists of columns 1, 2, and 4. The offspring are ordered as in Proposition 10, below.

In Proposition 10 we shall present another, more sequential way of writing down the mitosis offspring of a pipe dream. It uses a device invented by Bergeron and Billey.

Definition 8 (Bergeron and Billey [BB93]). A *chutable rectangle* is a connected $2 \times k$ rectangle C inside a pipe dream D such that $k \geqslant 2$ and all but the following three locations in C are crosses: the northwest, southwest, and southeast corners. Applying a *chute move* to D is accomplished by placing a '+' in the southwest corner of a chutable rectangle C and removing the '+' from the northeast corner of the same C.

Heuristically, a chute move therefore looks like:



The following basic fact about chute moves was discovered by Bergeron and Billey [BB93].

Lemma 9. *The set $\mathcal{RP}(w)$ of reduced pipe dreams for w is closed under chute moves.*

Proof. If two pipe intersect at the '+' in the northeast corner of a chutable rectangle C, then chuting that '+' only changes the crossing point of the two pipes to the southwest corner of C. No other pipes are affected. □

Proposition 10. *Let D be a pipe dream, and suppose j is the smallest column index such that $(i+1,j) \notin D$ and $(i,p) \in D$ for all $p \leqslant j$. Then $D_p \in \text{mitosis}_i(D)$ is obtained from D by*

1. *removing (i,j), and then*
2. *performing chute moves from row i to row $i+1$, each one as far left as possible, so that (i,p) is the last '+' removed.*

Proof. Immediate from Definitions 6 and 8. □

4. Intron mutation

Definition 11. Let D be a pipe dream and i a fixed row index. Order the boxes in rows i and $i+1$ of D as in the following diagram:

	1	2	3	4	...
i	1	3	5	7	...
$i+1$	2	4	6	8	

An *intron*[2] in these two adjacent rows is a $2 \times k$ rectangle C such that

1. the first and last boxes in C (the northwest and southeast corners) are elbows; and
2. no elbow in C is strictly northeast or strictly southwest of another elbow (so due north, due south, due east, or due west are all okay).
3. Ignoring all $\begin{array}{|c|}\hline + \\ \hline + \\ \hline\end{array}$ columns in rows i and $i+1$, an intron is just a sequence of $\begin{array}{|c|}\hline \cdot \\ \hline + \\ \hline\end{array}$ columns in rows i and $i+1$, followed by a sequence of $\begin{array}{|c|}\hline + \\ \hline \cdot \\ \hline\end{array}$ columns, possibly with one $\begin{array}{|c|}\hline \cdot \\ \hline \cdot \\ \hline\end{array}$ column in between. Columns with two crosses $\begin{array}{|c|}\hline + \\ \hline + \\ \hline\end{array}$ can be ignored for the purpose of proofs in what follows.
4. If an intron C satisfies the following extra condition, then C is called a *maximal* intron:
5. the elbow with largest index before C (if there is one) resides in row $i+1$, and the elbow with smallest index after C (if there is one) resides in row i.

Lemma 12. *For an intron C in a reduced pipe dream, a unique intron $\tau(C)$ satisfies*

1. *the sets of columns with exactly two crosses are the same in C and $\tau(C)$, and*
2. *the number c_i of crosses in row i of C equals the number of crosses in row $i+1$ of $\tau(C)$, and conversely.*

The involution τ, called intron mutation, *is always accomplished by a sequence of chute moves or inverse chute moves (because C is part of a reduced pipe dream).*

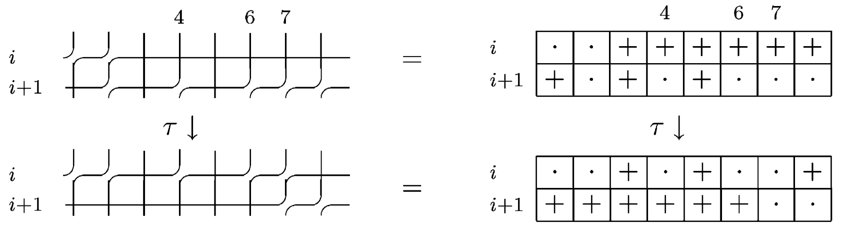
Proof. First assume $c_i > c_{i+1}$ and work by induction on $c = c_i - c_{i+1}$. If $c = 0$ then $\tau(C) = C$ and the lemma is obvious. If $c > 0$ then consider the leftmost $\begin{array}{|c|}\hline + \\ \hline \cdot \\ \hline\end{array}$ column. Moving to the left from this column there must be a column not equal to $\begin{array}{|c|}\hline + \\ \hline + \\ \hline\end{array}$, since the northwest entry of C is an elbow. The rightmost such column must be $\begin{array}{|c|}\hline \cdot \\ \hline \cdot \\ \hline\end{array}$, because its row i entry is an elbow (by construction) and its row $i+1$ entry cannot be a cross

[2] For the origin of this term, see [KM03a, Section 3.5].

(for then the pipes crossing there would also cross in the $\begin{smallmatrix}+\\ \cdot\end{smallmatrix}$ column). This means we can chute the '+' in $\begin{smallmatrix}+\\ \cdot\end{smallmatrix}$ into the $\begin{smallmatrix}\cdot\\ \cdot\end{smallmatrix}$ column, and proceed by induction.

Flip the argument 180° if $c_i < c_{i+1}$, so the chute move becomes an inverse chute. □

For example, here is an intron mutation accomplished by chuting the crosses in columns 4, 6, and then 7 of row i; the zigzag shapes formed by the dots in these introns are typical.



Proposition 13. *For each i there is an involution* $\tau_i : \mathcal{RP}(w) \to \mathcal{RP}(w)$ *such that* $\tau_i^2 = 1$, *and for all* $D \in \mathcal{RP}(w)$:

1. $\tau_i D$ *agrees with D outside rows i and* $i+1$.
2. $\mathrm{start}_i(\tau_i D) = \mathrm{start}_i(D)$, *and* $\tau_i D$ *agrees with D strictly west of this column.*
3. $\ell_i^i(\tau_i D) = \ell_{i+1}^i(D)$,

where $\ell_r^i(-)$ *is the number of crosses in row r that are east of or in column* $\mathrm{start}_i(-)$.

Proof. Let $D \in \mathcal{RP}(w)$. Consider the union of all columns in rows i and $i+1$ of D that are east of or coincide with column $\mathrm{start}_i(D)$. Since the first and last boxes in this region (numbered as in Definition 11) are elbows, this region breaks uniquely into a disjoint union of $2 \times k$ rectangles, each of which is either a maximal intron or completely filled with crosses. Indeed, this follows from (1) and Definition 11. Applying intron mutation to each maximal intron therein leaves a pipe dream that breaks up uniquely into maximal introns and solid crosses in the same way. Therefore, the lemma comes down to verifying that intron mutation preserves the property of being in $\mathcal{RP}(w)$, which comes from Lemmas 9 and 12. □

Remark 14. Intron mutation is precisely the involution (coplactic operation) σ_i defined by Lascoux on words (see the survey article [LLT97], for example) and extended to reduced pipe dreams in [Len02]. However, when all introns in rows i and $i+1$ are strung together, the involution τ_i does not agree with σ_i. In fact, Lascoux's involution is based on 'r-pairing', which is also used in the work of Bergeron [Ber92] and Lenart [Len02] to define combinatorial versions of divided difference operators. Intron mutation is therefore a different mechanism by which combinatorial divided differences can be defined on reduced pipe dreams.

5. Mitosis theorem

Theorem 15. *If* $\text{length}(ws_i) < \text{length}(w)$, *then the set* $\mathcal{RP}(ws_i)$ *of reduced pipe dreams for* ws_i *is the disjoint union* $\dot{\bigcup}_{D\in\mathcal{RP}(w)} \text{mitosis}_i(D)$. *Therefore*

$$\mathcal{RP}(w) = \text{mitosis}_{i_k} \cdots \text{mitosis}_{i_1}(D_0) \tag{2}$$

if $s_{i_1} \cdots s_{i_k}$ *is a reduced expression for* $w_0 w$.

Proof. Use the description of mitosis in Proposition 10 along with Lemmas 5 and 9 to conclude that $\text{mitosis}_i(D) \subseteq \mathcal{RP}(ws_i)$ whenever $D \in \mathcal{RP}(w)$. It follows directly from the definitions that $\text{mitosis}_i(D) \cap \text{mitosis}_i(D') = \varnothing$ if $D \neq D'$ are reduced pipe dreams for w. Thus it suffices to prove that $\text{mitosis}_i(\mathcal{RP}(w))$ has the same cardinality as $\mathcal{RP}(ws_i)$.

Fix $D \in \mathcal{RP}(w)$, write $\mathbf{x}^D = \prod_{(i,j)\in D} x_i$, and let $J = |\mathcal{J}_i(D)|$ be the number of mitosis offspring of D. The monomial $\mathbf{x}^D$ is a product $x_i^J \mathbf{x}^{D'}$, where D' is the pipe dream (*not* reduced) obtained from D by erasing the crosses in row i of $\mathcal{J}_i(D)$. Definition 6 implies that

$$\sum_{E\in\text{mitosis}_i(D)} \mathbf{x}^E = \sum_{d=1}^{J} x_i^{J-d} x_{i+1}^{d-1} \cdot \mathbf{x}^{D'} = \partial_i(x_i^J) \cdot \mathbf{x}^{D'}. \tag{3}$$

If $\tau_i D = D$, then $\mathbf{x}^{D'}$ is symmetric in x_i and x_{i+1} by Proposition 13, so that

$$\partial_i(x_i^J) \cdot \mathbf{x}^{D'} = \partial_i(x_i^J \cdot \mathbf{x}^{D'}) = \partial_i(\mathbf{x}^D)$$

in this case. On the other hand, if $\tau_i D \neq D$, then letting s_i act on polynomials by switching x_i and x_{i+1}, Proposition 13 implies that adding the sums in (3) for D and $\tau_i D$ yields

$$\partial_i(x_i^J) \cdot (\mathbf{x}^{D'} + s_i \mathbf{x}^{D'}) = \partial_i(x_i^J(\mathbf{x}^{D'} + s_i \mathbf{x}^{D'})) = \partial_i(\mathbf{x}^D + \mathbf{x}^{\tau_i D}).$$

Pairing off the elements of $\mathcal{RP}(w)$ not fixed by τ_i, we therefore conclude that

$$\sum_{E\in\text{mitosis}_i(\mathcal{RP}(w))} \mathbf{x}^E = \partial_i\left(\sum_{D\in\mathcal{RP}(w)} \mathbf{x}^D\right) = \partial_i(\mathfrak{S}_w(\mathbf{x})) = \mathfrak{S}_{ws_i}(\mathbf{x}) = \sum_{E\in\mathcal{RP}(ws_i)} \mathbf{x}^E$$

by Theorem 4 and the recursion for $\mathfrak{S}_w(\mathbf{x}) := \mathfrak{S}_w(x_1, \ldots, x_n)$ as in Definition 3. Plugging in $1, \ldots, 1$ for $\mathbf{x} = x_1, \ldots, x_n$ implies that $|\text{mitosis}_i(\mathcal{RP}(w))| = |\mathcal{RP}(ws_i)|$, as desired. $\square$

Finally, we come to the generation of Schubert coefficients by induction on weak Bruhat order via mitosis. For notation, if $v = s_{i_1} \cdots s_{i_k}$ is a reduced expression, set $\text{mitosis}_v = \text{mitosis}_{i_k} \cdots \text{mitosis}_{i_1}$.

Corollary 16. *For any permutation* $w \in S_n$ *we have*

$$\mathfrak{S}_w(x_1, \ldots, x_n) = \sum_{D \in \text{mitosis}_v(D_0)} \mathbf{x}^D \quad \textit{for } v = w_0 w,$$

where $\mathcal{RP}(w_0) = \{D_0\}$, *and* $\mathbf{x}^D = \prod_{(i,j) \in D} x_i$ *for any pipe dream* D.

Proof. Theorem 15 and Theorem 4. □

6. Mitosis poset

The next definition generalizes to arbitrary n the poset of pipe dreams for $n = 3$ in Fig. 2.

Definition 17. Theorem 15 defines a partial order, namely

$$D' \prec D \quad \text{if } D' \in \text{mitosis}_i(D) \text{ for some } i,$$

making the reduced pipe dreams for all of S_n into the *mitosis poset* $\mathcal{RP}_n = \bigcup_{w \in S_n} \mathcal{RP}(w)$.

The poset $\mathcal{RP}_n$, which is ranked by length = cardinality, fibers over the weak Bruhat order on S_n, with the preimage of $w \in S_n$ being $\mathcal{RP}(w)$. A reduced expression for $w_0 w$ can be thought of as the edge labels on a decreasing path beginning at w_0 and ending at w in the weak Bruhat order on S_n. The preimage in $\mathcal{RP}_n$ of such a path is a tree having $\mathcal{RP}(w)$ among its leaves (two reduced pipe dreams cannot share an offspring by the disjointness of the union in Theorem 15).

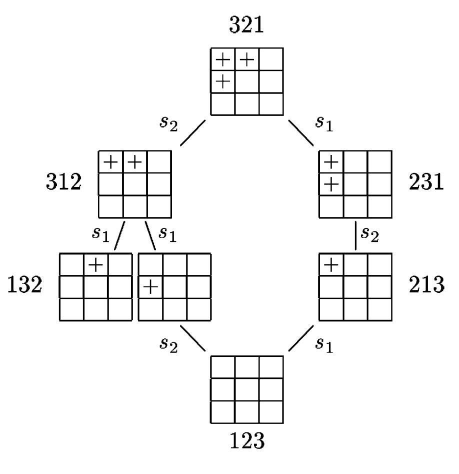


Fig. 2. Hasse diagram for $\mathcal{RP}_3$.

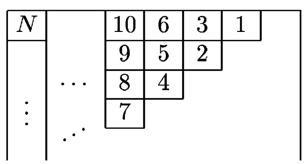
Definition 18. A path decreasing from w_0 to w in the weak order is *poptotic* if the leaves of its preimage in $\mathcal{RP}_n$ are precisely $\mathcal{RP}(w)$.

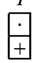
In other words, a path is poptotic if every reduced pipe dream lying over its interior has at least one offspring. For example, the right hand path in Fig. 2 from 321 to 123 is poptotic because only one reduced pipe dream appears at each stage, while the left path is apoptotic[3] because the first reduced pipe dream for 132 has no offspring under mitosis$_2$.

Proposition 19. *Poptotic paths from w_0 to w exist. In fact, the lexicographically first reduced expression for w_0w (in which $s_1 > s_2 > \cdots > s_{n-1}$) corresponds to a poptotic path.*

In particular, the lex first path from w_0 to id_n passes through *dominant* permutations, which by definition have exactly one reduced pipe dream (shaped like a Young diagram).

Proof. Number the boxes in the strict upper-left triangle, meaning all locations (q, p) such that $q + p \leqslant n$, as follows, where $N = \binom{n}{2}$.



The ordered sequence $(1, 2, 1, 3, 2, 1, 4, 3, 2, 1, \ldots)$ of row indices of boxes in this upper triangle gives rise to the lex first reduced expression $s_1s_2s_1s_3s_2s_1s_4s_3s_2s_1\cdots$ for the long word $w_0 = w_0\mathrm{id}_n$. In general, lex first reduced words for arbitrary w_0w correspond bijectively to the complements in the upper-left triangle of so-called *top* reduced pipe dreams [BB93], which are characterized (by definition) as having no  configurations. The reduced word corresponding to a top pipe dream is the ordered subsequence of row indices skipping the crosses atop each column.

Now suppose that $\mathrm{length}(ws_i) < \mathrm{length}(w)$, and that the lex first reduced expression for w_0ws_i ends in $s_{i'}s_i$. Under the bijection above between lex first reduced words and complements of top reduced pipe dreams, $s_{i'}$ and s_i correspond to the row indices i' and i of boxes numbered α' and α satisfying $\alpha' < \alpha$. [N.B. Either $i = i' - 1$, in which case $\alpha' = \alpha - 1$, or else $i > i'$, and α sits just above the main antidiagonal in some column to the left of α'.] Therefore, we shall assume by

[3] The word 'apoptosis' refers in biology to programmed cell death, where some cell in a multicellular organism commits suicide for the greater good of the organism. Thus *apoptotic* indicates that some reduced pipe dream dies without offspring, while *poptotic* indicates that all pipe dreams survive with offspring.

induction on length that

$$\text{every reduced pipe dream in } \mathcal{RP}(w) \text{ has crosses in boxes} > \alpha' \text{ and an elbow at } \alpha'. \qquad (*)$$

(the case $\text{length}(w_0 w) = 1$ is easy). The goal is to prove that $(*)$ holds with ws_i in place of w and α in place of α'. But first, note that Lemma 5, which holds with α in position (i,j) by assumption $(*)$, says that $\text{mitosis}_i(D)$ is nonempty for all $D \in \mathcal{RP}(w)$, as required.

More precisely, Lemma 5 says that removing the cross at α from each $D \in \mathcal{RP}(w)$ produces a pipe dream in $\mathcal{RP}(ws_i)$. Furthermore, either α lies in the top row or the box in D due north of α is a cross, so it is impossible for chute moves to end there after deleting the cross from α. Consequently, Proposition 10 implies that every pipe dream $D' \in \text{mitosis}_i(D)$ has crosses in boxes marked $> \alpha$, and an elbow joint in the box marked α. The proof is complete by Theorem 15. $\square$

Example 20. The three pipe dreams on the right in Example 7 are all reduced pipe dreams for $v = 13685742 = w \cdot s_3$, where $w = 13865742$ as in Example 2. Setting $i = 4$ and inspecting the inversions of v, we find that $\text{length}(vs_4) < \text{length}(v)$. On the other hand, mitosis_4 kills the first two of the three pipe dreams, whereas the last has two offspring. Thus any path from w_0 to vs_4 ending with $(\ldots, v, vs_4)$ is necessarily apoptotic.

Note that the lex first reduced expression for $w_0 v$, which corresponds to a poptotic path from w_0 to v by Proposition 19, equals $s_2 s_1 s_3 s_5 s_4 s_3 s_2 s_1 s_7 s_6 s_5 s_4 s_3 s_2 s_1$, while the lex first reduced expression for $w_0 vs_4$ equals $s_2 s_1 s_3 s_2 s_5 s_4 s_3 s_2 s_1 s_7 s_6 s_5 s_4 s_3 s_2 s_1$ (the s_2 in the fourth slot is new). These correspond to top reduced pipe dreams

$\cdot$	+	$\cdot$	+	+	$\cdot$	+	
$\cdot$	+	$\cdot$	+	+	$\cdot$		
$\cdot$	+	$\cdot$	+	$\cdot$			
$\cdot$	+	$\cdot$	+				
$\cdot$	+	$\cdot$					
$\cdot$	+						
$\cdot$							

v

and

$\cdot$	+	$\cdot$	+	+	$\cdot$	+	
$\cdot$	+	$\cdot$	+	$\cdot$	$\cdot$		
$\cdot$	+	$\cdot$	+	$\cdot$			
$\cdot$	+	$\cdot$	+				
$\cdot$	+	$\cdot$					
$\cdot$	+						
$\cdot$							

vs_4

in which the row indices of the dots give the lex first reduced expressions.

Whether or not a path from w_0 to w is poptotic, breadth-first search on the preimage tree (ordering the mitosis offspring as in Proposition 10) yields a total order on $\mathcal{RP}(w)$. It can be shown that *poptotic* total orders by breadth-first search are linear extensions of the partial order on reduced pipe dreams determined by chute operations.

Define the simplicial complex $\mathcal{L}_w$ with vertex set $[n] \times [n]$ to have as its facets the *complements* of the reduced pipe dreams for w:

$$\text{facets}(\mathcal{L}_w) = \{([n] \times [n]) \backslash D \mid D \in \mathcal{RP}(w)\}.$$

This is an example of a 'subword complex' [KM03a,KM03b], and is hence shellable by [KM03a, Theorems B and E.]. Through heuristic arguments and computer calculations in small symmetric groups, we are convinced of the following.

Conjecture 21. *Poptotic orders on* $\mathscr{RP}(w)$ *by breadth-first search yield shellings of* $\mathscr{L}_w$.

To emphasize: shellability is not in question, because shellings of $\mathscr{L}_w$ appear in [KM03a, Section 1.8]. The conjecture would just give more intuitive shellings than those known. It is conceivable that all of the apoptotic total orders are shellings, too, although this seems less likely.

Acknowledgments

The author is grateful to Allen Knutson, Cristian Lenart, Vic Reiner, and Anne Schilling for motivation and inspiration. Mitosis was discovered in part because of a superb AMS meeting on Modern Schubert Calculus organized by Frank Sottile and Nantel Bergeron (the latter also supplied LATEX macros for drawing pipe dreams).

References

[Ber92] N. Bergeron, A combinatorial construction of the Schubert polynomials, J. Combin. Theory Ser. A 60 (2) (1992) 168–182.

[BB93] N. Bergeron, S. Billey, RC-graphs and Schubert polynomials, Exp. Math. 2 (4) (1993) 257–269.

[BJS93] S.C. Billey, W. Jockusch, R.P. Stanley, Some combinatorial properties of Schubert polynomials, J. Algebraic Combin. 2 (4) (1993) 345–374.

[FK96] S. Fomin, A.N. Kirillov, The Yang-Baxter equation, symmetric functions, and Schubert polynomials, Discrete Math. 153 (1–3) (1996) 123–143, Proceedings of the Fifth Conference on Formal Power Series and Algebraic Combinatorics, Florence, 1993.

[FS94] S. Fomin, R.P. Stanley, Schubert polynomials and the nil-Coxeter algebra, Adv. Math. 103 (2) (1994) 196–207.

[KM03a] A. Knutson, E. Miller, Gröbner geometry of Schubert polynomials, Ann. of Math. 2 (2003), to appear.

[KM03b] A. Knutson, E. Miller, Subword complexes in Coxeter groups, Adv. Math. (2003), to appear.

[Koh91] A. Kohnert, Weintrauben, Polynome, Tableaux, Bayreuth. Math. Schr. (38) (1991) 1–97, Dissertation, Universität Bayreuth, Bayreuth, 1990.

[LLT97] A. Lascoux, B. Leclerc, J.-Y. Thibon, The plactic monoid, available at `http://www-igm.univ-mlv.fr/jyt/articles.html`, 1997.

[LS82] A. Lascoux, M.-P. Schützenberger, Polynômes de Schubert, C. R. Acad. Sci. Paris Sér. I Math. 294 (13) (1982) 447–450.

[Len02] C. Lenart, A unified approach to combinatorial formulas for Schubert polynomials, Preprint available at `http://www.math.albany.edu:8000/math/pers/lenart/articles/schubert.html`, 2002.

[LS02] C. Lenart, F. Sottile, Skew Schubert polynomials, Proc. Amer. Math. Soc., 2002, to appear.

[Mac91] I.G. Macdonald, Notes on Schubert polynomials, Publications du LACIM, Universitè du Québec à Montréal, 1991.
[Win99] R. Winkel, Diagram rules for the generation of Schubert polynomials, J. Combin. Theory Ser. A 86 (1) (1999) 14–48.
[Win02] R. Winkel, A short proof of Kohnert's algorithm for the generation of Schubert polynomials, in Sem. Loth. Comb. (2002), to appear.