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A note on an unusual type of polar decomposition

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Abstract

Motivated by applications in the theory of unitary congruence, we introduce the factorization of a square complex matrix A of the form $A = SU$, where S is complex symmetric and U is unitary. We call this factorization a symmetric–unitary polar decomposition or an SUPD. It is shown that an SUPD exists for every matrix A and is always nonunique. Even the symmetric factor S can be chosen in infinitely many ways. Nevertheless, we show that many properties of the conventional polar decomposition related to normal matrices have their counterparts for the SUPD, provided that normal matrices are replaced with conjugate–normal ones.

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1. Introduction

Let A be a square complex matrix. Recall (e.g., see [3, Section 7.3]) that a polar decomposition of A is its factorization

$$A = PU, \tag{1}$$

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where P is Hermitian positive semidefinite and U is unitary. The matrix P is always uniquely determined as the Hermitian positive semidefinite square root of AA^*, and U is uniquely determined by (1) for a nonsingular A:

$$U = P^{-1}A.$$

If A is singular, then A admits infinitely many representations of form (1). The polar decomposition can also be defined for rectangular matrices; however, in this paper, we restrict ourselves to the case of square matrices. More precisely, factorization (1) should be termed a left polar decomposition, because A can also be factored as

$$A = WQ, \tag{2}$$

where W is unitary and Q is Hermitian positive semidefinite. Again, Q is uniquely determined as the Hermitian positive semidefinite square root of A^*A, while W is uniquely determined when A is nonsingular. In fact, for a nonsingular A, the matrix W is identical to U (see the exercise preceding Theorem 7.3.6 in [3]). We call (2) a right polar decomposition of A.

Unitary and Hermitian matrices are special with respect to unitary similarities, which makes polar decompositions a useful tool in the theory of unitary similarity. However, if we are concerned with unitary congruences rather than similarities, then the choice of a Hermitian factor in (1) or (2) is no longer natural. Indeed, the property of being a Hermitian matrix is not preserved by unitary congruences. By contrast, both symmetry and unitarity are preserved.

In Section 3, we introduce a new type of polar decomposition, namely

$$A = SU \tag{3}$$

or

$$A = VR, \tag{4}$$

where U and V are unitary, while S and R are complex symmetric. Following the pattern given in [4, Section 6.4], we call (3) a left symmetric–unitary polar decomposition or, for brevity, a left SUPD. Accordingly, (4) is a right SUPD.

It is shown in Section 3 that an SUPD, both left and right, exists for every matrix A and is always nonunique. Even the symmetric factor can now be chosen in infinitely many ways. We also show how the factors of an SUPD transform when A undergoes a unitary congruence.

The properties of the conventional polar decomposition are most remarkable for a normal matrix A. In fact, the very property of being a normal matrix is equivalent to a number of commutativity relations connected with the polar decomposition. We summarize these relations in the following theorem (see [2, Conditions 37–39]).

Theorem 1. *Let* (1) *be a polar decomposition of the matrix A. Then, A is normal if and only if any of the following conditions is fulfilled*:

(a) $AP = PA$;
(b) $AU = UA$;
(c) $PU = UP$.

A square matrix A is said to be conjugate–normal if

$$AA^* = \overline{A^*A}. \tag{5}$$

Conjugate–normal matrices play the same role in the theory of unitary congruence as conventional normal matrices do with respect to unitary similarities. In Section 4, we state and prove a theorem that concerns SUPDs of a conjugate–normal matrix A. To a great extent, though not entirely, it resembles Theorem 1.

The auxiliary material needed for Sections 3 and 4 is presented in Section 2.

2. Preliminaries

The polar decomposition of A can easily be obtained from its singular value decomposition

$$A = X\Sigma Y^*. \tag{6}$$

For instance, this is the way in which the polar decomposition is introduced in [3, Section 7.3].

Recall that, in (6), X is a unitary matrix whose columns are eigenvectors of AA^*, the diagonal entries σ_i of Σ are the square roots of the corresponding eigenvalues, and Y is a unitary matrix whose columns are eigenvectors of A^*A. Now, rewrite (6) as

$$A = (X\Sigma X^*)(XY^*) \tag{7}$$

and observe that $U = XY^*$ is a unitary matrix, while $P = X\Sigma X^*$ is Hermitian positive semidefinite. Thus, (7) yields a left polar decomposition of A. Similarly, rewriting (6) as

$$A = (XY^*)(Y\Sigma Y^*), \tag{8}$$

we obtain the right polar decomposition (2). Note that

$$W = XY^* = U. \tag{9}$$

We stress again that, although X (i.e., an orthonormal basis of eigenvectors of AA^*) can be chosen in infinitely many ways, the matrix

$$P = X\Sigma X^* \tag{10}$$

being the Hermitian positive semidefinite square root of AA^*, is determined uniquely. In the next section, we will see that this is not the case with the symmetric factor in the SUPD.

There is an intimate relationship between conjugate–normal and ordinary normal matrices found in [1].

Theorem 2. *Every conjugate–normal matrix is unitarily congruent to a real normal matrix.*

The real normal matrix in Theorem 2 can be chosen as a block diagonal matrix with 1×1 and 2×2 blocks. For complex symmetric matrices, which are an especially nice subclass of conjugate–normal matrices, the congruent normal matrices are diagonal. This is nothing else than the classical Takagi theorem [3, Corollary 4.4.4].

Theorem 3. *A complex symmetric matrix A can be represented in the form*

$$A = U\Sigma U^{\mathrm{T}}, \tag{11}$$

where U is a unitary matrix and Σ is real diagonal. Decomposition (11) *becomes a singular value decomposition of A if Σ is chosen nonnegative.*

We say that matrices A and B concommute if

$$A\overline{B} = B\overline{A}.$$

It is easily verified that this concommutativity property is preserved if A and B undergo the same unitary congruence transformation. In other words, if

$$\widetilde{A} = F^{\mathrm{T}} A F, \quad \widetilde{B} = F^{\mathrm{T}} B F,$$

where F is unitary, then

$$\widetilde{A}\overline{\widetilde{B}} = \widetilde{B}\overline{\widetilde{A}}.$$

3. Symmetric–unitary polar decomposition

As in Section 2, we begin by considering the singular value decomposition (6) of A. Now, we rewrite (6) as

$$A = (X\Sigma X^{\mathrm{T}})(\overline{X}Y^*) \tag{12}$$

and observe that

$$S = X\Sigma X^{\mathrm{T}} \tag{13}$$

is a complex symmetric matrix, while

$$U = \overline{X}Y^* \tag{14}$$

is unitary. This simple consideration establishes the existence of a left SUPD for every matrix A. The existence of a right SUPD can be shown similarly. Rewrite (6) as

$$A = (XY^{\mathrm{T}})(\overline{Y}\Sigma Y^*) \tag{15}$$

and set

$$R = \overline{Y}\Sigma Y, \quad V = XY^{\mathrm{T}}. \tag{16}$$

Observe that

$$V = XY^{\mathrm{T}} = \overline{U} \tag{17}$$

(see (14)).

As noted in Section 2, different choices for the eigenvector matrix X yield the same matrix P in polar decomposition (1). This is different with the SUPD. For instance, replace X in (6) with

$$X_1 = XD,$$

where D is a diagonal matrix. Such a replacement amounts to a renormalization of the eigenvectors of AA^*. Now, formula (13) produces the matrix

$$S_1 = X_1\Sigma X_1^{\mathrm{T}} = X(\Sigma D^2)X^{\mathrm{T}},$$

which is different from S unless $D^2 = I$. We conclude that there are infinitely many left SUPDs for each matrix A. The same is true of right SUPDs.

We now show that each SUPD of A can be obtained through the construction described above for an appropriate eigenvector matrix X. Let

$$A = SU \tag{18}$$

be an arbitrary SUPD of A. Then

$$A^* = U^*\overline{S}$$

and

$$S\overline{S} = AA^*. \tag{19}$$

If

$$S = X\Lambda X^{\mathrm{T}}$$

is Takagi's factorization of S (Theorem 3), then, from (19), we obtain

$$X^* S\overline{S} X = (X^* S\overline{X})(X^{\mathrm{T}}\overline{S}X) = \Lambda\overline{\Lambda} = X^* AA^* X.$$

Since $\Lambda\overline{\Lambda}$ is diagonal, $X^*(AA^*)X$ is diagonal as well, which says that X is an eigenvector matrix for AA^*. The argument for the right SUPD is analogous.

Although different choices of X yield different matrices S, these symmetric matrices still have some important properties in common. Namely, each S is a solution to matrix equation (19), and each S has the same singular values equal to the singular values of A. In view of Takagi's theorem, the latter property means that any two matrices S (for the same A) are unitarily congruent.

Suppose that the matrix A in (1) undergoes the unitary similarity transformation

$$A \to \widetilde{A} = F^* AF, \tag{20}$$

where F is unitary. Then

$$\widetilde{A} = F^*(PU)F = (F^* PF)(F^* UF) = \widetilde{P}\widetilde{U}.$$

Thus, both factors in the polar decomposition are transformed by the same rule as in (20).

Now, instead of (20), consider the unitary congruence transformation

$$A \to \widetilde{A} = F^{\mathrm{T}} AF, \tag{21}$$

where F is again unitary. Let (3) be an SUPD of A. Then, we have

$$\widetilde{A} = F^{\mathrm{T}}(SU)F = (F^{\mathrm{T}} SF)(F^* UF).$$

Thus, the symmetric matrix

$$\widetilde{S} = F^{\mathrm{T}} SF \tag{22}$$

and the unitary matrix

$$\widetilde{U} = F^* UF \tag{23}$$

yield an SUPD of the transformed matrix $\widetilde{A}$. While (22) is a transformation of the same kind as (21), relation (23) is a unitary similarity transformation rather than a unitary congruence. In particular, this implies that, under unitary congruences, the unitary factor of an SUPD preserves its eigenvalues.

4. Conjugate–normal matrices

In this section, we prove the following theorem:

Theorem 4. *A square matrix A is conjugate–normal if and only if any of the following conditions is fulfilled*:

(a) *There exists an SUPD of A* (*see* (3)) *such that*

$$A\overline{S} = S\overline{A}. \tag{24}$$

(b) *There exists an SUPD of A such that*

$$U(A^* A) = (A^* A)U. \tag{25}$$

(c) *There exists an SUPD of A such that*

$$U(\overline{S}S) = (\overline{S}S)U. \tag{26}$$

Remark 1. For definiteness, Theorem 4 was formulated in terms of the left SUPD. A similar assertion can be stated for the right SUPD.

Proof. *Necessity.* Assume that A is a conjugate–normal matrix, and let F be a unitary matrix that transforms A into a real normal matrix $\widetilde{A}$ (see Theorem 2): $\widetilde{A} = F^{\mathrm{T}}AF$. Consider the left polar decomposition:

$$\widetilde{A} = \widetilde{P}\widetilde{U} \tag{27}$$

of $\widetilde{A}$. Since $\widetilde{A}$ is real, both factors in (27) may be chosen real. Since $\widetilde{A}$ is normal, we have by Theorem 1 the relations

$$\widetilde{A}\widetilde{P} = \widetilde{P}\widetilde{A} \tag{28}$$

and

$$\widetilde{U}\widetilde{P} = \widetilde{P}\widetilde{U},$$

which implies

$$\widetilde{U}\widetilde{P}^2 = \widetilde{P}^2\widetilde{U}. \tag{29}$$

Now, we reverse transformation (21) in order to return to the original matrix A. Setting

$$S = \overline{F}\widetilde{P}F^*, \quad U = F\widetilde{U}F^*, \tag{30}$$

we obtain an SUPD of A (see (27)):

$$A = SU.$$

Since

$$\overline{S}S = F\widetilde{P}^2F^*,$$

we deduce relation (26) from (29) and the second formula in (30). Next, we observe that, by (19) and (5)

$$\overline{S}S = \overline{S\overline{S}} = \overline{AA^*} = A^*A,$$

which says that (25) is the same relation as (26).

For the real matrices $\widetilde{A}$ and $\widetilde{P}$ in (28), commutativity and concommutativity are the same thing. However, concommutativity is preserved by unitary congruence transformations, which means that A and S must obey relation (24).

Sufficiency. We first assume that (a) is fulfilled. Let F be a unitary matrix that brings S to the real diagonal matrix Λ (see Theorem 3):

$$\Lambda = F^{\mathrm{T}}SF. \tag{31}$$

Without loss of generality, we can regard Λ as a block diagonal matrix of the form

$$\Lambda = \lambda_1 I_{k_1} \oplus \lambda_2 I_{k_2} \oplus \cdots \oplus \lambda_m I_{k_m}, \tag{32}$$

where

$$\lambda_1 > \lambda_2 > \cdots > \lambda_m \geqslant 0$$

and

$$k_1 + k_2 + \cdots + k_m = n$$

(the order of A). Set

$$\widetilde{A} = F^{\mathrm{T}} A F. \tag{33}$$

Then, the concommutativity relation

$$\widetilde{A}\Lambda = \Lambda\overline{\widetilde{A}}$$

implies the following. If we adopt for $\widetilde{A}$ the same partitioning as in (32), then all of the off-diagonal blocks $\widetilde{A}_{ij}$ are zero; thus

$$\widetilde{A} = \widetilde{A}_{11} \oplus \widetilde{A}_{22} \oplus \cdots \oplus \widetilde{A}_{mm}. \tag{34}$$

Setting

$$\widetilde{U} = F^{*} U F, \tag{35}$$

we have

$$\widetilde{A} = \Lambda\widetilde{U}. \tag{36}$$

If $\lambda_m > 0$ (i.e., if A is nonsingular), then (36), combined with (32) and (34), implies that $\widetilde{U}$ is a block diagonal matrix of the same type as (34) and

$$\widetilde{A}_{ii} = \lambda_i \widetilde{U}_{ii}, \quad i = 1, 2, \ldots, m. \tag{37}$$

Note that a scalar multiple of a unitary matrix is both a normal and a conjugate–normal matrix. Thus, being a direct sum of the conjugate–normal blocks $\widetilde{A}_{ii}$, the matrix $\widetilde{A}$ itself is conjugate–normal. The same is true of the unitarily congruent matrix A.

Suppose that $\lambda_m = 0$. As in the previous case, we deduce from (36) that

$$\widetilde{U}_{ij} = 0$$

for $i = 1, 2, \ldots, m-1$ and $j \neq i$. Since $\widetilde{U}$ is unitary, this immediately implies that

$$\widetilde{U}_{mj} = 0, \quad j = 1, 2, \ldots, m-1.$$

The rest of the argument is the same as above, the only distinction being that

$$\widetilde{A}_{mm} = \lambda_m U_{mm} = 0.$$

Now, assume that (c) is fulfilled. Again, we apply to A, S and U transformations (33), (31), and (35), respectively. Since

$$\Lambda^2 = \Lambda\Lambda = \overline{\Lambda}\Lambda = (F^{*}\overline{SF})(F^{\mathrm{T}} S F) = F^{*}(\overline{S}S)F,$$

relation (26) transforms into

$$\widetilde{U}\Lambda^2 = \Lambda^2\widetilde{U}.$$

It follows that $\widetilde{U}$ is block diagonal:

$$\widetilde{U} = \widetilde{U}_{11} \oplus \widetilde{U}_{22} \oplus \cdots \oplus \widetilde{U}_{mm}.$$

The rest of the proof is as in case (a).

Finally, assume that (b) is fulfilled. Substituting (3) into (25), we obtain

$$U(U^*\overline{S})(SU) = (U^*\overline{S})(SU)U$$

or

$$\overline{S}S = U^*(\overline{S}S)U$$

or

$$U\overline{S}S = (\overline{S}S)U.$$

Thus, (25) is the same relation as (26), which completes the proof of Theorem 4. □

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