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Boundary value problems for multi-term fractional differential equations

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ABSTRACT

Multi-term fractional diffusion-wave equation along with the homogeneous/non-homogeneous boundary conditions has been solved using the method of separation of variables. It is observed that, unlike in the one term case, solution of multi-term fractional diffusion-wave equation is not necessarily non-negative, and hence does not represent anomalous diffusion of any kind.

1. Introduction

The time fractional diffusion-wave equation [1–4] is obtained from the classical diffusion or wave equation by replacing the first- or second-order time derivative by a fractional derivative of order α with $0 < \alpha < 1$ or $1 < \alpha < 2$, respectively. In a seminal paper Schneider and Wyss [1] formulated fractional diffusion/wave equation (together with appropriate initial conditions) in terms of integrodifferential equations. They have shown that for $0 < \alpha < 1$, the Green's function is a probability density, and the mean square displacement is proportional to t^{α}, hence represents sub-diffusive behaviour. It has further been observed that for $1 < \alpha < 2$, only one-dimensional case represents probability density. Hence $\alpha \in (1, 2)$ represents enhanced diffusion, termed as superdiffusion, but only in one dimension. On the other hand in higher dimensions, for $\alpha > 1$, the solutions need not be non-negative and hence do not represent physical diffusion of any kind.

The time-fractional diffusion-wave equation has been studied widely in the literature, as it models a wide range of important physical phenomena. These equations represent propagation of mechanical waves in visco-elastic media [5], a non-Markovian diffusion process with memory [6], charge transport in amorphous semiconductors [7] and many more. There exists a large number of articles devoted to the study of one-dimensional anomalous diffusion [1–9], whereas a few have discussed multi-dimensional case [1,10]. Some researchers have used Riemann–Liouville derivative [2–4] while others have used Caputo derivative [1,10,11,13]. Various methods have been used to solve the fractional diffusion-wave equation. Schneider and Wyss [1] have used Green's function method. Metzler et al. [12] have introduced separation of variables method for solving fractional Fokker–Plank equation. Agrawal [11] has solved fractional diffusion equation using finite sine transform method, and Metzler and Klafter [6] have employed the method of images and Fourier–Laplace transform. Further Daftardar-Gejji and Jafari have used separation of variables method [13], while solving fractional boundary value problems for fractional diffusion-wave equations. Sokolov and Metzler [14] have used method of subordination to obtain first passage time density for Levý random processes, as methods like method of images are inadequate there. Iterative methods, such

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as Adomian decomposition method (ADM) has also been explored extensively to solve fractional diffusion-wave equation [15,16].

Present paper deals with multi-term generalisation of fractional diffusion-wave equation:

$$P(D)u(x,t) = k\frac{\partial^2 u(x,t)}{\partial x^2} + q(t), \quad 0 < x < \pi,\ t > 0,$$

where

$$P(D) = D_t^{\mu} - \sum_{i=1}^{r-1} \lambda_i D_t^{\mu_i}, \quad 0 < \mu_{r-1} < \mu_{r-2} < \cdots < \mu_1 < \mu \leqslant 2,$$

with homogeneous/non-homogeneous boundary conditions. Method of separation of variables is used and fractional boundary value problems are further explicitly solved in various cases. The paper has been organized as follows. In Section 2 preliminaries and notations are given. In Section 3 homogeneous fractional boundary value problems (BVPs) are solved. Section 4 deals with solution of non-homogeneous fractional BVPs. In Section 5, BVP in higher dimensions has been dealt with. Section 6 deals with non-homogeneous boundary conditions. Some numerical examples have been presented in Section 7, followed by conclusions in Section 8.

2. Preliminaries and notations

This section deals with some preliminaries and notations regarding fractional calculus [17–20].

Definition 2.1. A real function $f(x)$, $x > 0$, is said to be in space C_α, $\alpha \in \Re$, if there exists a real number $p\ (> \alpha)$, such that $f(x) = x^p f_1(x)$ where $f_1(x) \in C[0, \infty)$.

Definition 2.2. A real function $f(x)$, $x > 0$, is said to be in space C_α^m, $m \in \mathbb{N} \cup \{0\}$, if $f^{(m)} \in C_\alpha$.

Definition 2.3. Let $f \in C_\alpha$ and $\alpha \geqslant -1$, then the expression

$$I_t^{\mu} f(x,t) = \frac{1}{\Gamma(\mu)} \int_0^t (t-\tau)^{\mu-1} f(x,\tau)\, d\tau, \quad t > 0,$$

is called as the (left sided) Riemann–Liouville integral of order μ.

Definition 2.4. The (left sided) Riemann–Liouville fractional derivative of f, $f \in C_{-1}^m$, $m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, is defined as

$${}^R D_t^{\mu} f(x,t) = \frac{\partial^m}{\partial t^m}\big(I_t^{m-\mu} f(x,t)\big), \quad m-1 < \mu < m,\ m \in \mathbb{N},\ t > 0.$$

Definition 2.5. The (left sided) Caputo fractional derivative of f, $f \in C_{-1}^m$, $m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, is defined as

$$\begin{aligned} D_t^{\mu} f(x,t) &= \frac{\partial^m}{\partial t^m} f(x,t), \quad \mu = m, \\ &= I_t^{m-\mu} \frac{\partial^m f(x,t)}{\partial t^m}, \quad m-1 < \mu < m,\ m \in \mathbb{N}. \end{aligned}$$

Definition 2.6. Two parameter Mittag–Leffler function is defined as [18,19]

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\beta + \alpha k)}.$$

Note that

$$E_{\alpha,1}(z) = E_\alpha(z)$$

and

$$E_{1,1}(z) = \mathrm{Exp}(z).$$

Definition 2.7. Multivariate Mittag–Leffler function is defined as [17,18]

$$E_{(a_1,a_2,\ldots,a_n),b}(z_1,z_2,\ldots,z_n)=\sum_{k=0}^{\infty}\sum_{l_1+l_2+\cdots+l_n=k}(k;l_1,\ldots,l_n)\frac{\prod_{i=1}^{n} z_i^{l_i}}{\Gamma(b+\sum_{i=1}^{n} a_i l_i)},$$

where $l_1 \geqslant 0,\ldots,l_n \geqslant 0$ and multinomial coefficient

$$(k;l_1,\ldots,l_n)=\frac{k!}{l_1!l_2!\cdots l_n!}.$$

Theorem 2.1. *(See [17,18].) Let $\mu>0$, $m-1<\mu\leqslant m$, $m\in\mathbb{N}_0=\mathbb{N}\cup\{0\}$, $\lambda\in\mathbb{R}$. The initial value problem (IVP)*

$$\begin{aligned}&\big(D_x^{\mu} y\big)(x)-\lambda y(x)=g(x),\\ &y^{(k)}(0)=c_k\in\mathbb{R},\quad k=0,\ldots,m-1,\end{aligned}\tag{2.1}$$

where the function g is assumed to lie in C_{-1} if $\mu\in\mathbb{N}$, in C_{-1}^{1} if $\mu\notin\mathbb{N}$, and the unknown function $y(x)$ is to be determined in the space C_{-1}^{m}, has a solution, unique in the space C_{-1}^{m}, of the form

$$y=y_g+y_h.\tag{2.2}$$

Here y_g is a solution of the IVP (2.1) with zero initial conditions and is represented in the form

$$y_g(x)=\int_0^x t^{\mu-1}E_{\mu,\mu}\big(\lambda t^{\mu}\big)g(x-t)\,dt\tag{2.3}$$

and

$$y_h(x)=\sum_{k=0}^{m-1} c_k x^k E_{\mu,k+1}\big(\lambda x^{\mu}\big)\tag{2.4}$$

is a solution of the homogeneous part of Eq. (2.1) with the given initial conditions.

Theorem 2.2. *(See [17,18].) Let $\mu>\mu_1>\cdots>\mu_n\geqslant 0$, $m_i-1<\mu_i\leqslant m_i$, $m_i\in\mathbb{N}_0=\mathbb{N}\cup\{0\}$, $\lambda_i\in\mathbb{R}$, $i=1,\ldots,n$. The IVP*

$$\begin{aligned}&\big(D_x^{\mu} y\big)(x)-\sum_{i=1}^{n}\lambda_i\big(D_x^{\mu_i} y\big)(x)=g(x),\\ &y^{(k)}(0)=c_k\in\mathbb{R},\quad k=0,\ldots,m-1,\ m-1<\mu\leqslant m,\end{aligned}\tag{2.5}$$

where the function g is as in Theorem 2.1 above, has a solution, unique in the space C_{-1}^{m}, of the form

$$y(x)=y_g(x)+\sum_{k=0}^{m-1}c_k u_k(x),\quad x\geqslant 0.\tag{2.6}$$

Here

$$y_g(x)=\int_0^x t^{\mu-1}E_{(\mu-\mu_1,\ldots,\mu-\mu_n),\mu}\big(\lambda_1 t^{\mu-\mu_1},\ldots,\lambda_n t^{\mu-\mu_n}\big)g(x-t)\,dt\tag{2.7}$$

is a solution of the IVP (2.5) with zero initial conditions, and the system of functions

$$u_k(x)=\frac{x^k}{k!}+\sum_{i=l_k+1}^{n}\lambda_i x^{k+\mu-\mu_i}E_{(\mu-\mu_1,\ldots,\mu-\mu_n),k+1+\mu-\mu_i}\big(\lambda_1 x^{\mu-\mu_1},\ldots,\lambda_n x^{\mu-\mu_n}\big)\tag{2.8}$$

fulfills the initial conditions $u_k^{(l)}(0)=\delta_{kl}$, $k,l=0,\ldots,m-1$. The natural numbers l_k, $k=0,\ldots,m-1$, are determined from the conditions $m_{l_k}\geqslant k+1$, $m_{l_k+1}\leqslant k$.

Theorem 2.3. *Let $f\in C_{-1}^{m}$, $m\in\mathbb{N}$ and $m-1<\mu\leqslant m$. Then the Riemann–Liouville and the Caputo fractional derivatives are connected by the relation* [17–19]

$${}^{R}D_t^{\mu} f(x,t)=D_t^{\mu} f(x,t)+\sum_{k=0}^{m-1}\frac{\partial^k f}{\partial t^k}(x,0)\frac{t^{k-\mu}}{\Gamma(1+k-\mu)},\quad t>0.$$

Hence the following fractional diffusion equation involving Caputo derivative

$$D_t^{\mu} f(x,t) = \frac{\partial^2 f(x,t)}{\partial x^2}, \quad 0 < \mu < 1,$$

is equivalent to the fractional diffusion equation involving Riemann–Liouville derivative

$${}^R D_t^{\mu} f(x,t) - \frac{t^{-\mu}}{\Gamma(1-\mu)} f(x,0) = \frac{\partial^2 f(x,t)}{\partial x^2}, \quad 0 < \mu < 1.$$

3. Multi-term homogeneous fractional BVPs

3.1. Type I

In the present section we consider the following multi-term homogeneous fractional differential equation:

$$P(D)u(x,t) = k\frac{\partial^2 u(x,t)}{\partial x^2}, \quad 0 < x < \pi,\ t > 0, \tag{3.1}$$

along with the boundary conditions

$$u(0,t) = u(\pi,t) = 0, \quad t \geqslant 0, \tag{3.2}$$

$$u(x,0) = f(x), \quad 0 < x < \pi, \tag{3.3}$$

where

$$P(D) = D_t^{\mu} - \sum_{i=1}^{r-1} \lambda_i D_t^{\mu_i}, \quad 0 < \mu_{r-1} < \mu_{r-2} < \cdots < \mu_1 < \mu \leqslant 1,$$

k and λ_i are constants. Assume $u(x,t) = X(x)T(t)$, then (3.1) along with conditions (3.2) yield

$$X''(x) + \theta X(x) = 0, \qquad X(0) = X(\pi) = 0, \tag{3.4}$$

and

$$\big(P(D) + \theta k\big)T(t) = 0, \tag{3.5}$$

where θ is a separation constant. The Sturm–Liouville problem given by (3.4) has eigenvalues $\theta_n = n^2$ and the corresponding eigenfunctions $X_n(x) = \sin nx$ $(n = 1, 2, \ldots)$. Thus (3.6) takes the form

$$\big(P(D) + n^2 k\big)T(t) = 0 \quad (n = 1, 2, \ldots). \tag{3.6}$$

For any fixed positive integer n, the solution of (3.6) is (except for a constant factor) (cf. Theorem 2.2) $T_n(t) = 1 - n^2 k t^{\mu} E_{(\mu-\mu_1,\ldots,\mu-\mu_{r-1},\mu),1+\mu}(\lambda_1 t^{\mu-\mu_1}, \ldots, \lambda_{r-1} t^{\mu-\mu_{r-1}}, -n^2 k t^{\mu})$. The formal solution of the boundary value problem is, therefore

$$u(x,t) = \sum_{n=1}^{\infty} B_n \sin nx\big(1 - n^2 k t^{\mu} E_{(\mu-\mu_1,\ldots,\mu-\mu_{r-1},\mu),1+\mu}\big(\lambda_1 t^{\mu-\mu_1}, \ldots, \lambda_{r-1} t^{\mu-\mu_{r-1}}, -n^2 k t^{\mu}\big)\big), \tag{3.7}$$

where the coefficients B_n need to be determined so that

$$f(x) = u(x,0) = \sum_{n=1}^{\infty} B_n \sin nx, \quad 0 \leqslant x \leqslant \pi. \tag{3.8}$$

In view of (3.8)

$$B_n = \frac{2}{\pi} \int_0^{\pi} f(x) \sin nx \, dx \quad (n = 1, 2, \ldots). \tag{3.9}$$

Hence the solution of fractional boundary value problem (3.1)–(3.3) is given by (3.7) where B_n's are as given in (3.9).

3.2. Type II

Consider the multi-term homogeneous fractional differential equation

$$P(D)u(x,t) = k\frac{\partial^2 u(x,t)}{\partial x^2}, \quad 0 < x < \pi,\ t > 0, \tag{3.10}$$

along with the boundary conditions

$$\begin{aligned} &u(0,t) = u(\pi,t) = 0, \quad t \geqslant 0,\\ &u(x,0) = f(x), \quad 0 < x < \pi,\\ &u_t(x,0) = g(x), \quad 0 < x < \pi, \end{aligned} \tag{3.11}$$

where

$$P(D) = D_t^{\mu} - \sum_{i=1}^{r-1} \lambda_i D_t^{\mu_i}, \quad 1 < \mu_{r-1} < \mu_{r-2} < \cdots < \mu_1 < \mu \leqslant 2,$$

k and λ_i are constants. Assume $u(x,t) = X(x)T(t)$, then (3.10) along with conditions (3.11) yield

$$X''(x) + \theta X(x) = 0, \qquad X(0) = X(\pi) = 0, \tag{3.12}$$

and

$$\big(P(D) + \theta k\big)T(t) = 0, \tag{3.13}$$

where θ is a separation constant. The Sturm–Liouville problem given by (3.12) has eigenvalues $\theta_n = n^2$ and the corresponding eigenfunctions $X_n(x) = \sin nx$ $(n = 1, 2, \ldots)$. Thus (3.13) takes the form:

$$\big(P(D) + n^2 k\big)T(t) = 0 \quad (n = 1, 2, \ldots). \tag{3.14}$$

For any fixed positive integer n, the general solution of (3.14) is (cf. Theorem 2.2) $T_n(t) = A_n T_{0n} + B_n T_{1n}$, where

$$T_{0n} = 1 - n^2 k t^{\mu} E_{(\mu-\mu_1,\ldots,\mu-\mu_{r-1},\mu),1+\mu}\big(\lambda_1 t^{\mu-\mu_1}, \ldots, \lambda_{r-1} t^{\mu-\mu_{r-1}}, -n^2 k t^{\mu}\big), \tag{3.15}$$

$$T_{1n} = t - n^2 k t^{1+\mu} E_{(\mu-\mu_1,\ldots,\mu-\mu_{r-1},\mu),2+\mu}\big(\lambda_1 t^{\mu-\mu_1}, \ldots, \lambda_{r-1} t^{\mu-\mu_{r-1}}, -n^2 k t^{\mu}\big) \tag{3.16}$$

are independent solutions of (3.14) satisfying $T_{in}^{(j)}(0) = \delta_{ij}$, $i, j = 1, 2$. The general solution of the boundary value problem is, therefore

$$u(x,t) = \sum_{n=1}^{\infty} (A_n T_{0n} + B_n T_{1n}) \sin nx, \tag{3.17}$$

where the coefficients A_n and B_n need to be determined so that

$$f(x) = u(x,0) = \sum_{n=1}^{\infty} A_n \sin nx, \quad 0 \leqslant x \leqslant \pi, \tag{3.18}$$

$$g(x) = u_t(x,0) = \sum_{n=1}^{\infty} B_n \sin nx, \quad 0 \leqslant x \leqslant \pi. \tag{3.19}$$

In view of (3.18) and (3.19)

$$\begin{aligned} A_n &= \frac{2}{\pi}\int_0^{\pi} f(x) \sin nx\, dx \quad (n = 1, 2, \ldots),\\ B_n &= \frac{2}{\pi}\int_0^{\pi} g(x) \sin nx\, dx \quad (n = 1, 2, \ldots). \end{aligned} \tag{3.20}$$

The solution of fractional boundary value problem (3.10)–(3.11) is (3.17), where A_n and B_n's are as given in (3.20).

3.3. Type III

Consider

$$P(D)u(x,t)=K\frac{\partial^2 u(x,t)}{\partial x^2},\quad 0<x<\pi,\ t>0, \tag{3.21}$$

along with the boundary conditions

$$\begin{aligned}
&u(0,t)=u(\pi,t)=0,\quad t\geqslant 0,\\
&u(x,0)=f(x),\quad 0<x<\pi,\\
&u_t(x,0)=g(x),\quad 0<x<\pi,
\end{aligned} \tag{3.22}$$

where

$$P(D)=D_t^{\mu}-\sum_{i=1}^{r-1}\lambda_i D_t^{\mu_i},\quad 0<\mu_{r-1}<\cdots<\mu_{r-k}\leqslant 1<\mu_{r-k-1}<\cdots<\mu_1<\mu\leqslant 2,$$

$1\leqslant k\leqslant r-1$, K and λ_i are constants. Assume $u(x,t)=X(x)T(t)$, then (3.21) along with conditions (3.22) yield

$$X''(x)+\theta X(x)=0,\qquad X(0)=X(\pi)=0, \tag{3.23}$$

$$\big(P(D)+\theta K\big)T(t)=0, \tag{3.24}$$

where θ is a separation constant. The Sturm–Liouville problem given by (3.23) has eigenvalues $\theta_n=n^2$ and the corresponding eigenfunctions $X_n(x)=\sin nx$ $(n=1,2,\ldots)$. Thus (3.24) takes the form

$$\big(P(D)+n^2K\big)T(t)=0\quad (n=1,2,\ldots). \tag{3.25}$$

For any fixed positive integer n, the general solution of (3.25) is (cf. Theorem 2.2) $T_n(t)=A_nT_{0n}+B_nT_{1n}$, where

$$\begin{aligned}
T_{0n}&=1-n^2Kt^{\mu}E_{(\mu-\mu_1,\ldots,\mu-\mu_{r-1},\mu),1+\mu}\big(\lambda_1t^{\mu-\mu_1},\ldots,\lambda_{r-1}t^{\mu-\mu_{r-1}},-n^2Kt^{\mu}\big),\\
T_{1n}&=t+\sum_{i=r-k}^{r-1}\lambda_i t^{1+\mu-\mu_i}E_{(\mu-\mu_1,\ldots,\mu-\mu_{r-1},\mu),2+\mu-\mu_i}\big(\lambda_1t^{\mu-\mu_1},\ldots,\lambda_{r-1}t^{\mu-\mu_{r-1}},-n^2Kt^{\mu}\big)\\
&\quad-n^2Kt^{1+\mu}E_{(\mu-\mu_1,\ldots,\mu-\mu_{r-1},\mu),2+\mu}\big(\lambda_1t^{\mu-\mu_1},\ldots,\lambda_{r-1}t^{\mu-\mu_{r-1}},-n^2Kt^{\mu}\big)
\end{aligned}$$

are independent solutions of (3.25) satisfying $T_{in}^{(j)}(0)=\delta_{ij}$, $i,j=1,2$. The general solution of the boundary value problem is, therefore

$$u(x,t)=\sum_{n=1}^{\infty}(A_nT_{0n}+B_nT_{1n})\sin nx. \tag{3.26}$$

Using conditions (3.22) we can observe that the coefficients A_n and B_n are as given in (3.20).

4. Non-homogeneous case

Present section deals with the non-homogeneous fractional differential equation

$$P(D)u(x,t)=k\frac{\partial^2 u(x,t)}{\partial x^2}+q(t),\quad 0<x<\pi,\ t>0, \tag{4.1}$$

along with the boundary conditions

$$\begin{aligned}
&u(0,t)=u(\pi,t)=0,\quad t\geqslant 0,\\
&u(x,0)=f(x),\quad 0<x<\pi,\\
&u_t(x,0)=g(x),\quad 0<x<\pi,
\end{aligned} \tag{4.2}$$

where

$$P(D)=D_t^{\mu}-\sum_{i=1}^{r-1}\lambda_i D_t^{\mu_i},\quad 1<\mu_{r-1}<\mu_{r-2}<\cdots<\mu_1<\mu\leqslant 2,$$

k and λ_i denote constant coefficients, $q(t)$ is assumed to be a continuous function of t. Since (4.1) is non-homogeneous, we use the method of variation of parameters [21], where in first the corresponding homogeneous equation is solved (putting $q(t) \equiv 0$ in (4.1)), together with the boundary conditions, by separation of variables method. Assume $u(x,t) = X(x)T(t)$, then (4.1) along with conditions (4.2) yield

$$X''(x) + \theta X(x) = 0, \qquad X(0) = X(\pi) = 0, \tag{4.3}$$

$$\big(P(D) + \theta k\big)T(t) = 0 \quad (n = 1, 2, \ldots) \tag{4.4}$$

where θ is a separation constant. The Sturm–Liouville problem given by (4.3) has eigenvalues $\theta_n = n^2$ and the corresponding eigenfunctions $X_n(x) = \sin nx$ $(n = 1, 2, \ldots)$. For any fixed positive integer n, (4.4) is solved in Section 3.2. Now we seek a solution of the non-homogeneous problem which is of the form

$$u(x,t) = \sum_{n=1}^{\infty} B_n(t) \sin nx. \tag{4.5}$$

We assume that the series (4.5) can be differentiated term by term and note [21]

$$1 = \sum_{n=1}^{\infty} \frac{2[1-(-1)^n]}{n\pi} \sin nx, \quad 0 < x < \pi. \tag{4.6}$$

Hence, in view of (4.1), we get

$$\sum_{n=1}^{\infty} \big[P(D)B_n(t) + n^2 k B_n(t)\big] \sin nx = \sum_{n=1}^{\infty} \frac{2[1-(-1)^n]}{n\pi} q(t) \sin nx. \tag{4.7}$$

By identifying the coefficients in the sine series on each side of (4.7), we get

$$P(D)B_n(t) + n^2 k B_n(t) = \frac{2[1-(-1)^n]}{n\pi} q(t), \quad n = 1, 2, \ldots. \tag{4.8}$$

This non-homogeneous equation has general solution (cf. Theorem 2.2)

$$B_n(t) = y_q(t) + a_n y_{0n}(t) + b_n y_{1n}(t), \tag{4.9}$$

where

$$y_q(t) = \frac{2[1-(-1)^n]}{n\pi} \int_0^t \tau^{\mu-1} E_{(\mu-\mu_1,\ldots,\mu-\mu_{r-1},\mu),\mu}\big(\lambda_1 \tau^{\mu-\mu_1}, \ldots, \lambda_{r-1}\tau^{\mu-\mu_{r-1}}, -n^2 k \tau^{\mu}\big) q(t-\tau)\, d\tau \tag{4.10}$$

is solution of (4.8) with zero initial conditions, y_{0n} and y_{1n} are two independent solutions of homogeneous part of (4.8) having the same expressions as T_{0n} and T_{1n}, respectively, as given in (3.15)–(3.16). Observe that

$$B_n(0) = y_q(0) + a_n y_{0n}(0) + b_n y_{1n}(0) = a_n, \tag{4.11}$$

$$B_n'(0) = y_q'(0) + a_n y_{0n}'(0) + b_n y_{1n}'(0) = b_n. \tag{4.12}$$

The boundary conditions (4.2) yield

$$\sum_{n=1}^{\infty} B_n(0) \sin(nx) = f(x), \tag{4.13}$$

$$\sum_{n=1}^{\infty} B_n'(0) \sin(nx) = g(x), \tag{4.14}$$

From (4.11)–(4.14) it is clear that a_n and b_n have same expressions as A_n and B_n respectively given in Section 3.2.

5. Multi-term fractional differential equation in higher dimensions

Consider the fractional differential equation

$$P(D)u = a^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right), \quad 0 < x,\ y < \pi,\ t > 0, \tag{5.1}$$

along with the following boundary conditions:

$$u(x,0,t)=u(x,\pi,t)=u(0,y,t)=u(\pi,y,t)=0,\quad t\geqslant 0,$$
$$u(x,y,0)=f(x,y),\quad 0\leqslant x,y\leqslant\pi,$$
$$u_t(x,y,0)=g(x,y),\quad 0\leqslant x,y\leqslant\pi, \tag{5.2}$$

where 'a' denotes a constant coefficient, and

$$P(D)=D_t^{\mu}-\sum_{i=1}^{r-1}\lambda_i D_t^{\mu_i},\quad 1<\mu_{r-1}<\mu_{r-2}<\cdots<\mu_1<\mu\leqslant 2.$$

We assume that the partial derivatives $f_x(x,y)$, $f_y(x,y)$, $g_x(x,y)$ and $g_y(x,y)$ are also continuous. Substituting $U=X(x)Y(y)T(t)$ in (5.1), we get

$$\frac{P(D)T(t)}{a^2T(t)}=\frac{X''(x)}{X(x)}+\frac{Y''(y)}{Y(y)}=-\theta, \tag{5.3}$$

where θ is a separation constant. Eq. (5.3) implies

$$\frac{Y''(y)}{Y(y)}=-\theta-\frac{X''(x)}{X(x)}=-\xi, \tag{5.4}$$

where ξ is another separation constant. In view of (5.1) and (5.2), we get

$$X''(x)+(\theta-\xi)X(x)=0,\qquad X(0)=0,\qquad X(\pi)=0, \tag{5.5}$$

and

$$Y''(y)+\xi Y(y)=0,\qquad Y(0)=0,\qquad Y(\pi)=0. \tag{5.6}$$

(5.3) gives

$$P(D)T(t)+\theta a^2T(t)=0. \tag{5.7}$$

The Sturm–Liouville problem given in (5.6) has eigenvalues $\xi_m=m^2$ $(m=1,2,\ldots)$ and the corresponding eigenfunctions are $Y_m(y)=\sin my$. Similarly the Sturm–Liouville problem given in (5.5) has eigenvalues $\theta_n-\xi_n=n^2$ $(n=1,2,\ldots)$ and the corresponding eigenfunctions are $X_n(x)=\sin nx$. Thus (5.7) takes the form:

$$P(D)T(t)+a^2\big(m^2+n^2\big)T(t)=0,\quad m=1,2,\ldots,\ n=1,2,\ldots, \tag{5.8}$$

which is equivalent to (3.13). For any fixed positive integers m and n, the general solution of (5.8) is therefore

$$T_{mn}(t)=A_{mn}v_{mn}(t)+B_{mn}w_{mn}(t),$$

where A_{mn}, B_{mn} are arbitrary constants and $v_{mn}(t)$, $w_{mn}(t)$ are independent solutions of (5.8) given by (cf. Theorem 2.2)

$$v_{mn}(t)=1-a^2\big(m^2+n^2\big)t^{\mu}E_{(\mu-\mu_1,\ldots,\mu-\mu_{r-1},\mu),1+\mu}\big(\lambda_1t^{\mu-\mu_1},\ldots,\lambda_{r-1}t^{\mu-\mu_{r-1}},-a^2\big(m^2+n^2\big)t^{\mu}\big),$$
$$w_{mn}(t)=t-a^2\big(m^2+n^2\big)t^{1+\mu}E_{(\mu-\mu_1,\ldots,\mu-\mu_{r-1},\mu),2+\mu}\big(\lambda_1t^{\mu-\mu_1},\ldots,\lambda_{r-1}t^{\mu-\mu_{r-1}},-a^2\big(m^2+n^2\big)t^{\mu}\big).$$

The solution of the boundary value problem is, therefore

$$u(x,y,t)=\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\sin nx\sin my\,T_{mn}(t). \tag{5.9}$$

The boundary conditions (5.2) yield

$$A_{mn}=\frac{4}{\pi^2}\int_0^{\pi}\sin my\int_0^{\pi}f(x,y)\sin nx\,dx\,dy, \tag{5.10}$$

$$B_{mn}=\frac{4}{\pi^2}\int_0^{\pi}\sin my\int_0^{\pi}g(x,y)\sin nx\,dx\,dy. \tag{5.11}$$

6. Non-homogeneous boundary conditions

In the present section we consider the following homogeneous multi-term fractional differential equation:

$$P(D)u = k\frac{\partial^2 u}{\partial x^2}, \quad 0 < x < 1,\ t > 0, \tag{6.1}$$

along with the non-homogeneous boundary conditions

$$u(0,t) = 0, \qquad Ku_x(1,t) = A, \quad t > 0, \tag{6.2}$$

$$u(x,0) = 0, \qquad u_t(x,0) = 0, \quad 0 < x < 1, \tag{6.3}$$

where

$$P(D) = D_t^{\mu} - \sum_{i=1}^{r-1} \lambda_i D_t^{\mu_i}, \quad 1 < \mu_{r-1} < \mu_{r-2} < \cdots < \mu_1 < \mu \leqslant 2.$$

Substituting $u(x,t) = U(x,t) + \Phi(x)$ in (6.1)–(6.3), we obtain

$$P(D)U = k\left[\frac{\partial^2 U}{\partial x^2} + \Phi''(x)\right], \quad 0 < x < 1,\ t > 0, \tag{6.4}$$

$$U(0,t) + \Phi(0) = 0, \qquad K\left[U_x(1,t) + \Phi'(1)\right] = A, \qquad U(x,0) + \Phi(x) = 0, \qquad U_t(x,0) = 0. \tag{6.5}$$

Assume $\Phi''(x) = 0$ and $\Phi(0) = 0$, $K\Phi'(1) = A$. Hence $\Phi(x) = \frac{A}{K}x$. The boundary value problem (6.4)–(6.5) now reduces to the homogeneous case

$$P(D)U = k\frac{\partial^2 U}{\partial x^2} \quad (0 < x < 1,\ t > 0), \tag{6.6}$$

$$U(0,t) = 0, \qquad U_x(1,t) = 0, \qquad U(x,0) = -\frac{A}{K}x, \qquad U_t(x,0) = 0. \tag{6.7}$$

Substituting $U = X(x)T(t)$ in (6.6) and using conditions (6.7), we get

$$X'' + \theta X = 0, \qquad X(0) = 0, \qquad X'(1) = 0, \tag{6.8}$$

$$P(D)T(t) + k\theta T(t) = 0, \qquad T'(0) = 0, \tag{6.9}$$

where θ is separation constant. The Sturm–Liouville problem given in (6.8) has eigenvalues $\theta_m = m^2$ $(m = 1, 2, \ldots)$ and the corresponding eigenfunctions are $X_m(x) = \sin(\frac{2m-1}{2}\pi x)$. Eq. (6.9) is equivalent to Eq. (3.14) and has solution

$$T_m(t) = 1 - k\frac{(2m-1)^2\pi^2}{4} t^{\mu} E_{(\mu-\mu_1,\ldots,\mu-\mu_{r-1},\mu),1+\mu}\left(\lambda_1 t^{\mu-\mu_1}, \ldots, \lambda_{r-1} t^{\mu-\mu_{r-1}}, -k\frac{(2m-1)^2\pi^2}{4} t^{\mu}\right) \tag{6.10}$$

up to an arbitrary constant. Thus the boundary value problem (6.6)–(6.7) has solution

$$U(x,t) = \sum_{m=1}^{\infty} A_m T_m(t) \sin\frac{(2m-1)\pi x}{2}, \tag{6.11}$$

where A_m is an arbitrary constant to be determined so that $U(x,0) = -\frac{A}{K}x$. Thus

$$A_m = -2\int_0^1 \frac{A}{K} x \sin\frac{(2m-1)\pi x}{2}\, dx = \frac{(-1)^m 8A}{K(2m-1)^2\pi^2}. \tag{6.12}$$

Since $u(x,t) = U(x,t) + \Phi(x)$, the boundary value problem (6.1)–(6.3) has solution

$$u(x,t) = \frac{A}{K}\left[x + 8\sum_{m=1}^{\infty} \frac{(-1)^m}{(2m-1)^2\pi^2} T_m(t) \sin\frac{(2m-1)\pi x}{2}\right]. \tag{6.13}$$

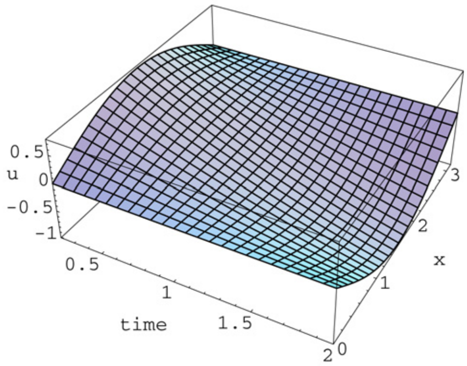

Fig. 1. Example 1.

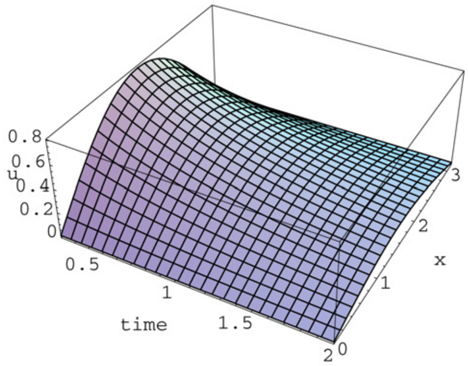

Fig. 2. Example 2

7. Illustrative examples

Example 1. Consider the fractional homogeneous differential equation

$$\begin{aligned} &\left(D_t^{0.9} - \lambda_1 D_t^{0.2}\right)u = \frac{\partial^2 u}{\partial x^2}, \quad 0 < x < \pi,\ t > 0, \\ &u(0,t) = u(\pi,t) = 0, \quad t \geqslant 0, \\ &u(x,0) = \sin(x), \quad 0 < x < \pi. \end{aligned} \tag{7.1}$$

By virtue of (3.7) solution of this equation is

$$u(x,t) = \sum_{n=1}^{\infty} B_n\left(1 - n^2 t^{0.9} E_{(0.7,0.9),1.9}\left(\lambda_1 t^{0.7}, -n^2 t^{0.9}\right)\right)\sin(nx), \quad n = 1, 2, \ldots,$$

where

$$B_n = \frac{2}{\pi}\int_0^{\pi} \sin x \sin nx\, dx \quad (n = 1, 2, \ldots).$$

Hence $B_1 = 1$ and $B_n = 0$ $(n = 2, 3, \ldots)$, and

$$\begin{aligned} u(x,t) &= \left(1 - t^{0.9} E_{(0.7,0.9),1.9}\left(\lambda_1 t^{0.7}, -t^{0.9}\right)\right)\sin(x) \\ &= \sin(x)\left(1 - t^{0.9}\sum_{k=0}^{\infty}\sum_{m=0}^{k} \frac{\lambda_1^m (-1)^{k-m} t^{0.9k-0.2m}}{\Gamma(1 + 0.9 + 0.9k - 0.2m)}\right) \\ &= \sin(x)\left(1 - t^{0.9}\sum_{k=0}^{\infty}\left(\frac{(-1)^k t^{0.9k}}{\Gamma(0.9(k+1)+1)} + \sum_{m=1}^{k} \frac{\lambda_1^m (-1)^{k-m} t^{0.9k-0.2m}}{\Gamma(1 + 0.9 + 0.9k - 0.2m)}\right)\right). \end{aligned} \tag{7.2}$$

$u(x,t)$ given in (7.2) for the case $\lambda_1 = 1$ is plotted in Fig. 1.

Example 2. Consider the fractional homogeneous differential equation

$$\begin{aligned} &\left(D_t^{0.9}\right)u = \frac{\partial^2 u}{\partial x^2}, \quad 0 < x < \pi,\ t > 0, \\ &u(0,t) = u(\pi,t) = 0, \quad t \geqslant 0, \\ &u(x,0) = \sin(x), \quad 0 < x < \pi. \end{aligned} \tag{7.3}$$

Putting $\lambda = 0$ in (7.2) we get the solution of BVP (7.3) as

$$u(x,t) = \sin(x)\left(1 - t^{0.9}\sum_{k=0}^{\infty} \frac{(-1)^k t^{0.9k}}{\Gamma(0.9(k+1)+1)}\right) = \sin(x)\sum_{k=0}^{\infty}\frac{(-t^{0.9})^k}{\Gamma(1+0.9k)} = \sin(x) E_{0.9}\left(-t^{0.9}\right). \tag{7.4}$$

$u(x,t)$ is plotted in Fig. 2.

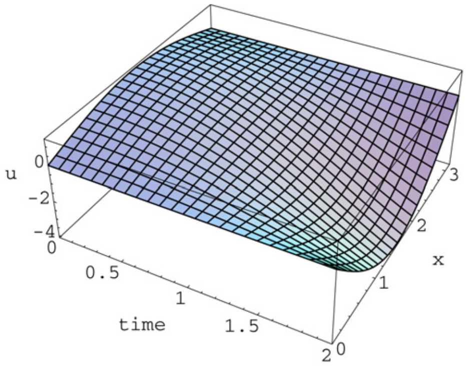

Fig. 3. Example 3: $\mu = 1.9$, $\mu_1 = 1.4$.

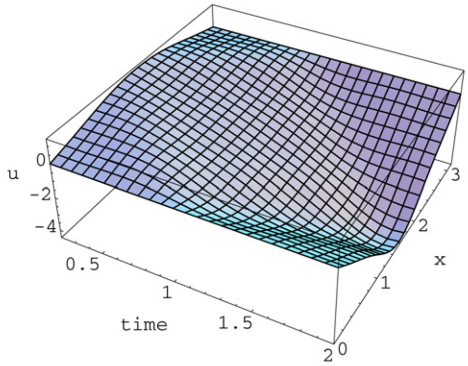

Fig. 4. Example 3: $\mu = 1.8$, $\mu_1 = 1.5$.

Comment. Solution of Example 2 is non-negative as expected (cf. Fig. 2) whereas two-term equation in Example 1 has solution which is not non-negative (cf. Fig. 1).

Example 3. Consider the following two-term homogeneous fractional differential equation along with the boundary conditions given below

$$
\begin{aligned}
&\left(D_t^{\mu} - D_t^{\mu_1}\right)u = \frac{\partial^2 u}{\partial x^2}, \quad 1 < \mu_1 < \mu \leqslant 2,\ t > 0,\\
&u(0,t) = u(\pi,t) = 0, \quad t \geqslant 0,\\
&u(x,0) = f(x) = \frac{\pi}{2} - \left|\frac{\pi}{2} - x\right|, \quad 0 < x < \pi,\\
&u_t(x,0) = 0, \quad 0 < x < \pi.
\end{aligned}
$$

This has solution (cf. Section 3.2)

$$
u(x,t) = \sum_{n=1}^{\infty} \frac{4}{n^2\pi} \sin\left(\frac{n\pi}{2}\right)\left(1 - n^2 t^{\mu} E_{(\mu-\mu_1,\mu),1+\mu}\left(t^{\mu-\mu_1}, -n^2 t^{\mu}\right)\right)\sin(nx).
$$

In Figs. 3 and 4 $u(x,t)$ is plotted for different values of μ and μ_1.

Example 4. Consider fractional differential equation

$$
\begin{aligned}
&\left(D_t^{1.3} - D_t^{0.7}\right)u = \frac{\partial^2 u}{\partial x^2}, \quad 0 < x < \pi,\ t > 0,\\
&u(0,t) = u(\pi,t) = 0, \quad t \geqslant 0,\\
&u(x,0) = 2\pi - |2\pi - 4x|, \qquad u_t(x,0) = x^2, \quad 0 < x < \pi.
\end{aligned}
$$

In view of (3.26) solution of given boundary value problem is

$$
\begin{aligned}
u(x,t) = \sum_{n=1}^{\infty} \sin nx \Bigg[&\frac{16}{n^2\pi} \sin\left(\frac{n\pi}{2}\right)\left(1 - n^2 t^{1.3} E_{(0.6,1.3),2.3}\left(t^{0.6}, -n^2 t^{1.3}\right)\right)\\
&+ \frac{2}{n^3\pi}\left(-2 + \left(2 - n^2\pi^2\right)\cos n\pi\right)\left(t + t^{1.6} E_{(0.6,1.3),2.6}\left(t^{0.6}, -n^2 t^{1.3}\right) - n^2 t^{2.3} E_{(0.6,1.3),3.3}\left(t^{0.6}, -n^2 t^{1.3}\right)\right)\Bigg].
\end{aligned}
$$

Example 5. Consider fractional non-homogeneous differential equation

$$
\begin{aligned}
&\left(D_t^{1.7} - D_t^{1.2}\right)u = \frac{\partial^2 u}{\partial x^2} + t^2, \quad 0 < x < \pi,\ t > 0,\\
&u(x,0) = \frac{\pi}{2} - \left|\frac{\pi}{2} - x\right|, \quad 0 < x < \pi,\\
&u_t(x,0) = 0, \quad 0 < x < \pi.
\end{aligned}
$$

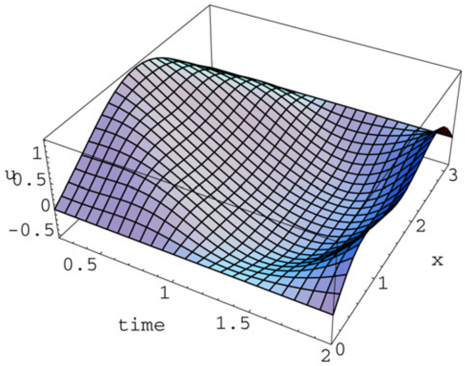

Fig. 5. Example 5.

This boundary value problem has solution (cf. Section 3.2)

$$u(x,t)=\sum_{n=1}^{\infty}\sin nx\left[\frac{2[1-(-1)^n]}{n\pi}\int_0^t \tau^{0.7}E_{(0.5,1.7),1.7}\left(\tau^{0.5},-n^2\tau^{1.7}\right)(t-\tau)^2\,d\tau + \frac{4}{n^2\pi}\sin\frac{n\pi}{2}\left(1-n^2t^{1.7}E_{(0.5,1.7),2.7}\left(t^{0.5},-n^2t^{1.7}\right)\right)\right].$$

The solution $u(x,t)$ is plotted in Fig. 5.

8. Conclusions

In view of the illustrative examples and Figs. 1–5, it is clear that in multi-term case solutions need not be non-negative. Hence multi-term generalisation of fractional diffusion-wave equation does not represent sub/super diffusion in any dimension.

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