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Supertwistors and cubic string field theory for open $N = 2$ strings

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Abstract

The known Lorentz invariant string field theory for open $N = 2$ strings is combined with a generalization of the twistor description of anti-self-dual (super-)Yang–Mills theories. We introduce a Chern–Simons-type Lagrangian containing twistor variables and derive the Berkovits–Siegel covariant string field equations of motion via the twistor correspondence. Both the purely bosonic and the maximally space–time supersymmetric cases are considered.

1. Introduction

It was recently shown by Witten [1] that B-type open topological string theory with the supertwistor space $\mathbb{C}P^{3|4}$ as a target space is equivalent to holomorphic Chern–Simons (hCS) theory on the same space (for related works see [2–7]). This hCS theory in turn is equivalent to supersymmetric $\mathcal{N} = 4$ anti-self-dual Yang–Mills (ASDYM) theory in four dimensions. The $\mathcal{N} = 4$ super-ASDYM model is governed by the Siegel Lagrangian [8]. Its truncation to the bosonic sector describes $\mathcal{N} = 0$ ASDYM theory with an auxiliary field of helicity -1 [8,9].

It may be of interest to generalize the twistor correspondence to the level of string field theory (SFT). This could be done using the approach proposed in [5] or in the more general setting of [6]. Alternatively, one could concentrate on (an appropriate extension of) SFT for $N = 2$ string theory. At tree level, open $N = 2$ strings are known to reduce to the ASDYM model in a Lorentz noninvariant gauge [10]; their SFT formulation [11] is based on the $N = 4$ topological string description [12,13]. The latter contains twistors from the outset: the coordinate $\zeta \in \mathbb{C}P^1$, the linear system, the integrability and the classical solutions with the help of twistor methods were all incorporated into $N = 2$ open string field theory in [14,15]. Since this theory [11] generalizes the Wess–Zumino–Witten-type model [16] for ASDYM theory and thus describes only anti-self-dual gauge fields (having helicity $+1$), it is not Lorentz invariant. Its maximally supersymmetric extension, $\mathcal{N} = 4$ super-ASDYM the-

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ory, however, does admit a Lorentz-invariant formulation [8,9]. This theory and its truncation to $\mathcal{N}=0$ features pairs of fields of opposite helicity. In [17] it was proposed to lift the corresponding Lagrangians to SFT.

In the present Letter the twistor description of both the purely bosonic and the $\mathcal{N}=4$ supersymmetric ASDYM models [8,9] is raised to the SFT level. In contrast to previous proposals [1–7], we allow the string to vibrate only in part of the supertwistor space. The remaining coordinates of this space are not promoted to word-sheet fields but kept as non-dynamical string field parameters. Concretely, we propose a cubic action containing an integration over the supertwistor space $\mathbb{C}P^{3|4}$ and show that its hCS-type equations of motion are equivalent to the covariant string field equations introduced in [17]. This model may be regarded as a specialization of Witten's supertwistor SFT and may even be equivalent to it. In any case, it is directly related with $\mathcal{N}=4$ super-ASDYM theory in four dimensions. We also consider its proper truncation to the bosonic sector, which yields a twistor SFT related to non-supersymmetric ASDYM theory.

2. Covariant string field theory for open $N=2$ strings

Open $N=2$ strings

From the worldsheet point of view critical open $N=2$ strings in a flat four-dimensional space–time of signature $(--++)$ or $(++++)$ are nothing but $N=2$ supergravity on a two-dimensional (pseudo) Riemannian surface with boundaries, coupled to two chiral $N=2$ matter multiplets (X,ψ). The latter's components are complex scalars (the four embedding coordinates) and Dirac spinors (their four NSR partners) in two dimensions. In the ghost-free formulation of the $N=2$ string one employs the extension of the $c=6$, $N=2$ superconformal algebra to the "small" $N=4$ superconformal algebra[2]

$$T=\partial_z X^{\alpha\dot\beta}\partial_z X_{\alpha\dot\beta}+\psi^{\dot\alpha\ddot\beta}\partial_z\psi_{\dot\alpha\ddot\beta},$$

$$G_\alpha{}^{\ddot\beta}=\psi^{\dot\gamma\ddot\beta}\partial_z X_{\alpha\dot\gamma},\qquad J^{\ddot\alpha\ddot\beta}=\psi^{\dot\gamma\ddot\alpha}\psi_{\dot\gamma}{}^{\ddot\beta},\tag{1}$$

where $\alpha,\beta=1,2$ and $\dot\alpha,\dot\beta=\dot 1,\dot 2$ are space–time spinor indices and $\ddot\alpha,\ddot\beta=\ddot 1,\ddot 2$ denote the world-sheet internal indices associated with the group $SU(1,1)''$ (Kleinian space $\mathbb{R}^{2,2}$) or $SU(2)''$ (Euclidean space $\mathbb{R}^{4,0}$) of R symmetries. For the reality structures imposed on target space coordinates and superconformal algebra generators see [11,12,15,18]. After twisting this algebra,

$$\bar D_\alpha:=G_\alpha{}^{\ddot 1}\tag{2}$$

become two fermionic spin-one operators which subsequently serve as BRST-like currents since they are nilpotent [11,12],

$$(\bar D_1)^2=0=(\bar D_2)^2\quad\text{and}\quad\{\bar D_1,\bar D_2\}=0.\tag{3}$$

Furthermore, $\psi^{\dot\alpha\ddot 1}$ is now conformal spin zero while $\psi^{\dot\alpha\ddot 2}$ is conformal spin one.

Covariant string field theory

Following Berkovits and Siegel [17], we introduce two Lie-algebra valued fermionic string fields $A_\alpha[X,\psi]$ and three Lie-algebra valued bosonic string fields $G^{\alpha\beta}[X,\psi]$ (symmetric in α and β). Although we suppress it in our notation, string fields are always multiplied using Witten's star product (midpoint gluing prescription) [19].[3] The index structure reveals that the fields $G_{\alpha\beta}$ parametrize the self-dual[4] tensor $G_{\alpha\dot\alpha,\beta\dot\beta}=\epsilon_{\dot\alpha\dot\beta}G_{\alpha\beta}$ on the target space.

The Lorentz invariant string field theory action [17] reads

$$S_{\mathrm{BS}}=\big\langle\mathrm{tr}\big(G^{\alpha\beta}\,F_{\alpha\beta}\big)\big\rangle,\tag{4}$$

where $\langle\cdots\rangle$ means integration over all modes of X and ψ, the trace "tr" is taken over the Lie algebra indices and

$$F_{\alpha\beta}:=\bar D_\alpha A_\beta+\bar D_\beta A_\alpha+\{A_\alpha,A_\beta\}.\tag{5}$$

[2] We raise and lower indices with $\epsilon^{12}=-\epsilon^{21}=-1$, $\epsilon_{12}=-\epsilon_{21}=1$, and similarly for $\epsilon^{\dot\alpha\dot\beta}$, $\epsilon_{\dot\alpha\dot\beta}$ and $\epsilon^{\ddot\alpha\ddot\beta}$, $\epsilon_{\ddot\alpha\ddot\beta}$.

[3] The star product was concretized in oscillator language for bosons in [20] and for twisted fermions in [21].

[4] Self-duality can always be interchanged with anti-self-duality by flipping the orientation of the four-dimensional target space. For the choice of the orientation made in [1,22] these $G_{\alpha\beta}$ parametrize a self-dual tensor.

Note that the action of $\bar{D}_\alpha$ on any string field B is defined in conformal field theory language as taking the contour integral [11]

$$(\bar{D}_\alpha B)(z) = \oint_z \frac{\mathrm{d}w}{2\pi\mathrm{i}} \bar{D}_\alpha(w) B(z). \tag{6}$$

The covariant string field equations of motion following from the action (4) read

$$F_{\alpha\beta} = 0 \quad \text{and} \quad \bar{D}_\alpha G^{\alpha\beta} + \left[A_\alpha, G^{\alpha\beta}\right] = 0. \tag{7}$$

For a supersymmetric generalization of the action (4) Berkovits and Siegel [17] introduce a multiplet of string fields

$$\left(A_\alpha, \tilde{\chi}_i, \phi_{ij}, \chi^{\alpha i}, G^{\alpha\beta}\right) \qquad \text{with } i, j = 1, 2, 3, 4 \tag{8}$$

imitating the $\mathcal{N} = 4$ ASDYM multiplet [8]. Here, A_α and $\chi^{\alpha i}$ are fermionic while $\tilde{\chi}_i$, ϕ_{ij} and $G^{\alpha\beta}$ are bosonic. Ref. [17] proposes the following action for this super SFT:

$$\widehat{S}_{\mathrm{BS}} = \left\langle \mathrm{tr}\left(G^{\alpha\beta} F_{\alpha\beta} + 2\chi^{\alpha i} \nabla_\alpha \tilde{\chi}_i + \frac{1}{8}\epsilon^{ijkl} \phi_{ij} \nabla_\alpha \nabla^\alpha \phi_{kl} + \frac{1}{2}\epsilon^{ijkl} \phi_{ij} \tilde{\chi}_k \tilde{\chi}_l \right)\right\rangle \tag{9}$$

with

$$\nabla_\alpha B := \bar{D}_\alpha B + A_\alpha B - (-1)^{|B|} B A_\alpha, \tag{10}$$

where the Grassmann parity $|B|$ equals 0 or 1 for bosonic or fermionic fields B, respectively. Due to the large number of string fields this model seems unattractive. However, as we shall see in the coming section, all these fields appear as components of *one* string field living in a twistor extended target space.

3. Cubic string field theory for open $N = 2$ strings

Supertwistor space notation

In Appendix A we describe the supertwistor space $\mathcal{P}_\epsilon^{3|4}$ of the space $(\mathbb{R}^4, g_\epsilon)$ with the metric $g_\epsilon = \mathrm{diag}(-\epsilon, -\epsilon, 1, 1)$ and $\epsilon = \pm 1$. It is fibered over the real two-dimensional space Σ_ϵ with $\Sigma_{-1} = \mathbb{C}P^1$ and $\Sigma_{+1} = H^2$ covered by two patches $U_\pm^\epsilon$. The space $\mathcal{P}_\epsilon^{3|4}$ is parametrized by four even complex coordinates $(x^{\alpha\dot{\alpha}}) \in \mathbb{C}^4$ subject to the reality conditions $x^{2\dot{2}} = \bar{x}^{1\dot{1}}$ and $x^{2\dot{1}} = \epsilon\bar{x}^{1\dot{2}}$, complex coordinates $\zeta_\pm \in U_\pm^\epsilon$ and odd (Grassmann) coordinates $\theta_\pm^i$, $i = 1, \ldots, 4$. The space $\mathcal{P}_\epsilon^{3|4}$ is a Calabi–Yau supermanifold [1]. From now on we shall work on the patch U_+^ϵ of Σ_ϵ, and for notational simplicity we shall omit the subscript "+" in $\zeta_+ \in U_+^\epsilon$, θ_+^i etc. For further use we introduce

$$(\zeta_\alpha) = \begin{pmatrix} 1 \\ \zeta \end{pmatrix}, \qquad (\zeta^\alpha) = \begin{pmatrix} -\zeta \\ 1 \end{pmatrix},$$
$$\left(\hat{\zeta}^\alpha\right) = \begin{pmatrix} \epsilon \\ -\bar{\zeta} \end{pmatrix} \quad \text{and} \quad \nu = (1 - \epsilon\zeta\bar{\zeta})^{-1}. \tag{11}$$

BRST operator

Let us introduce the operator

$$\bar{D} := \zeta^\alpha \bar{D}_\alpha = \psi^{\dot{\beta}\dot{1}} \partial_z \left(\zeta^\alpha X_{\alpha\dot{\beta}}\right) \tag{12}$$

taking values in the holomorphic line bundle $\mathcal{O}(1)$. We notice that the operators $\zeta^\alpha \partial_z X_{\alpha\dot{\beta}}$ act as derivatives on string fields. Their zero mode parts, $\zeta^\alpha \frac{\partial}{\partial x^{\alpha\dot{\beta}}}$, form two type $(0, 1)$ vector fields on the bosonic twistor space $\mathcal{P}_\epsilon^3$ fibered over Σ_ϵ (see Appendix A for more details). Recall that $\mathcal{P}_\epsilon^3$ being an open subset of $\mathbb{C}P^3$ is the twistor space of $(\mathbb{R}^4, g_\epsilon)$. In order to obtain a general type $(0, 1)$ vector field along the twistor space one should therefore extend the operator (12) by adding the type $(0, 1)$ derivative along Σ_ϵ,

$$\bar{\partial} := \mathrm{d}\bar{\zeta} \frac{\partial}{\partial \bar{\zeta}}. \tag{13}$$

Assuming that string fields now depend on the extra variable $\zeta \in \Sigma_\epsilon$, we define the operator

$$Q := \bar{D} + \bar{\partial} \tag{14}$$

acting on string fields via (6) for the $\bar{D}$ part and by ordinary differentiation with respect to $\bar{\zeta}$. It is easy to see that $Q^2 = 0$ due to (3) and the facts that $\bar{D}$ does not depend on $\bar{\zeta}$ and that $\{\mathrm{d}\bar{\zeta}, \psi^{\dot{\alpha}\ddot{\beta}}\} = 0$ (cf. [23]). We take this nilpotent operator as the BRST operator of our SFT extended to $\Sigma_\epsilon^{1|4} \hookrightarrow \mathcal{P}_\epsilon^{3|4}$.

String fields

We now consider a fermionic (odd) string field $\mathcal{A}[X, \psi, \theta^i, \zeta, \bar{\zeta}]$ depending not only on $X(\sigma)$ and $\psi(\sigma)$ but also on θ^i and on the parameter $\zeta \in \Sigma_\epsilon$.

It is important to realize that θ^i and ζ do not depend on σ here but may be considered as zero modes of world-sheet fields. Since the operator Q has the split form (14) it is natural to assume the same splitting of the string field $\mathcal{A}$,

$$\mathcal{A} = \mathcal{A}_{\bar{D}}\big[X,\psi,\theta^i,\zeta,\bar{\zeta}\big] + \mathcal{A}_{\bar{\partial}}\big[X,\psi,\theta^i,\zeta,\bar{\zeta}\big] \quad \text{with } \mathcal{A}_{\bar{\partial}} = \mathcal{A}_{\bar{\zeta}}\,\mathrm{d}\bar{\zeta}, \tag{15}$$

where $\mathcal{A}_{\bar{D}}$ gauges $\bar{D}$ and $\mathcal{A}_{\bar{\partial}}$ gauges $\bar{\partial}$. Note also that $\bar{D}$ takes values in $\mathcal{O}(1)$ and $\bar{\partial}$ in $\mathcal{O}(0)$; therefore, $\mathcal{A}_{\bar{D}}$ and $\mathcal{A}_{\bar{\partial}}$ are $\mathcal{O}(1)$ and $\mathcal{O}(0)$ valued, respectively. Since $\mathrm{d}\bar{\zeta}$ is the basis section of the bundle $\bar{\mathcal{O}}(-2)$ complex conjugate to $\mathcal{O}(2)$ and anticommutes with spinors $\psi^{\dot{\alpha}\ddot{\beta}}$ we reason that $\mathcal{A}_{\bar{\zeta}}$ is bosonic (even) and takes values in $\bar{\mathcal{O}}(-2)$. It is also assumed that a term $\mathcal{A}_\zeta\,\mathrm{d}\zeta$ is absent in the splitting (15), i.e., $\mathcal{A}$ is a string field of the $(0,1)$-form type (cf. [1] for the B-model argument). Note that by definition the string field $\mathcal{A}_{\bar{D}}$ does not contain $\mathrm{d}\bar{\zeta}$ and is fermionic (odd).

Cubic action

Having $\mathrm{d}\zeta$ and $\mathrm{d}\theta^i$ we introduce the action

$$S = \int \mathrm{d}\zeta \int \mathrm{d}\theta^1\,\mathrm{d}\theta^2\,\mathrm{d}\theta^3\,\mathrm{d}\theta^4 \left\langle \mathrm{tr}\left(\mathcal{A}Q\mathcal{A} + \frac{2}{3}\mathcal{A}^3\right)\right\rangle, \tag{16}$$

where $\langle\cdots\rangle$ is the same integration over (X,ψ) modes as in (4). Note that $\int \mathrm{d}\zeta$ acts as integration over Σ_ϵ for terms containing $\mathrm{d}\bar{\zeta}$ and as a contour integral around $\zeta=0$ for other terms. The Lagrangian $\mathcal{L}$ in (16) can be split into two parts,

$$\mathcal{L} = \mathrm{tr}\left(\mathcal{A}Q\mathcal{A} + \frac{2}{3}\mathcal{A}^3\right) = \mathcal{L}_1 + \mathcal{L}_2, \quad \text{with} \tag{17}$$

$$\mathcal{L}_1 = \mathrm{tr}\big(\mathcal{A}_{\bar{D}}\bar{\partial}\mathcal{A}_{\bar{D}} + 2\mathcal{A}_{\bar{D}}\bar{D}\mathcal{A}_{\bar{\partial}} + 2\mathcal{A}_{\bar{\partial}}\mathcal{A}_{\bar{D}}^2\big), \tag{18}$$

$$\mathcal{L}_2 = \mathrm{tr}\left(\mathcal{A}_{\bar{D}}\bar{D}\mathcal{A}_{\bar{D}} + \frac{2}{3}\mathcal{A}_{\bar{D}}^3\right), \tag{19}$$

where we used the cyclicity under the trace and omitted total derivatives.

It is important to note that $\mathcal{L}_1$ takes values in $\mathcal{O}(2)$, which is compensated by the holomorphic measure $\mathrm{d}\zeta\,\mathrm{d}\theta^1\,\mathrm{d}\theta^2\,\mathrm{d}\theta^3\,\mathrm{d}\theta^4$ being $\mathcal{O}(-2)$ valued.[5] At the same time, $\mathcal{L}_2$ takes values in $\mathcal{O}(3)$ which causes it to drop out of the action by virtue of Cauchy's theorem applied to the ζ contour integral. Thus, the action (16) can be rewritten as

$$S = \int \mathrm{d}\zeta \int \mathrm{d}\theta^1\,\mathrm{d}\theta^2\,\mathrm{d}\theta^3\,\mathrm{d}\theta^4 \times \big\langle \mathrm{tr}\big(\mathcal{A}_{\bar{D}}\bar{\partial}\mathcal{A}_{\bar{D}} + 2\mathcal{A}_{\bar{D}}\bar{D}\mathcal{A}_{\bar{\partial}} + 2\mathcal{A}_{\bar{\partial}}\mathcal{A}_{\bar{D}}^2\big)\big\rangle. \tag{20}$$

Moreover, both forms (16) and (20) of the action lead to the same Chern–Simons-type equation of motion,

$$Q\mathcal{A} + \mathcal{A}^2 = 0 \tag{21}$$

which decomposes into

$$\bar{D}\mathcal{A}_{\bar{\partial}} + \bar{\partial}\mathcal{A}_{\bar{D}} + \{\mathcal{A}_{\bar{D}},\mathcal{A}_{\bar{\partial}}\} = 0 \quad \text{and} \tag{22}$$

$$\bar{D}\mathcal{A}_{\bar{D}} + \mathcal{A}_{\bar{D}}^2 = 0. \tag{23}$$

Component analysis

Recall that $\mathcal{A}_{\bar{D}}$ and $\mathcal{A}_{\bar{\zeta}}$ take values in the bundles $\mathcal{O}(1)$ and $\bar{\mathcal{O}}(-2)$, respectively. Together with the fact that the θ^i are nilpotent and $\mathcal{O}(1)$ valued, this determines the dependence of $\mathcal{A}_{\bar{D}}$ and $\mathcal{A}_{\bar{\partial}} = \mathcal{A}_{\bar{\zeta}}\mathrm{d}\bar{\zeta}$ on θ^i, ζ and $\bar{\zeta}$. Namely, this dependence has the form (cf. [22])

$$\begin{aligned}
\mathcal{A}_{\bar{D}} &= \zeta^\alpha A_\alpha + \theta^i\tilde{\chi}_i + \frac{\nu}{2!}\theta^{ij}\hat{\zeta}^\alpha\phi_{\alpha ij} \\
&\quad + \frac{\nu^2}{3!}\theta^{ijk}\hat{\zeta}^\alpha\hat{\zeta}^\beta\chi_{\alpha\beta ijk} \\
&\quad + \frac{\nu^3}{4!}\theta^{ijkl}\hat{\zeta}^\alpha\hat{\zeta}^\beta\hat{\zeta}^\gamma G_{\alpha\beta\gamma ijkl}, \\
\mathcal{A}_{\bar{\zeta}} &= \frac{\nu^2}{2!}\theta^{ij}\phi_{ij} + \frac{\nu^3}{3!}\theta^{ijk}\hat{\zeta}^\alpha\chi_{\alpha ijk} \\
&\quad + \frac{\nu^4}{4!}\theta^{ijkl}\hat{\zeta}^\alpha\hat{\zeta}^\beta G_{\alpha\beta ijkl},
\end{aligned} \tag{24}$$

where ζ^α, $\hat{\zeta}^\alpha$ and ν are given in (11) and $\theta^{i_1 i_2\cdots i_k} := \theta^{i_1}\theta^{i_2}\cdots\theta^{i_k}$. The expansion (24) is defined up to a gauge transformation generated by a group-valued function which may depend on ζ and $\bar{\zeta}$. All string fields appearing in the expansion (24) depend only on $X(\sigma)$ and $\psi(\sigma)$. From the properties of $\mathcal{A}_{\bar{D}}$, $\mathcal{A}_{\bar{\partial}}$ and θ^i it follows that the fields with an odd number of spinor indices are fermionic (odd) while those with an even number of spinor indices are bosonic (even). Moreover, due to the symmetry of the $\hat{\zeta}^\alpha$ products and

[5] The choice of four Grassmann coordinates θ^i is dictated by the Calabi–Yau condition: the contribution of the coordinates (X,ζ,θ) to the first Chern number is $(2,2,-4)$, respectively.

the skew symmetry of the θ^i products all component fields are automatically symmetric in their spinor indices and antisymmetric in their Latin indices.

Substituting (24) into (22), we obtain the equations[6]

$$\phi_{\alpha ij} = -\nabla_\alpha \phi_{ij} \quad \text{and} \quad \chi_{\alpha\beta ijk} = \frac{1}{2}\nabla_{(\alpha}\chi_{\beta)ijk} \quad \text{and}$$
$$G_{\alpha\beta\gamma ijkl} = -\frac{1}{3}\nabla_{(\alpha}G_{\beta)\gamma ijkl} \tag{25}$$

showing that $(\phi_{\alpha ij}, \chi_{\alpha\beta ijk}, G_{\alpha\beta\gamma ijkl})$ is a set of auxiliary fields. The other nontrivial equations following from (22) and (23) after substituting (24) read[7]

$$F_{\alpha\beta} \equiv \bar{D}_\alpha A_\beta + \bar{D}_\beta A_\alpha + \{A_\alpha, A_\beta\} = 0,$$
$$\nabla_\alpha \tilde{\chi}_i = 0,$$
$$\epsilon^{\alpha\beta}\nabla_\alpha \chi^i_\beta + 2\epsilon\big[\phi^{ij}, \tilde{\chi}_j\big] = 0,$$
$$\nabla_\alpha \nabla^\alpha \phi_{ij} + 2\epsilon\big[\tilde{\chi}_i, \tilde{\chi}_j\big] = 0,$$
$$\epsilon^{\alpha\beta}\nabla_\alpha G_{\beta\gamma} + 2\epsilon\big[\tilde{\chi}_i, \chi^i_\gamma\big] + \epsilon\big[\nabla_\gamma \phi_{ij}, \phi^{ij}\big] = 0, \tag{26}$$

where we introduced

$$\phi^{ij} := \frac{1}{2!}\epsilon^{ijkl}\phi_{kl}, \qquad \chi^i_\alpha := \frac{1}{3!}\epsilon^{ijkl}\chi_{\alpha jkl} \quad \text{and}$$
$$G_{\alpha\beta} := \frac{1}{4!}\epsilon^{ijkl}G_{\alpha\beta ijkl}. \tag{27}$$

Up to constant field rescalings

$$G_{\alpha\beta} \to -G_{\alpha\beta}, \qquad \tilde{\chi}_i \to \frac{1}{2}\tilde{\chi}_i,$$
$$\phi_{ij} \to \frac{1}{2}\phi_{ij} \quad \text{and} \quad \chi^i_\alpha \to \chi^i_\alpha \tag{28}$$

Eq. (26) for $\epsilon = 1$ coincide with the equations of motion following from the action (9) proposed by Berkovits and Siegel [17]. In the zero mode sector they reduce to the anti-self-dual $\mathcal{N} = 4$ super-Yang–Mills equations of motion. Hence, we have established that maximally supersymmetric ASDYM theory can be obtained from the standard cubic SFT for a single string field $\mathcal{A}$ after extending the setting to the supertwistor space.

[6] Round brackets denote symmetrization with respect to enclosed indices.

[7] Recall that $\epsilon = 1$ for signature $(--++)$ and $\epsilon = -1$ for signature $(++++)$.

4. Bosonic truncation of open string field theory

In order to make contact with non-supersymmetric ASDYM theory, we subject our string field $\mathcal{A}$ from (15) and (24) to the truncation conditions

$$\int d\theta^1\, d\theta^2\, d\theta^3\, d\theta^4 \begin{Bmatrix} \theta^i \\ \theta^{ij} \\ \theta^{ijk} \end{Bmatrix} \mathcal{A} = 0. \tag{29}$$

These conditions imply that $\mathcal{A}$ depends only on the combination

$$\theta := \theta^1\theta^2\theta^3\theta^4 \quad \text{i.e.,} \quad \mathcal{A} = \mathcal{A}[X, \psi, \theta, \zeta, \bar{\zeta}]. \tag{30}$$

Obviously, the even nilpotent variable θ belongs to the bundle $\mathcal{O}(4)$ and the integration measure in (29) to $\mathcal{O}(-4)$.

The properties of the truncated string field

$$\mathcal{A} = \mathcal{A}_{\bar{D}}[X, \psi, \theta, \zeta, \bar{\zeta}] + \mathcal{A}_{\bar{\partial}}[X, \psi, \theta, \zeta, \bar{\zeta}] \tag{31}$$

are the same as before the truncation, except for the restricted dependence on the Grassmann variables. The operators Q, $\bar{D}$ and $\bar{\partial}$, the actions (16) and (20), the Lagrangians (17)–(19) and the equations of motion (21)–(23) are unchanged. However, the expansion (24) now simplifies to

$$\mathcal{A}_{\bar{D}} = \zeta^\alpha A_\alpha[X, \psi] + \theta\nu^3 \hat{\zeta}^\alpha \hat{\zeta}^\beta \hat{\zeta}^\gamma G_{\alpha\beta\gamma}[X, \psi],$$
$$\mathcal{A}_{\bar{\partial}} = \theta\nu^4 \hat{\zeta}^\alpha \hat{\zeta}^\beta G_{\alpha\beta}[X, \psi]\, d\bar{\zeta}, \tag{32}$$

where (see (27))

$$G_{\alpha\beta} = \frac{1}{4!}\epsilon^{ijkl}G_{\alpha\beta ijkl} \quad \text{and}$$
$$G_{\alpha\beta\gamma} := \frac{1}{4!}\epsilon^{ijkl}G_{\alpha\beta\gamma ijkl}. \tag{33}$$

From the properties of $\mathcal{A}_{\bar{D}}$ and $\mathcal{A}_{\bar{\partial}}$ it follows that the string fields A_α and $G_{\alpha\beta\gamma}$ are odd and the $G_{\alpha\beta}$ are even.

Substituting the expansion (32) into the equations of motion (22) and (23), we recover for A_α and $G_{\alpha\beta}$ the bosonic string field equations

$$F_{\alpha\beta} = 0 \quad \text{and} \quad \bar{D}_\alpha G^{\alpha\beta} + \big[A_\alpha, G^{\alpha\beta}\big] = 0 \tag{34}$$

displayed already in (7) and for $G_{\alpha\beta\gamma}$ the dependence

$$G_{\alpha\beta\gamma} = -\frac{1}{3}\nabla_{(\alpha}G_{\beta)\gamma} \tag{35}$$

as expected. The same result occurs when putting to zero in (26) the string fields $\tilde{\chi}_i$, χ^i_α and ϕ_{ij} as the truncation (29) demands. All other equations following from (22) and (23) are satisfied automatically, due to (34) and the Bianchi identities.

Hence, we have proven that the cubic supertwistor SFT defined by the action (16) together with the geometric truncation conditions (29) is equivalent to the Berkovits–Siegel SFT given by the action (4). Moreover, (4) and (9) derive from (16) simply by substituting there the expansion (32) or (24), respectively, and integrating over the Grassmann and twistor variables. All this is similar to the field theory case [1,22] where in the supertwistor reformulation of $\mathcal{N}=4$ ASDYM theory as hCS theory the dependence of all fields on the twistor variable ζ is fixed (up to a gauge transformation) by the topology of the supertwistor space and one can integrate over it, descending from six to four real dimensions.

5. Conclusions

The basic result of this Letter can be summarized in the equations

$$\begin{aligned} S &= \int \mathrm{d}\zeta \int \mathrm{d}\theta^1\,\mathrm{d}\theta^2\,\mathrm{d}\theta^3\,\mathrm{d}\theta^4 \Big\langle \mathrm{tr}\Big(\mathcal{A}Q\mathcal{A}+\frac{2}{3}\mathcal{A}^3\Big)\Big\rangle \\ &= \int_{\Sigma_\epsilon} \mathrm{d}\zeta\wedge \mathrm{d}\bar\zeta \int \mathrm{d}\theta^1\,\mathrm{d}\theta^2\,\mathrm{d}\theta^3\,\mathrm{d}\theta^4 \\ &\quad \times \big\langle \mathrm{tr}\big(-\mathcal{A}_{\bar D}\partial_{\bar\zeta}\mathcal{A}_{\bar D}+2\mathcal{A}_{\bar D}\bar D\mathcal{A}_{\bar\zeta}+2\mathcal{A}_{\bar D}^2\mathcal{A}_{\bar\zeta}\big)\big\rangle \\ &= c\Big\langle \mathrm{tr}\Big(G^{\alpha\beta}F_{\alpha\beta}+2\epsilon\chi^{\alpha i}\nabla_\alpha\tilde\chi_i \\ &\qquad + \frac{1}{4}\epsilon\phi^{ij}\nabla_\alpha\nabla^\alpha\phi_{ij}+\phi^{ij}\tilde\chi_k\tilde\chi_l\Big)\Big\rangle, \end{aligned} \tag{36}$$

where c is an inessential numerical constant. The first step demands a split $\mathcal{A}=\mathcal{A}_{\bar D}+\mathcal{A}_{\bar\zeta}\,\mathrm{d}\bar\zeta$ of the basic (supertwistor) string field. The second step requires integrating over $\Sigma_\epsilon^{1|4}$ and rescaling the field as in (28). Truncating $\mathcal{A}$ to its lowest and highest Grassmann components (the $O(\theta^0)$ and $O(\theta^4)$ parts) projects the above action to $\langle\mathrm{tr}(G^{\alpha\beta}F_{\alpha\beta})\rangle$, which governs bosonic $N=2$ open SFT. Finally, reducing to the string zero modes one recovers the twistor description of $\mathcal{N}=4$ and $\mathcal{N}=0$ ASDYM on the field theory level.

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Appendix A. Supertwistor space

The twistor space of $\mathbb{R}^{4,0}$

Let us consider the Riemann sphere $\mathbb{C}P^1\cong S^2$ with homogeneous coordinates $(\mu_\alpha)\in\mathbb{C}^2$. It can be covered by two patches,

$$U_+=\{(\mu_1,\mu_2):\ \mu_1\neq 0\}\quad\text{and}\quad U_-=\{(\mu_1,\mu_2):\ \mu_2\neq 0\} \tag{A.1}$$

with coordinates $\zeta_+:=\mu_2/\mu_1$ on U_+ and $\zeta_-:=\mu_1/\mu_2$ on U_-. On the intersection $U_+\cap U_-$ we have $\zeta_+=\zeta_-^{-1}$.

The holomorphic line bundle $\mathcal{O}(n)$ over $\mathbb{C}P^1$ is defined as a two-dimensional complex manifold with the holomorphic projection

$$\pi:\mathcal{O}(n)\to\mathbb{C}P^1 \tag{A.2}$$

such that it is covered by two patches $\tilde U_+$ and $\tilde U_-$ with coordinates (w_+,ζ_+) on $\tilde U_+$ and (w_-,ζ_-) on $\tilde U_-$ related by $w_+=\zeta_+^n w_-$ and $\zeta_+=\zeta_-^{-1}$ on $\tilde U_+\cap\tilde U_-$. A global holomorphic section of $\mathcal{O}(n)$ exists only for $n\geqslant 0$. Over $U_\pm\subset\mathbb{C}P^1$ it is represented by polynomials $p_\pm^{(n)}$ in $\zeta_\pm$ of degree n with $p_+^{(n)}=\zeta_+^n p_-^{(n)}$ on $U_+\cap U_-$.

Recall that the Riemann sphere

$$\mathbb{C}P^1\cong\frac{SO(4)}{U(2)} \tag{A.3}$$

parametrizes the space of all translational invariant (constant) complex structures on the Euclidean space $\mathbb{R}^{4,0}$, and the space $\mathcal{P}_{\mathrm{E}}^3:=\mathbb{R}^4\times\mathbb{C}P^1$ is called the twistor space of $\mathbb{R}^{4,0}$ [24]. As a complex manifold $\mathcal{P}_{\mathrm{E}}^3$ is a rank 2 holomorphic vector bundle $\mathcal{O}(1)\oplus\mathcal{O}(1)$ over $\mathbb{C}P^1$:

$$\mathcal{P}_{\mathrm{E}}^3=\mathcal{O}(1)\oplus\mathcal{O}(1). \tag{A.4}$$

For more details and references see, e.g., [22,24,25].

The twistor space of $\mathbb{R}^{2,2}$

In the Kleinian space $\mathbb{R}^{2,2}$ of signature $(--++)$ constant complex structures are parametrized by the two-sheeted hyperboloid

$$H^2 = H_+ \cup H_- \cong \frac{SO(2,2)}{U(1,1)}, \tag{A.5}$$

where

$$H_+ = \{\zeta_+ \in U_+ \colon |\zeta_+| < 1\} \cong \frac{SU(1,1)}{U(1)} \quad \text{and}$$
$$H_- = \{\zeta_- \in U_- \colon |\zeta_-| < 1\} \cong \frac{SU(1,1)}{U(1)}. \tag{A.6}$$

In fact, under the action of the group $SU(1,1)$ the Riemann sphere is decomposed into three orbits, $\mathbb{C}P^1 = H_+ \cup S^1 \cup H_-$, where the boundary of both H_+ and H_- is given by

$$S^1 = \{\zeta \in \mathbb{C}P^1 \colon |\zeta| = 1\} \cong \frac{SU(1,1)}{B_+} \quad \text{with}$$
$$B_+ = \begin{pmatrix} a_1 + \mathrm{i}a_2 & a_3 - \mathrm{i}a_2 \\ a_3 + \mathrm{i}a_2 & a_1 - \mathrm{i}a_2 \end{pmatrix} \tag{A.7}$$

with $a_{1,2,3} \in \mathbb{R}$ and $a_1^2 - a_2^2 = 1$.

The twistor space of $\mathbb{R}^{2,2}$ is the space $\mathcal{P}_K^3 := \mathbb{R}^4 \times H^2$ which as a complex manifold coincides with the restriction of the rank 2 holomorphic vector bundle (A.4) to the bundle over $H^2 \subset \mathbb{C}P^1$. Equivalently, it can be described as a space $\mathcal{P}_K^3 = \mathcal{P}_{\mathrm{E}}^3 \setminus \mathcal{T}^3$, where $\mathcal{T}^3$ is a real three-dimensional subspace of $\mathcal{P}_{\mathrm{E}}^3$ stable under an anti-linear involution (real structure) which can be defined on $\mathcal{P}_{\mathrm{E}}^3$. For more details see, e.g., [26].

Vector fields of type $(0,1)$

For considering both signatures together, we denote by Σ_ϵ the space of complex structures on $(\mathbb{R}^4, g_\epsilon)$ with the metric $g_\epsilon = \mathrm{diag}(-\epsilon, -\epsilon, 1, 1)$ and $\epsilon = \pm 1$, so that

$$\Sigma_{-1} = \mathbb{C}P^1 \quad \text{and} \quad \Sigma_{+1} = H^2. \tag{A.8}$$

Therefore, Σ_ϵ is covered by two patches $U_\pm^\epsilon$ with $U_\pm^{-1} = U_\pm$ and $U_\pm^{+1} = H_\pm$. Analogously, we denote by $\mathcal{P}_\epsilon^3$ the twistor space of $(\mathbb{R}^4, g_\epsilon)$ with $\mathcal{P}_{-1}^3 = \mathcal{P}_{\mathrm{E}}^3$ and $\mathcal{P}_{+1}^3 = \mathcal{P}_K^3$. The complex manifold $\mathcal{P}_\epsilon^3$ is covered by two patches $\mathcal{V}_\pm^\epsilon$ with complex coordinates $(w_\pm^{\dot\alpha}, \zeta_\pm)$ on $\mathcal{V}_\pm^\epsilon$. We introduce

$$(\zeta_\alpha^\pm) = \begin{pmatrix} 1 \\ \zeta_\pm \end{pmatrix}, \qquad (\zeta_\pm^\alpha) = \begin{pmatrix} -\zeta_\pm \\ 1 \end{pmatrix},$$
$$\nu_+ = (1 - \epsilon\zeta_+\bar\zeta_+)^{-1} \quad \text{and}$$
$$\nu_- = -\epsilon(1 - \epsilon\zeta_-\bar\zeta_-)^{-1} \tag{A.9}$$

for $\zeta_\pm \in U_\pm^\epsilon$. Note that in terms of $(\zeta_\alpha^\pm)$ or $(\zeta_\pm^\alpha)$ a section of the bundle $\mathcal{O}(n)$ over $U_\pm^\epsilon$ can be written as

$$p_\pm^{(n)} = p^{\alpha_1\cdots\alpha_n}\zeta_{\alpha_1}^\pm\cdots\zeta_{\alpha_n}^\pm = p_{\alpha_1\cdots\alpha_n}\zeta_\pm^{\alpha_1}\cdots\zeta_\pm^{\alpha_n}. \tag{A.10}$$

Recall that $(\mathbb{R}^4, g_\epsilon)$ can be parametrized by coordinates $(x^{\alpha\dot\alpha}) \in \mathbb{C}^4$ with the reality conditions $x^{2\dot2} = \bar x^{1\dot1}$ and $x^{2\dot1} = \epsilon\bar x^{1\dot2}$ [22]. On the twistor space $\mathcal{P}_\epsilon^3 \cong \mathbb{R}^4 \times \Sigma_\epsilon$ we have coordinates $(w_\pm^{\dot\alpha}, w_\pm^{\dot3}) = (x^{\alpha\dot\alpha}\zeta_\alpha^\pm, \zeta_\pm)$ or $(x^{\alpha\dot\alpha}, \zeta_\pm, \bar\zeta_\pm)$. The antiholomorphic vector fields $\partial/\partial\bar w_\pm^{\dot\alpha}$ and $\partial/\partial\bar w_\pm^{\dot3}$ can be rewritten in the coordinates $(x^{\alpha\dot\alpha}, \zeta_\pm, \bar\zeta_\pm)$ as

$$\frac{\partial}{\partial\bar w_\pm^{\dot1}} = \nu_\pm\zeta_\pm^\alpha\frac{\partial}{\partial x^{\alpha\dot2}}, \qquad \frac{\partial}{\partial\bar w_\pm^{\dot2}} = \epsilon\nu_\pm\zeta_\pm^\alpha\frac{\partial}{\partial x^{\alpha\dot1}} \quad \text{and}$$
$$\frac{\partial}{\partial\bar w_\pm^{\dot3}} = \frac{\partial}{\partial\bar\zeta_\pm} - \epsilon\, x^{1\dot\alpha}\zeta^\alpha\frac{\partial}{\partial x^{\alpha\dot\alpha}}. \tag{A.11}$$

The vector fields

$$\bar v_{\dot\alpha}^\pm = \zeta_\pm^\alpha\frac{\partial}{\partial x^{\alpha\dot\alpha}} \quad \text{and} \quad \bar v_{\dot3}^\pm = \frac{\partial}{\partial\bar\zeta_\pm} \tag{A.12}$$

can be taken as a basis of vector fields of type $(0,1)$ on $\mathcal{P}_\epsilon^3$.

Supertwistors

Let us now add four odd variables θ^i such that

$$\theta^i\theta^j + \theta^j\theta^i = 0 \quad \text{for } i, j = 1, 2, 3, 4, \tag{A.13}$$

and each θ^i takes its value in the line bundle $\mathcal{O}(1)$ over $\mathbb{C}P^1$ (cf. [1]). For describing them formally we introduce a Grassmann parity changing operator Π which, when acting on a vector bundle, flips the Grassmann parity of the fibre coordinates. Hence, we consider the bundle $\Pi\mathbb{C}^4 \otimes \mathcal{O}(1) \to \mathbb{C}P^1$ which is parametrized by complex variables $\zeta_\pm \subset U_\pm \subset \mathbb{C}P^1$ and fibre Grassmann coordinates $\theta_\pm^i$ such that $\theta_+^i = \zeta_+\theta_-^i$ on the intersection of the two patches covering the total space of this vector bundle.

With the Grassmann variables θ^i one can introduce the supertwistor space $\mathcal{P}_{\mathrm{E}}^{3|4}$ as a holomorphic vector bundle over $\mathbb{C}P^1$, namely,

$$\mathcal{P}_{\mathrm{E}}^{3|4} = \mathbb{C}^2 \otimes \mathcal{O}(1) \oplus \Pi\mathbb{C}^4 \otimes \mathcal{O}(1). \tag{A.14}$$

The supertwistor space $\mathcal{P}_K^{3|4}$ is defined as a restriction of the bundle $\mathcal{P}_{\mathrm{E}}^{3|4} \to \mathbb{C}P^1$ to the bundle over the two-sheeted hyperboloid $H^2 \subset \mathbb{C}P^1$.

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