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Vanishing structure set of 3-manifolds

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A B S T R A C T

In this short note we update a result proved in Roushon (2007) [17]. This will complete our program of Roushon (2000) [13] showing that the structure set vanishes for compact aspherical 3-manifolds.

1. Introduction

This paper is to note that the program we started in [13] is now complete.

Let us first recall that a compact manifold M with boundary is called *topologically rigid* if any homotopy equivalence $f : (N, \partial N) \to (M, \partial M)$ from another compact manifold with boundary, so that $f|_{\partial N} : \partial N \to \partial M$ is a homeomorphism is homotopic to a homeomorphism relative to boundary.

Let M be a compact connected 3-manifold whose fundamental group is torsion free.

We prove the following theorem.

Theorem 1.1. *If M is aspherical then $M \times \mathbb{D}^n$ is topologically rigid for $n \geqslant 2$. Here $\mathbb{D}^n$ denotes the n-dimensional disc.*

In [13,14] we proved Theorem 1.1 under various conditions. In [13] we proved it for the nonempty boundary case and for the situation when the manifold contains an incompressible square root closed torus. In [14] we assumed the manifold has positive first Betti number. Due to some recent developments in Geometric Topology (see [1,2,15–17]) we are now able to deduce Theorem 1.1. Also the main ideas from [13,14] go behind the proof of this general case.

The first step to prove Theorem 1.1 is to show that the Whitehead group of $\pi_1(M)$ is trivial. We deduce the following for this purpose.

Theorem 1.2. *Let G be isomorphic to the fundamental group of M then*

$$Wh(G) = K_{-i}(G) = \tilde{K}_0(G) = 0$$

for all $i \geqslant 2$.

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2. Proofs of Theorems 1.1 and 1.2

For terminologies concerning 3-manifolds used in the proofs see [7] or [13].

Proof of Theorem 1.2. By (Kneser–Milnor) prime decomposition theorem G is isomorphic to the free product of a free group and finitely many groups $G_1, G_2, \ldots, G_n$ where for each i, G_i is isomorphic to the fundamental group of an aspherical irreducible 3-manifold M_i (see [14, Lemma 3.1]). Since the Whitehead group of a free product is the direct sum of the Whitehead groups of the individual factors of the free product (see [19]) it is enough to prove that the Whitehead group vanishes for G_i. Now by the Geometrization Theorem (conjectured by Thurston and proved by Perelman) M_i is either Seifert fibered, Haken or hyperbolic. The hyperbolic case follows from some more general result of Farrell and Jones in [4], for Haken case it follows from Waldhausen's result in [20]. For non-Haken Seifert fibered space the vanishing result is due to Plotnick (see [9]). For the reduced projective class groups $\tilde{K}_0(-)$ and for the negative K-groups $K_{-i}(-)$ the same sequence of arguments and references work. For details see [3]. In fact, more generally it is shown in [3] that G is K-flat, i.e., $Wh(G \times \mathbb{Z}^n) = 0$ for all non-negative integer n.

This completes the proof of Theorem 1.2. □

Below we recall the statement of the Fibered Isomorphism Conjecture of Farrell and Jones. For details about this conjecture see [5]. Here we follow the formulation given in [6, Appendix].

Let $\mathcal{F}$ be one of the three functors from the category of topological spaces to the category of spectra: (a) the stable topological pseudo-isotopy functor $\mathcal{P}()$; (b) the algebraic K-theory functor $\mathcal{K}()$; and (c) the L-theory functor $L^{\langle -\infty \rangle}()$. The L-theory functor also includes an orientation data, that is a homomorphism $\omega : \pi_1(X) \to \mathbb{Z}_2$. If the topological space is an oriented manifold then this homomorphism is zero.

Let $\mathcal{M}$ be a category whose objects are continuous surjective maps $p : E \to B$ between topological spaces E and B. And a morphism between two maps $p : E_1 \to B_1$ and $q : E_2 \to B_2$ is a pair of continuous maps $f : E_1 \to E_2$, $g : B_1 \to B_2$ such that the following diagram commutes.

$$\begin{array}{ccc} E_1 & \xrightarrow{f} & E_2 \\ \downarrow p & & \downarrow q \\ B_1 & \xrightarrow{g} & B_2 \end{array}$$

There is a functor defined by Quinn in [10] from $\mathcal{M}$ to the category of Ω-spectra which associates to the map p a spectrum $\mathbb{H}(B, \mathcal{F}(p))$ with the property that $\mathbb{H}(B, \mathcal{F}(p)) = \mathcal{F}(E)$ if B is a single point space. For an explanation of $\mathbb{H}(B, \mathcal{F}(p))$ see [5, Section 1.4]. Also the map $\mathbb{H}(B, \mathcal{F}(p)) \to \mathcal{F}(E)$ induced by the morphism: $\mathrm{id} : E \to E$; $B \to *$ in the category $\mathcal{M}$ is called the *Quinn assembly map*.

Let Γ be a discrete group and $\mathcal{E}$ be a Γ-space which is universal for the class of all virtually cyclic subgroups of Γ and denote $\mathcal{E}/\Gamma$ by $\mathcal{B}$. For definition and properties of universal space see [5, Appendix]. Let X be a space on which Γ acts freely and properly discontinuously and $p : X \times_\Gamma \mathcal{E} \to \mathcal{E}/\Gamma = \mathcal{B}$ be the map induced by the projection onto the second factor of $X \times \mathcal{E}$.

The *Fibered Isomorphism Conjecture* for Γ states that the map

$$\mathbb{H}\big(\mathcal{B}, \mathcal{F}(p)\big) \to \mathcal{F}(X \times_\Gamma \mathcal{E}) = \mathcal{F}(X/\Gamma)$$

is an (weak) equivalence of spectra. The equality in the above display is induced by the map $X \times_\Gamma \mathcal{E} \to X/\Gamma$ and using the fact that $\mathcal{F}$ is homotopy invariant. If X is simply connected then this is called the *Isomorphism Conjecture* for Γ.

In this paper we consider the case when $\mathcal{F}() = L^{\langle -\infty \rangle}()$. We have already mentioned that this L-theory functor contains the orientation data $\omega : \Gamma \to \mathbb{Z}_2$ so as to include the case of nonorientable manifolds.

Let us now deduce the following theorem which is an immediate consequence of [17, 3(a) of Theorem 2.2] and some recent results from [1,2].

Theorem 2.1. *Let G be isomorphic to the fundamental group of a 3-manifold. Then the Farrell–Jones Fibered Isomorphism Conjecture in $L^{\langle -\infty \rangle}$-theory is true for $G \wr H$ where H is some finite group.*

Proof. The theorem follows from [17, 3(a) of Theorem 2.2] provided we show that the conjecture is true for $\Gamma \wr H$ where H is some finite group and Γ belongs to the following classes of groups:

(1) $\mathbb{Z}^2 \rtimes_\sigma \mathbb{Z}$ for all actions σ of $\mathbb{Z}$ on $\mathbb{Z}^2$.
(2) Fundamental groups of closed nonpositively curved Riemannian 3-manifolds.
(3) $\Gamma \simeq \lim_{i \in I} \Gamma_i$ where $\{\Gamma_i\}$ is a directed system of groups so that for each $i \in I$ the conjecture is true for $\Gamma_i \wr K$ where K is some finite group.

We now note the following to complete the proof of the theorem.

(1) follows from [2] where the conjecture is proved for virtually polycyclic groups.

(2) follows from [1] where the conjecture is proved for finite-dimensional $CAT(0)$-groups.

And (3) follows from [6, Theorem 7.1]. □

Proof of Theorem 1.1. If $\partial M \neq \emptyset$ then the theorem follows from [13, Theorem 1.1]. Therefore we can assume that M is closed. Now recall that the combination of Theorems 1.2 and 2.1 implies the isomorphism of the classical assembly map in L-theory. Namely, the map $H_k(BG, \mathbb{L}_0) \to L_k(G)$ is an isomorphism for all k. Since M aspherical it is a model of BG, thus we have the isomorphism $H_k(M, \mathbb{L}_0) \to L_k(G)$. See the proof of [8, Theorem 1.28] or [18, Corollary 5.3] for a detailed argument.

Next we recall the definition of structure set and the surgery exact sequence.

Let M be a compact manifold with boundary (may be empty) so that $Wh(\pi_1(M)) = 0$. Consider all objects $(N, \partial N, f)$, where N is a manifold with boundary ∂N and $f : N \to M$ is a homotopy equivalence such that $f|_{\partial N} : \partial N \to \partial M$ is a homeomorphism. Two such objects $(N_1, \partial N_1, f_1)$ and $(N_2, \partial N_2, f_2)$ are equivalent if there is a homeomorphism $g : N_1 \to N_2$ such that the obvious diagram commutes up to homotopy relative to the boundary. The equivalence classes of these objects is the homotopy-topological structure set $\mathcal{S}(M, \partial M)$.

In [11,12] Ranicki defined homotopy functors $\mathcal{S}_k(X)$ from the category of topological spaces to the category of abelian groups which fit into the following exact sequence:

$$\cdots \to \mathcal{S}_k(X) \to H_k(X, \mathbb{L}_0) \to L_k\big(\pi_1(X)\big) \to \mathcal{S}_{k-1}(X) \to \cdots.$$

Also it is shown in [11,12] that there is a bijection between $\mathcal{S}(M \times \mathbb{D}^k, \partial(M \times \mathbb{D}^k))$ and $\mathcal{S}_{k+dim\,M}(M)$ provided $dim\,M + k \geqslant 5$.

The proof of the theorem is now complete since we have already proved the isomorphism $H_k(M, \mathbb{L}_0) \to L_k(G)$ for all k. □

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