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Unsolved problems presented at the Julius Petersen Graph Theory Conference

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A problem session was held during the Julius Petersen Graph Theory Conference, Hindsgavl, Denmark, July 1–6, 1990. The following is a list of the problems presented in connection with the meeting. We thank all contributors. Special thanks to Herbert Fleischner, who chaired the problem session and later collected the problems in writing from the participants.

Problem 1 (Adrian Bondy). *Path double cover*

Conjecture. Any k-regular graph G can be double-covered by paths of length k so that each vertex is the end of exactly two paths.

Editors' comment. The formulation of the problem is based on Bondy's oral presentation at the meeting. Bondy remarked that the conjecture is true for $k = 3$, and that it is related to Kostochka's conjecture (Problem 15 below).

Problem 2 (Nathaniel Dean). *Convex hull*

From among all rectilinear drawings of K_n (i.e., every edge drawn as a straight-line segment) choose one D with the fewest number of crossings. The convex hull of D must be a polygon (if $n \geqslant 3$).

Conjecture. The convex hull of D is a triangle.

Problem 3 (Nathaniel Dean). *Restricted Steiner tree*

Suppose we are given n points at fixed locations in the plane $\mathbb{R}^2$.

(a) Find locations for k more points so as to achieve a smallest possible minimum spanning tree (MST) on the $n + k$ points.

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(b) For every set S of k points in $\mathbb{R}^2$ we get a labelled MST. Call two such sets equivalent if they yield the same labelled tree. This induces an equivalence relation. In the case $k = 1$ each class corresponds to a region of the plane. What can be said about these regions? Some small examples show that they are not necessarily convex. What happens for $k > 1$?

Problem 4 (Nathaniel Dean). *Generalization of Menger's theorem*

Conjecture. Let G be a k-connected graph. Then for every set $S = \{v_1, v_2, \ldots, v_k\}$ of vertices of G there are k trees $T_1, \ldots, T_k$ with T_i rooted at v_i $(i = 1, \ldots, k)$ such that for every vertex $x \in V(G) - S$ the paths $P_1, P_2, \ldots, P_k$, where P_i is the unique xv_i-path of T_i, are pairwise disjoint except at x.

Note. Case $k = 1$ is trivial, and case $k = 2$ follows easily from the existence of an st-numbering.

Editors' comment. An st-numbering of a graph on n vertices is a labelling of the vertices with distinct integers $1, 2, \ldots, n$ so that, except for two vertices s and t which are labelled 1 and n, every vertex has a neighbour with a smaller label and one with a larger label (see the book S. Even: *Graph Algorithms*, Computer Science Press 1979, p. 180).

The conjecture is still open, but Dean told us by e-mail on October 11, 1990 that he just learned that it has been settled in the 3-connected case in [A. Zehavi and A. Itai: Three Tree-Paths, *J. Graph Theory* 13 (1989), p. 175–188]. In fact a conjecture in that paper seems equivalent to the conjecture of Dean.

Conjecture (Zehavi and Itai). If a graph G is k-connected and $v \in G$, then there exists k spanning trees of G such that the paths from any vertex x to v on the spanning trees are internally disjoint.

Problem 5 (Paul Erdős).

Denote by $h(n)$ the largest integer for which every $G(n)$ (i.e. a graph of n vertices) contains an induced trivial graph (i.e. a complete graph or an empty graph) of $h(n)$ vertices. It is well known that

$$\frac{\log n}{2 \log 2} \leqslant h(n) \leqslant \frac{2 \log n}{\log 2}.$$

The exact determination of $h(n)$ or even the proof that $h(n)/\log n \to c$ are well known and difficult problems.

As far as I remember, Fajtlowicz, Stanton and I posed the following problem: Let $f(n)$ be the largest integer for which every $G(n)$ contains an induced regular subgraph on $f(n)$ vertices. Determine or estimate $f(n)$ as well as you can. As far as we know it is not even known if $f(n) > h(n)$ holds for some n. Small values of n would also be of interest.

Bollobás observed $f(n) < cn^{\frac{1}{2}}$, but probably $f(n)$ is much smaller, $f(n)/\log n \to \infty$ would be very interesting.

Problem 6 (Ralph Faudree).

For a fixed integer $k > 1$, let G be a graph of order n that is (nearly) regular of degree n/k and has a 2-factor. Does G have a 2-factor with $\leqslant k-1$ cycles (or $\leqslant f(k)$ cycles for some function f independent of n)?

Problem 7 (Herbert Fleischner).

An A-trail of a plane eulerian graph (or any eulerian graph embedded in some surface) is an eulerian trail in which consecutive edges are always neighbours in the cyclic ordering of the edges incident with $v \in V(G)$, for all $v \in V(G)$, where this cyclic ordering is given by the embedding. In the case of 4-regular graphs, A-trails and non-intersecting eulerian trails are equivalent concepts. However, while non-intersecting eulerian trails exist, this is not the case in general with respect to A-trails, even if G is planar and 3-connected.

Conjecture. Every 4-connected plane eulerian graph has an A-trail.

Problem 8 (Herbert Fleischner).

Consider the n-dimensional hypercube W_n, and denote by f_n the number of acyclic orientations (of the edges) of W_n.

Question. What is the asymptotic behaviour of f_n?

Editors' comment. Fleischner has informed us by e-mail on Feb. 7, 1992, that the problem can be solved using [W. Goddard, V. King and L. Schulman: Optimal randomized algorithms for local sorting and set-maxima, *Theory of Computing, Proceedings of the 22nd annual ACM Symposium,* ACM Press, 1990, p. 45–53].

Problem 9 (András Frank).

Given a graph $G = (V, E)$, determine the maximum cardinality of a subset F of edges for which $|F \cap C| \leqslant |C|/2$ for every cut C of G. This is solved for planar graphs, but not known for example for the complete bipartite graph $K_{n,n}$.

Problem 10 (András Frank).

Characterize those connected graphs (multiple edges allowed) which can be reduced to a single node by a sequence of 'odd' contractions. The contraction of two adjacent nodes u, v is called *odd* if there is an odd number of parallel edges connecting u and v. (By a theorem of Lovász a planar graph has this property if and only if its planar dual is factor-critical.)

Problem 11 (John Gimbel).

Given a graph G, let $z(G)$ be the smallest order of all partitions of $V(G)$ where each part induces a complete or empty graph. Is there a constant $c > 0$ with the property that if $\chi(G) = n$ then G has a subgraph H where $z(H) \geqslant cn/\log n$?

Editors' comment. $z(G)$ is known as the co-chromatic number of G. In the oral presentation of the problem, Gimbel mentioned that Erdős and he have proved that a graph of large chromatic number has a subgraph of large co-chromatic number, but that the best possible order of magnitude is not known.

Problem 12 (Roland Häggkvist).

Does every graph on $4n$ vertices, each of degree at least $2n$, contain every 3-regular bipartite graph on at most $4n$ vertices (n large)?

Comments. (1) I actually believe that every sufficiently large n-order Dirac graph (i.e. $\delta \geqslant n/2$) contains every k-regular bipartite graph on at most n vertices, k fixed. No proof is in sight.

(2) The problem as stated *should* have a positive answer and *should* be tractable by probabilistic methods.

Editors' comment. In his oral presentation at the meeting Häggkvist mentioned the special case of 'every 3-regular bipartite graph on $2n$ vertices' as unsolved, but maybe provable.

Problem 13 (Bill Jackson).

Conjecture. Let G be a 2-edge connected graph. Then G has an even connected subgraph H such that $|H| \geqslant 2$ and for each component F of $G - V(H)$ there are at most three edges between F and H.

Editors' comment. A graph is called even if all vertices have even degree. In his oral presentation at the meeting Jackson called the statement a kind of nonplanar analog of Tutte's bridge theorem for planar graphs.

Problem 14 (Francois Jaeger). *Petersen-colouring*

About ten years ago the following problem was posed. Let G be a cubic graph and P the Petersen graph. A Petersen-colouring of G maps the edges of G into the edges of P such that any three edges of G incident with a common vertex are mapped onto three edges of P incident with a common vertex.

Conjecture. Every cubic bridgeless graph G has a Petersen-colouring.

Note. If G is 3-edge-colourable the conjecture is of course true for G. The conjecture in general implies the Cycle Double Cover Conjecture and Fulkerson's conjecture on 6-regular graphs (*if the edges of a simple* 3-*regular bridgeless graph are doubled, then the resulting* 6-*regular graph is* 6-*edge-colourable*).

Editors' comment. The above formulation of the problem is based on Jaeger's oral presentation at the meeting.

Problem 15 (A. Kostochka).

Conjecture. If G is a $2k$-edge-connected graph with $\delta(G) \geq 2k+1$ then there exists a partition $\mathscr{P}$ of $E(G)$ into paths of length at least $2k+1$.

Editors' comment. If G is $(2k+1)$-regular, then each path in the conjectured partition has length equal to $2k+1$ and each vertex of G is the end of exactly one of the paths. In his oral presentation Kostockha mentioned the conjecture to be true for $k=1$ (reduce first to the 3-regular case, then use Petersen's theorem making the edges of a 1-factor the middle edges of the paths of length 3; Adrian Bondy remarked that this last idea has been discovered by several people).

Problem 16 (Jenö Lehel and Horst Sachs). *A problem in the theory of polyhedra*
Consider the set of all finite polyhedra in 3-dimensional space. Two polyhedra which have isomorphic graphs are considered equivalent. Prove of disprove:

> In every equivalence class there is a polyhedron whose edges touch a given sphere. (*)

Remark 1. Let S be a sphere, Π an equivalence class, and P a polyhedron from Π whose edges touch S. For the sake of convenience, we do not distinguish between an edge and the straight line which contains it.

Replace every edge e of P by a straight line e' which is perpendicular to e and touches S at the same point as e. Then the new lines form a polyhedron P' whose edges touch S and whose equivalence class is the dual $\bar{\Pi}$ of Π.

Remark 2. (*), if true, has the following consequences (details to be checked):

(i) Let G be a 3-connected simple planar finite abstract graph and $\bar{G}$ its dual. G and $\bar{G}$ can simultaneously be realized as coin graphs on the sphere as well as in the plane (apply a stereographic projection) such that the boundary circles of the coins (disks) realizing G and those of the coins realizing $\bar{G}$ intersect perpendicularly.

(ii) (Here the results of some additional considerations are used—please, check.) The simultaneous representation of Π and $\bar{\Pi}$, and of G and $\bar{G}$, as described above, is unique up to a group Γ of circle-preserving transformations (in the plane: $w=(az+b)/(cz+d)$; this group has 6 real parameters, 3 of which are needed for congruence transformations). Thus we have found a polyhedron representation which is canonic modulo Γ.

Remark 3. For simple polyhedra (in every vertex there meet precisely 3 faces) and their duals (all faces are triangles) and the corresponding graphs the truth of (*), (i) and (ii) follows from Thurston's solution to Ringel's coin problem (see; Yves Colin de Verdière, Empilements de cercles: Convergence d'une méthode de point fixe, *Forum Math.* 1 (1989), 395–402).

Editors' comment. Branko Grünbaum has informed us that the problem (*) was raised also by him and Shephard in [1] and mentioned by Schulte in [5]. Moreover the problem was completely solved by Schramm [4].

In a letter to us, dated September 9, 1991, Sachs said: "*Die Geschichte des von mir in Hindsgavl formulierten Problems* (Problem 16) (*das Branko schon früher publiziert hatte*) *hat eine recht kuriose Wendung genommen.*" Sachs explained how he, during the meeting in honour of L. Collatz in Hamburg in July 1990, discovered how the coin problem of Ringel and Jackson can be solved using the theory of conformal mappings, and he continued: "*Später sprach ich darüber mit Rainer Kühnau aus Halle* (*wir waren vor* 30 *Jahren beide Assistenten bei Herbert Grötzsch an der Universität Halle*), *und der sagte mir, dass das ja alles schon Paul Koebe um* 1935 *bekannt gewesen sei* (*bei ihm liegt also die Priorität!*)—*für mich natürlich enttäuschend, aber doch nicht zu sehr, denn schliesslich freut man sich über jedes schöne Resultat um des Resultates willen.*"

In a very interesting note [3] Sachs has put the whole story straight. He mentions that a footnote in Koebe's paper [2] clearly indicates that Koebe also anticipated (*).

References

[1] B. Grünbaum and G.C. Shephard, Some problem on polyhedra, J. Geom. 29 (1987) 182–190.

[2] P. Koebe, Kontaktprobleme der konformen Abbildung, Berichte über die Verhandlungen der Sächsischen Akademie der Wissenschaften zu Leipzig, Math.-Phys. Klasse 88 (1936) 141–164.

[3] H. Sachs, Coin graphs, polyhedra and conformal mapping, manuscript 1991, submitted to the proceedings of the Workshop on Algebraic and Topological Methods in Graph Theory, Yugoslavia, June 1991.

[4] O. Schramm, How to cage an egg, University of California at San Diego, preprint 1990.

[5] E. Schulte, Analogues of Steinitz's theorem about non-inscribable polytopes, in: Intuitive Geometry, Siófok 1985; Colloq. Math. Soc. János Bolyai 48 (1987) 503–516.

Problem 17 (Michael Lomonosov). *Questions from network reliability*

Let $G = (V, E)$ be an undirected graph. A probability distribution on the set of its *cocircuits* can be defined as follows: for every ordering $\pi: e_1, e_2, \ldots, e_k, e_{k+1}, \ldots, e_{|E|}$ of E consider the number k such that $G_k = (V, \{e_1, \ldots, e_k\})$ is disconnected and G_{k+1} is connected. Then $\{e_{k+1}, \ldots, e_{|E|}\}$ contains a unique cocircuit $C(\pi)$ of G. Put $\operatorname{Prob}(C) = \#\{\pi: C(\pi) = C\}/|E|!$ for every cocircuit C.

[For calculations: $\operatorname{Prob}(C) = |C| \int_0^\infty e^{-|C|t} R(G - C, t)\, dt$, where $R(H, t)$ denotes the probability that all components of H remain connected after the edges of H are randomly and independently erased with probability e^{-t}, $t \geq 0$].

This distribution is disbalanced to the side of small cuts (see, e.g. the Erdős–Renyi theorem on graph evolution).

Questions. For arbitrary G find:

(1) Lower bound on the probability of the minimum cuts of G.

(2) Lower bound on max probability of a laminar family of cuts of G (i.e. such that no two cuts from the family are crossing).

(3) Lower bound on max probability of a cactus-type family of cuts of G (a cactus is a graph any two of whose circuits have at most one vertex in common; the cuts of a cactus form what is called a cactus-type family of cuts; the minimum cuts of any G form a cactus-type family (Dinic, Karzanov, Lomonosov (1974), in Russian)).

(4) Is it true that for any $\varepsilon > 0$ there exists a polynomial-length sublist of cocircuits whose probability is $\geqslant 1 - \varepsilon$?

Problem 18 (Jarik Nešetřil).

Is it true that for every perfect graph G there exists a perfect graph H with the following property:

For every partition $E(H) = E_1 \cup E_2$ there exists an induced subgraph G' of H, G' isomorphic to G, such that either $E(G') \subseteq E_1$ or $E(G') \subseteq E_2$.

If true, this is a Ramsey type theorem for perfect graphs. Note that the analogous result for partitions of vertices is true.

Problem 19 (László Pyber).

Let G be a bridgeless graph. The cycle depth of G is the smallest number k, such that $E(G)$ has a covering by cycles for which evey vertex of G is covered by at most k cycles. We denote this number by $cd(G)$.

Conjecture A. $cd(C) \leqslant \Delta(G)$ where Δ is the maximum degree.

Conjecture B. $cd(G) \leqslant \frac{2}{3}\Delta(G) + 1$.

Comments. $cd(G) \leqslant \frac{3}{2}\Delta(G)$ follows from Jaeger's 8-flow theorem.

$cd(G) \leqslant \Delta(G)$ would be a consequence of the cycle double cover conjecture.

$cd(K_{3,n-3}) \geqslant \frac{2}{3}(n-3)$.

For the Petersen graph P we have $cd(P) = 3 = \frac{2}{3}\Delta(P) + 1$.

Problem 20 (Horst Sachs). *Colouring ball packings*

This is a problem of long standing, often repeated at problem sessions. An n-packing B is a finite collection of unit balls in n-dimensional space, where any two balls of B are allowed to touch, but not to overlap (i.e., the interiors of any two balls of B are disjoint).

Let χ_n denote the minimum number of colours that suffice for colouring the balls in any n-packing B such that any two balls of B which touch must have different colours. It is known that $\chi_2 = 4$ and $5 \leqslant \chi_3 \leqslant 10$.

What is χ_n? In particular, what is χ_3?

Editors' comment. The paper [B. Jackson and G. Ringel: Colorings of circles, *American Math. Monthly* 91 (1984), 42–49] gives an exposition of problems related to Sachs' problem. Jackson and Ringel report there that $\chi_3 \leq 9$.

Problem 21 (Bjarne Toft).

At the excursion on lake Balaton at the Erdős' 60th birthday colloquium in Keszthély (Hungary), 1973, Nešetřil and Rödl, asked if a large critical k-chromatic graph contains a large critical $(k-1)$-chromatic subgraph. More formally: Let $G_1, G_2, G_3, \ldots$ be an infinite sequence of k-critical graphs, and let $G_1', G_2', G_3', \ldots$ be largest possible $(k-1)$-critical subgraphs of $G_1, G_2, G_3, \ldots$.

Does $|V(G_i)| \to \infty$ imply $|V(G_i')| \to \infty$?

This is still open for $k \geq 5$.

For graphs G_i of low connectivity, e.g. 2, I suspect the answer to be negative. This leads to the following much weaker question: Does $|\kappa(G_i)| \to \infty$ imply $|V(G_i')| \to \infty$? (κ denotes the vertex-connectivity).

Problem 22 (Zsolt Tuza). *Covering the cliques with vertices*

Let G be a chordal graph on n vertices (i.e. G contains no induced cycle of length greater than 3). Suppose that G has no isolated vertices, and every edge of G is contained in a complete subgraph of order 4 in G. Does there exist a set of at most $n/4$ vertices that meets all cliques (= complete subgraphs maximal under inclusion) of G?

References

Zs. Tuza, Covering all cliques of a graph, Discrete Math. 86 (1990) 117–126.

T. Andreae, M. Schughart and Zs. Tuza, Clique-transversal sets of line graphs and complements of line graphs, Discrete Math. 88 (1991) 11–20.

P. Erdős, T. Gallai and Zs. Tuza, Covering the cliques of a graph with vertices, Discrete Math., to appear.

Editors' comment. In his oral presentation Tuza mentioned that the answer to this rather particular question in a certain sense 'completes the picture'.

Problem 23 (Zsolt Tuza). *Covering the edges with perfect subgraphs*

Let G be a graph on n vertices. Can the edge set of G be covered with at most $O(\log n/\log\log n)$ perfect subgraphs of G?

Reference

Zs. Tuza, Perfect graph decompositions, Graphs Combin. 7 (1991) 89–93.

Editors' comment. In his oral presentation Tuza mentioned that the above order of magnitude would be best possible.

Problem 24 (Zsolt Tuza). *P-intersection graphs*

Derive the following class $I(P)$ of graphs from the Petersen-graph P. First, take any subdivision of the edges of P (i.e. replace its edges by internally disjoint paths of arbitrary lengths). Next, take a collection of $\mathcal{G}$ of *connected* subgraphs of the subdivision. The vertices of the *intersection graph* $I(\mathcal{G})$ of $\mathcal{G}$ are the graphs $G \in \mathcal{G}$, and two such vertices G and G' are adjacent if and only if they share a vertex. We call $I(\mathcal{G})$ a *P-intersection graph.*

Problem. Characterize P-intersection graphs.

Reference

M. Biró, M. Hujter and Zs. Tuza, Precoloring extension I. Interval graphs, Discrete Math. 100 (1992).

Problem 25 (Noga Alon, Yair Caro and Zsolt Tuza). *Rainbow arithmetic progressions*

Given a natural number k, find the smallest $n = n(k)$ with the following property. If the integers $1, 2, \ldots, n$ are colored in such a way that each color is assigned to at most k of those numbers, then there is a 3-term arithmetic progression of three distinct colors.

Reference

N. Alon, Y. Caro and Zs. Tuza, Sub-Ramsey numbers for arithmetic progressions, Graphs Combin. 5 (4) (1989) 303–306.

Problem 26 (Heinz-Jürgen Voss).

Let $r \geqslant 4$, $t \geqslant 3$ be arbitrary integers. The sets of all vertices of a graph G having valency $\geqslant r$ or $<r$ are denoted by $V_{\geqslant r}(G)$ and $V_{<r}(G)$, respectively. In [1] I have proved that if G is a graph of girth $\geqslant t$ with $V_{<r}(G) = \emptyset$ then G contains a cycle of length $\geqslant (r-1)^{t/4-7}$. In [1] a graph is described all of whose cycles have length $\leqslant (r-1)^{t+1}$

For the weaker condition $|V_{<r}(G)| \leqslant |V_{\geqslant r}(G)|$ I present the following

Conjecture. If G is a graph of girth $\geqslant t$ with $|V_{<r}(G)| \leqslant |V_{\geqslant r}(G)|$, then G contains a cycle of length $\geqslant (r-2)^{\alpha t+\beta}$, where α, β are constants. Probably α can be chosen as $\alpha = \frac{1}{8}$.

Reference

[1] H.-J. Voss, Cycles and Bridges in Graphs (Deutscher Verlag der Wissenschaften 1991) and (Kluwer Academic Publishers, Dordrecht, 1991).

Problem 27 (Peter Winkler). *Greedy clique decomposition of graphs*

In the famous paper [2] of Erdős, Goodman and Posa, it is shown that the edges of any graph on n vertices can be partitioned into at most $n^2/4$ cliques.

Later, Chung [1] and independently Györi and Kostochka [3] strengthened this result by showing that the edges can be partitioned in such a way that the sum of the numbers of vertices of the cliques is at most $n^2/2$.

We conjecture that both results can be strengthened to apply to greedy decompositions. Specifically the following.

Conjecture 1. If maximal cliques are removed one by one from any n vertex graph, then the graph will be empty after at most $n^2/4$ steps.

Conjecture 2. If maximal cliques are removed one by one from any n vertex graph, then the graph will be empty after the sum of the numbers of vertices in the cliques has reached at most $n^2/2$.

One of the nice features of the second conjecture is that if it is true it has many extremal cases.

References

[1] F.R.K. Chung, On the decomposition of graphs, SIAM J. Algebraic Discrete Methods 2 (1) (1981) 1–12.

[2] P. Erdős, A.W. Goodman and L. Posa, The representation of a graph by set intersection, Canad. J. Math. 18 (1966) 106–112.

[3] E. Gyori and A.V. Kostochka, On a problem of G.O.H. Katona and T. Tarjan, Acta Math. Acad. Sci. Hung. 34 (1979) 321–327.

Editors' comment. This problem was submitted by e-mail on September 9, 1990. Conjecture 1 was solved in the affirmative with an interesting simple proof by Sean McGuinness (*The greedy clique decomposition of a graph,* manuscript, Odense University, September 1991, 5 pages, submitted to *J. Graph Theory*), and some progress on Conjecture 2 has also been made by McGuinness (*Restricted greedy clique decompositions and greedy clique decompositions of K_4-free graphs,* manuscript, Odense University 1992, 16 pages, submitted), indicating that Conjecture 2 is true with 'maximal' replaced by 'maximum'.