THREE PROBLEMS CONCERNING IDEALS OF DIFFERENTIABLE FUNCTIONS*

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In this paper, we study the validity of the following two statements in the internal logic of the toposes of Synthetic Differential Geometry:

1. The integral of $f$ is non-negative if $f$ is non-negative;
2. If $f=0$ in the set of non-negative reals, and $f=0$ in the set of non-positive reals, then $f=0$.

We find statements (1) and (2) to be true in the toposes considered. We also prove that

3. For $n$ greater than two, the arrow $t^n$ from the line to itself is not a stable effective epic.

This answers a question raised by Quê-Moerdijk-Reyes.

Introduction

We are dealing here with questions of internal mathematics in the models of Synthetic Differential Geometry. Since the models usually considered involve ideals of differentiable functions, the validity of internal statements means the validity of certain properties concerning ideals of differentiable functions. As it happens, even very elementary internal statements may lead to interesting problems concerning ideals of differentiable functions.

In this paper, we show the following two statements to hold in the Dubuc topos $\mathcal{G}$ and in two other smooth toposes $\mathcal{F}$ and $\mathcal{G}$:

1. $\forall f \in R^{[0,1]} \left( f \geq 0 \Rightarrow \left( \begin{array}{c} 1 \\ 0 \end{array} \right) f \geq 0 \right)$;
2. $\forall f \in R \left( f|_{R_{\geq 0}} = 0 \land f|_{R_{<0}} = 0 \right) \Rightarrow f \equiv 0$

($f|_{R_{\geq 0}}$ means $f$ restricted to $R_{\geq 0}$).

On the other hand, we consider a third question:

3. Are the arrows $t^n : R \rightarrow R$ (n odd) and $t^n : R \rightarrow R_{\geq 0}$ (n even) stable effective epics in the sites of definition of the toposes $\mathcal{G}$, $\mathcal{F}$ and $\mathcal{G}$?

In [8], van Quê-Moerdijk-Reyes show this to be true in either of the three sites

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for \( n = 2 \) and raise question (3). The surprising fact is that for \( n > 2 \) this is no longer true in either case. This happens to be particularly interesting in the case of the site of the topos \( \mathcal{S} \). The paper is preceded by a Section 0 where we recall some results and fix the notations.

**Section 0**

Let \( C^\infty(\mathbb{R}^k) \) be the ring of all differentiable (of class \( C^\infty \)) functions \( \mathbb{R}^k \to \mathbb{R} \). The \( C^\infty \)-CO (\( C^\infty \)-compact open) topology in \( C^\infty(\mathbb{R}^k) \) is the topology of uniform convergence on compacts of functions and all derivatives. Let \( I \subseteq C^\infty(\mathbb{R}^k) \) be an ordinary ideal. \( I \) is said to be a closed ideal if \( I \) is a \( C^\infty \)-CO closed subset of \( C^\infty(\mathbb{R}^k) \). By \( \overline{\cdot} \) we denote the closure operator in \( C^\infty(\mathbb{R}^k) \). Thus, \( I \) is closed iff \( \overline{I} = I \). Following Malgrange [7] we say that \( f \in C^\infty(\mathbb{R}^k) \) is pointwise in an ideal \( I \subseteq C^\infty(\mathbb{R}^k) \) iff for every \( x_0 \in \mathbb{R}^k \) there exists \( h \in I \) such that the Taylor series expansion of \( f \) at \( x_0 \) equals the Taylor series expansion of \( h \) at \( x_0 \), i.e. \( T_{x_0}(f) = T_{x_0}(h) \). Concerning closed ideals we have the following well-known theorem by H. Whitney:

**0.1. Theorem** (see [7]). Let \( I \subseteq C^\infty(\mathbb{R}^k) \) be an ideal. A function \( f \) is pointwise in \( I \) iff it is in \( \overline{I} \).

On the other hand, an ideal \( I \subseteq C^\infty(\mathbb{R}^k) \) is said to be of local nature (or of local character, or germ determined) iff for every \( f \in C^\infty(\mathbb{R}^k) \), \( f \in I \) iff there exists an open covering \( \{ U_\alpha \}_\alpha \) of \( \mathbb{R}^k \) such that \( f|_{U_\alpha} \in I|_{U_\alpha} \) (where \( I|_{U_\alpha} = \text{ideal generated in } C^\infty(\mathbb{U}_{\alpha}) \) by \( \{ h|_{U_\alpha} : h \in I \} \) (see [5])). If \( I \subseteq C^\infty(\mathbb{R}^k) \) is any ideal, there exists an ideal \( \overline{I} \) which is the smallest local nature ideal containing \( I \). Following [8] we call
- \( \mathcal{L} \) = category dual to that of finitely generated \( C^\infty \)-rings \( C^\infty(\mathbb{R}^k)/I \) presented by any ideal \( I \).
- \( \mathcal{G} \) = category dual to that of finitely generated \( C^\infty \)-rings \( C^\infty(\mathbb{R}^k)/I \) presented by an ideal \( I \) of local character.
- \( \mathcal{F} \) = category dual to that of finitely generated \( C^\infty \)-rings \( C^\infty(\mathbb{R}^k)/I \), presented by a closed ideal \( I \).

For instance, the typical object of \( \mathcal{L} \) is \( C^\infty(\mathbb{R}^k)/I \), and an arrow \( f : C^\infty(\mathbb{R}^k)/I \to C^\infty(\mathbb{R}^l)/J \) is a \( C^\infty \)-ring morphism \( f : C^\infty(\mathbb{R}^l)/J \to C^\infty(\mathbb{R}^k)/I \), which has to be evaluation at certain \( 'f' \), \( f : \mathbb{R}^k \to \mathbb{R}^l \), such that for every \( h \in J \), \( h \circ f \in I \) (see [5] for details).

**0.2. Example.** Denoting \( R = \overline{C^\infty(\mathbb{R})} \) we have, for each \( n \in \mathbb{N} \), the arrow \( t^n : R \to R \), which corresponds to \( \text{evaluation at } t^n \). This is the arrow mentioned in the introduction.

As usually (see [5, 8]) we equip these categories \( \mathcal{L}, \mathcal{G} \) and \( \mathcal{F} \) with open cover
topologies as described in [8]. Again following [8], we call \( \mathcal{J}, \mathcal{G} \) (the Dubuc topos, also called \( \mathcal{D} \) in [2, 3]) and \( \mathcal{F} \) the corresponding categories of sheaves.

Let us introduce the following notation:

0.3. Notation. (i) Let \( I \subseteq C^\infty(\mathbb{R}^k) \) be any ideal. We denote
\[
Cl_I(I) = I, \quad Cl_G(I) = \tilde{I}, \quad Cl_F(I) = \bar{I}
\]
(also \( Cl_\mathcal{J}(I) = I, Cl_\mathcal{G}(I) = \tilde{I}, Cl_\mathcal{F}(I) = \bar{I} \)). An ideal \( I \) is said to be \( C \)-closed (\( C = \mathbb{L}, \mathbb{S}, \mathbb{F} \) or \( C = \mathcal{J}, \mathcal{G}, \mathcal{F} \)) if \( Cl_C(I) = I \). Thus 'to be \( C \)-closed' means 'to be of local character', 'to be \( \mathcal{F} \)-closed' means 'to be \( C^\infty \)-CO closed' and 'to be \( \mathcal{J} \)-closed' means nothing: every ideal is \( \mathcal{J} \)-closed.

(ii) We think of the functions of \( C^\infty(\mathbb{R}^k) \) and \( C^\infty(\mathbb{R}^{k+l}) \) as functions of the variables \( \bar{x} = (x_1, \ldots, x_k) \) and \( (\bar{x}, \bar{t}) = (x_1, \ldots, x_k, t_1, \ldots, t_l) \) respectively. If \( I \subseteq C^\infty(\mathbb{R}^k) \) is any ideal, we may assume that \( I \subseteq C^\infty(\mathbb{R}^{k+l}) \), since a function of \( \bar{x} \) is also a function of \( (\bar{x}, \bar{t}) \) which does not depend on \( \bar{t} \). Of course, \( I \) is not an ideal of \( C^\infty(\mathbb{R}^{k+l}) \) but it generates an ideal that we call \( I(\bar{x}, \bar{t}) \). With this notation, and the one introduced in point (i), we have that the Cartesian product in the category \( C (C = \mathbb{L}, \mathbb{S}, \mathbb{F}, \mathcal{J}, \mathcal{G}, \mathcal{F}) \) is given by
\[
\frac{C^\infty(\mathbb{R}^k)/I \times C^\infty(\mathbb{R}^l)/J}{C^\infty(\mathbb{R}^{k+l})/Cl_C(I(\bar{x}, \bar{t}) + J(\bar{t}, \bar{x}))).
\]

A usual notation we will use is

0.4. Notation. Let \( X \subseteq \mathbb{R}^k \) be any closed set. \( m_X \) stands for the ideal of all functions of \( C^\infty(\mathbb{R}^k) \) vanishing on \( X \). Similarly, \( m_X^\infty \) is the ideal of all functions of \( C^\infty(\mathbb{R}^k) \) such that \( f \) and all its derivatives vanish on \( X \). Such an \( f \) is said to be flat on \( X \). Thus, \( m_X^\infty = \{ f \in C^\infty(\mathbb{R}^k) : f \text{ is flat on } X \} \).

Let us recall the following lemmas:

0.5. Lemma (see [10]). Let \( f_i (i \in \mathbb{N}) \) be a sequence of functions of \( m_X^\infty \). Then there exists \( g \in m_X^\infty, g > 0 \) in \( \mathbb{R}^k \setminus X \), such that for every \( i \in \mathbb{N}, f_i \in g \cdot m_X^\infty \). □

The following is not more than a version of a well-known lemma by E. Borel:

0.6. Lemma. Given a power series \( S \in C^\infty(\mathbb{R}^k)[[t]], \) say \( S(\bar{x}, t) = \sum_{i=0}^\infty A_i(\bar{x})(t^i/i!) \) for certain \( A_i \in C^\infty(\mathbb{R}^k) \), there exist functions \( \eta_i \in C^\infty(\mathbb{R}) \) such that

(i) \( \eta_i(t) = 1 \) in a neighborhood of \( t = 0 \);

(ii) \( \eta_i \) is of compact support;

(iii) The series \( s(\bar{x}, t) = \sum_{i=0}^\infty A_i(\bar{x}) \eta_i(t)(t^i/i!) \) converges in the \( C^\infty \)-CO topology, and so, \( s(\bar{x}, t) \in C^\infty(\mathbb{R}^{k+1}) \), moreover,
\[
\frac{\partial^{\left| \alpha \right| + j}}{\partial x^\alpha \partial t^j} s(\bar{x}, 0) = \frac{\partial^{\left| \alpha \right|}}{\partial x^\alpha} A_j(\bar{x})
\]
for every multi-index \( \alpha \in (\mathbb{N} \cup \{0\})^k \);

(iv) If needed (it will be in 3.3.3) we can assume the \( \eta_i \) to be even functions, so that \( s(\xi, t) \) will be even in the variable \( t \) if all \( A_i \) with odd \( i \) vanish.

**Proof.** For \( j \in \mathbb{N} \cup \{0\} \), \( \alpha \in (\mathbb{N} \cup \{0\})^k \) consider the function

\[
B_{\alpha,j}(\xi, t) = \frac{\partial^{|\alpha|} A_j}{\partial \xi^\alpha}(\xi) \cdot |t|^{j/2}
\]

which vanishes for \( t = 0 \). By continuity, given \( j \in \mathbb{N} \cup \{0\} \) we can choose \( \varepsilon_j \in \mathbb{R} \), \( 0 < \varepsilon_j < 1 \) such that for every \( (x, t) \in \mathbb{R}^{k+1} \) with \( |x| \leq j \) and \( -\varepsilon_j < t < \varepsilon_j \) and for every \( \alpha \in (\mathbb{N} \cup \{0\})^k \) with \( |\alpha| \leq j \) we have \( |B_{\alpha,j}(\xi, t)| < 1 \).

Now take an even function \( \eta \in C^\infty(\mathbb{R}) \) such that \( \eta([-\frac{1}{2}, \frac{1}{2}]) = 1 \) and \( \eta(t) = 0 \) for \( |t| \geq 1 \) and define \( \eta_j(t) = \eta(t/\varepsilon_j) \). It is easily seen that the series \( S(\xi, t) = \sum_{i=0}^{\infty} A_i(\xi) \eta_i(t) \cdot (t'/i!) \) has the required properties. \( \square \)

Finally we recall a well-known lemma. By the way, we remark that it is this lemma which implies that the congruence associated in the standard way to an ideal \( I \subseteq C^\infty(\mathbb{R}^n) \) is a \( C^\infty \)-ring congruence (see [4]).

0.7. **Lemma** (see [4]). For every \( h \in C^\infty(\mathbb{R}^{k+1}) \) and for every integer \( m \geq 0 \) there exist

\[
h_\alpha \in C^\infty(\mathbb{R}^k), \quad \{ \alpha = (\alpha_1, \ldots, \alpha_l) : \sum \alpha_i \leq m \},
\]

\[
l_\alpha \in C^\infty(\mathbb{R}^{k+1}), \quad \{ \alpha = (\alpha_1, \ldots, \alpha_l) : \sum \alpha_i = m + 1 \}
\]

such that \( h(\xi, \eta) = \sum \alpha h_\alpha(\xi) \xi^\alpha + \sum \alpha l_\alpha(\xi, \eta) \xi^\alpha \). (As usual, we set \( \eta(\xi) = \eta^{(\alpha)} \), \( \eta^{(\alpha)} = \eta(t_0^{(\alpha)}), \ldots, t^{(\alpha)} \), and \( |\alpha| = \alpha_1 + \cdots + \alpha_l \). Of course, \( h_\alpha(\xi) = (1/\alpha!)(\partial^{\alpha}| h/\partial \xi^\alpha)(\xi, 0) \). \( \square \)

We are going to work on certain statements in the logic of the toposes \( \mathcal{Z}, \mathcal{G} \) and \( \mathcal{F} \).

The details about internal logic in a topos may be found in [1, 6].

**Section 1**

In this section we show that in the topos \( C = \mathcal{G}, \mathcal{F}, \mathcal{Z} \), the following formula holds internally:

\[
\forall f \in R^{[0,1]} \left[ f \geq 0 \Rightarrow \left( \int_0^1 f \right) \geq 0 \right]
\]

(1)

where

(i) \( [0, 1] = C^\infty(\mathbb{R})/m_{[0, 1]} \);

(ii) \( \int_0^1 : R^{[0,1]} \to R \) is the arrow described as follows: if \( f : C^\infty(\mathbb{R}^k)/I \to R^{[0,1]} \), \( f \) corresponds to a \( [g] \in C(\mathbb{R}^{k+1})/\text{Cl}\mathcal{C}(I(\xi, t) + m_{[0, 1]}(t, \xi)) \). Then \( \int_0^1 f = \int_0^1 f : \)
$C^\infty(\mathbb{R}^k)/I \to R$ is the arrow corresponding to $[[\int_0^1 g(\xi, t)\,dt]] \in C^\infty(\mathbb{R}^k)/I$. This definition is independent of the choice of $g$, as may be verified;

(iii) Let us call $R_{\geq 0} = C^\infty(\mathbb{R})/m_{\mathbb{R}^\geq 0}$. An arrow $f : X \to R_{[0,1]}$ 'is $\geq 0'$ iff $f$ factors through $R_{[0,1]} \to R_{[0,1]}$.

(iv) Analogously, an arrow $x : X \to R$ 'is $\geq 0'$ iff it factors through $R_{\geq 0} \to R$.

1.1. Lemma. Formula (1) holds in the topos $C (C = \mathcal{Z}, \mathcal{F} or \mathcal{G})$ iff the following condition holds:

(1') For every $C$-closed ideal $I \subseteq C^\infty(\mathbb{R}^k)$ and $f \in C^\infty(\mathbb{R}^k+1)$ we have:

\[ \forall q \in m_{\mathbb{R}^\geq 0}, q(f(\xi, t)) \in Cl_C(I(\xi, t) + m_{[0,1]}(t, \xi)) \]

implies

\[ \forall q \in m_{\mathbb{R}^\geq 0} q\left(\int_0^1 f(\xi, t)\,dt\right) \in I. \]

Proof. Easy. \;

1.2. Theorem. Condition (1') holds, and so, (1) holds internally for $C = \mathcal{Z}, \mathcal{F}, \mathcal{G}$. 

To prove Theorem 1.2 we need the following lemmas:

1.3. Lemma. Let $f_i (i \in \mathbb{N})$ be a sequence of elements of $m_{\mathbb{R}^\geq 0}$. Then, there exists $\phi \in m_{\mathbb{R}^\geq 0}$ such that

(i) $\phi(t) > 0$ for $t < 0$, $\phi'(t) < 0$ for $t < 0$, $\phi''(t) > 0$ for $t < 0$;

(ii) For every $i \in \mathbb{N}$, $f_i \in \phi \cdot m_{\mathbb{R}^\geq 0}$.

Proof. For each $l, m, i \in \mathbb{N}$ write $d^l f_i / dt^l$ as a product of $m$ functions $p_j^{l,m,i} \in m_{\mathbb{R}^\geq 0}$ ($1 \leq j \leq m$), i.e., $d^l f_i / dt^l = \prod_{j=1}^m p_j^{l,m,i}$ (this can be done by Lemma 0.5).

Also by Lemma 0.5, we can get a $\psi \in m_{\mathbb{R}^\geq 0}$, $\psi(t) > 0$ for $t < 0$ such that for all $l, m, i, j$,

\[ \frac{d^2 p_j^{l,m,i}}{dt^2} \in \psi m_{\mathbb{R}^\geq 0}. \]

Now call

\[ \phi(t) = \int_0^t \left( \int_0^u \psi(w) \,dw \right) \,du. \]

This $\phi$ satisfies (i). To see that $\phi$ satisfies (ii) call

\[ g_i(t) = \begin{cases} \frac{f_i(t)}{\phi(t)} & \text{for } t < 0, \\ 0 & \text{otherwise.} \end{cases} \]
We have to show that \( g \in C^\infty(\mathbb{R}) \). It suffices to show that for all \( k \in \mathbb{N} \)
\[
\lim_{t \to 0} \frac{d^k}{dt^k} \left( f_n(t) \phi(t) \right) = 0.
\]

This derivative is a sum of terms of the form \( (f^{(i)}(t) \cdot \phi^{(j)}(t))/\phi(t) \) and this is equal to \( \prod_{j=1}^k (P_j \phi)/\phi(t) \cdot \phi^{(j)}(t) \). Now, each of the quotients \( P_j \phi/\phi \) tends to zero as \( t \) tends to zero, as follows from two applications of L'Hospital rule.

The following lemma is well-known as well as easy to prove:

1.4. Lemma. Let \( \phi \in C^2(\mathbb{R}^2) \) be such that \( \phi'' \geq 0 \) and (say) \( f \in C^0([0,1]) \). Then
\[
\phi\left( \int_0^1 f(t) \, dt \right) \leq \int_0^1 \phi(f(t)) \, dt.
\]

Proof of Theorem 1.2. Take \( Q \in \mathbb{R}^m \) and write \( Q \) and each of its derivatives as a product of \( m \) functions of \( \mathbb{R}^{\approx o} \), for every \( m \in \mathbb{N} \), say, \( Q(t) = \prod_{j=1}^m P_j^m \).

By Lemma 1.3, we can choose \( \phi \in m_{\approx o} \) satisfying Lemma 1.3(i) and such that \( P_j^m \phi_{\approx o} \).

We define \( h : R^k \to R \) by
\[
h(x) = \begin{cases} \phi\left( \int_0^1 f(x, t) \, dt \right) & \text{if } \int_0^1 f(x, t) \, dt < 0, \\ 0 & \text{otherwise.} \end{cases}
\]

(Notice that \( \int_0^1 f(x, t) \, dt < 0 \) implies \( \int_0^1 \phi(f(x, t)) \, dt > 0 \). We assert that \( h \in C^\infty(\mathbb{R}^k) \).

To see this we prove the following sublemma:

Sublemma. If \( \bar{x}_j \to \bar{x}_0 \) as \( j \to \infty \), \( \int_0^1 f(x, t) \, dt < 0 \) and \( \int_0^1 f(x_0, t) \, dt = 0 \). then
\[
\left. \frac{\partial^{(1)}}{\partial \bar{x}^0} \left( \phi\left( \int_0^1 f(x, t) \, dt \right) \right) \right|_{x = \bar{x}_j} \to 0.
\]

Proof. The derivative we are considering is equal to a sum of quotients of the form
\[
\phi^{(1)} \left( \int_0^1 f(x_j, t) \, dt \right) \cdot \phi(x_j)
\]
for a certain function \( A \in C^\infty(\mathbb{R}^k) \). The absolute value of this quotient is, by Lemma 1.4, less than or equal to
\[
\frac{\left| \phi^{(1)} \left( \int_0^1 f(x_j, t) \, dt \right) \cdot \phi(x_j) \right|}{\phi \left( \int_0^1 f(x_j, t) \, dt \right)^m} = \prod_{i=1}^m \left| \frac{P_i^m \left( \int_0^1 f(x_j, t) \, dt \right)}{\phi \left( \int_0^1 f(x_j, t) \, dt \right)^m} \right| \cdot |A(x_j)|
\]
which tends to zero since \( \int_0^1 f(x_j, t) \, dt \) tends to zero.

Let us return to the proof of Theorem 1.2. We had to prove that \( h \in C^\infty(\mathbb{R}^k) \).
But we know that $h$ is $C^{\infty}$ in $G = \{ \bar{x} : \int_{0}^{1} f(\bar{x}, t) dt < 0 \}$, and by the sublemma, any derivative of $h(\bar{x})$ tends to zero as $\bar{x} \in G$ tends to a point of $C^{\infty}$. This shows that $h \in C^{\infty}(\mathbb{R}^{k})$. Now, from the definition of $h$ it follows that $\varrho(\int_{0}^{1} f(\bar{x}, t) dt) = h(\bar{x}) \cdot \int_{0}^{1} \varphi(f(\bar{x}, t)) dt$. And, by hypothesis we know that $\varphi(f(\bar{x}, t)) \in Cl_{C}(I(\bar{x}, t) + m_{[0,1]}(t, \bar{x})) (C = \mathcal{F}, \mathcal{G}, \mathcal{H})$.

The proof now follows easily:

(a) $C = \mathcal{F}$: $\varphi(f(\bar{x}, t)) = \sum_{i} A_{i}(\bar{x}, t) h_{i}(\bar{x}) + \sum_{j} B_{j}(\bar{x}, t) m_{j}(t)$ for certain $h_{j} \in I$, $m_{j} \in m_{[0,1]}$. This shows that $\int_{0}^{1} \varphi(f(\bar{x}, t)) dt = \sum_{i} h_{i}(\bar{x}) \int_{0}^{1} A_{i}(\bar{x}, t) dt \in I$.

(b) $C = \mathcal{G}$: In this case the ideal $I$ is assumed to be of local nature, and our hypothesis say that $\varphi(f(\bar{x}, t))$ is locally a linear combination of elements of $I$ and elements of $m_{[0,1]}$. By compactness of the real interval $[0, 1]$ and using an appropriate partition of unity we see that for every $\bar{x}_{0} \in \mathbb{R}^{k}$ there exists a neighborhood $U$ of $\bar{x}_{0}$ and functions $A_{j}, B_{j} \in C^{\infty}(\mathbb{R}^{k+1}), h_{j} \in I$ and $m_{j} \in m_{[0,1]}$ such that in $U \times [0, 1]$ we have $\varphi(f(\bar{x}, t)) = \sum_{j} A_{j}(\bar{x}, t) h_{j}(\bar{x}) + \sum_{j} B_{j}(\bar{x}, t) \cdot m_{j}(t)$. Then, for $\bar{x} \in U$ we have $\int_{0}^{1} \varphi(f(\bar{x}, t)) dt = \sum_{j} h_{j}(\bar{x}) \int_{0}^{1} A_{j}(\bar{x}, t) dt \in I$, i.e., $\int_{0}^{1} \varphi(f(\bar{x}, t)) dt$ is locally in $I$ and therefore is in $I$, since $I$ is of local character.

(c) $C = \mathcal{H}$: Similar to (a) and (b).

This finishes the proof of Theorem 1.2. □

### Section 2.

In this section we show that the following formula holds internally in $C (C = \mathcal{F}, \mathcal{G})$:

$$V f \in R^{k}[f|_{R_{\geq 0}} = 0 \land f|_{R_{= 0}} = 0 = f = 0]$$

(2)

(where $R_{\geq 0} = C^{\infty}(\mathbb{R}) / m_{R_{\geq 0}}, R_{= 0} = C^{\infty}(\mathbb{R}) / m_{R_{= 0}}$ and $f|_{R_{= 0}}$ is $f \circ i$ where $i : \mathbb{R}_{\geq 0} \to R$ is the obvious ‘inclusion’).

2.1. Lemma. Formula (2) is internally valid in the topos $C (C = \mathcal{F}, \mathcal{G})$ iff the following condition holds:

For every $C$-closed ideal $I \subseteq C^{\infty}(\mathbb{R}^{k})$,

$$Cl_{C}(I(\bar{x}, t) + m_{R_{= 0}}(t, \bar{x})) \cap Cl_{C}(I(\bar{x}, t) + m_{R_{= 0}}(t, \bar{x})) = Cl_{C}(I(\bar{x}, t)).$$

(2')

Proof. Easy. □

2.2. Theorem. Condition (2') holds, and so, (2) holds internally $(C = \mathcal{F}, \mathcal{G})$.

Proof. We first prove that (2') holds in the case $C = \mathcal{F}$, i.e., $(I(\bar{x}, t) + m_{R_{= 0}}(t, \bar{x})) \cap (I(\bar{x}, t) + m_{R_{= 0}}(t, \bar{x})) = I(\bar{x}, t)$. The inclusion $\subseteq$ is obvious. To prove the converse inclusion, take $f(\bar{x}, t) \in (I(\bar{x}, t) + m_{R_{= 0}}(t, x)) \cap (I(\bar{x}, t) + m_{R_{= 0}}(t, x))$. This means that there exist functions $\varphi_{1} \in I (1 \leq i \leq n), \varphi_{2} \in I (n + 1 \leq j \leq m), A_{i} \in C^{\infty}(\mathbb{R}^{k+1}) (1 \leq i \leq n), A_{j} \in C^{\infty}(\mathbb{R}^{k+1}) (n + 1 \leq j \leq m), \varphi_{1} \in m_{R_{= 0}}(t, \bar{x}), \varphi_{2} \in m_{R_{= 0}}(t, \bar{x})$ such that
We have set $-A_i$ in the first sum for convenience in the notation below. Then,

(a) \[ q_1 - q_2 = \sum_{i=1}^{m} A_i(\bar{x}, t) \varphi_i(\bar{x}) \in I(\bar{x}, t). \]

We have to prove that $q_1 \notin I(\bar{x}, t)$. Consider the formal power series

\[ S_i(\bar{x}, t) = \sum_{j=0}^{\infty} \frac{\partial^j A_i}{\partial t^j}(\bar{x}, 0) \frac{t^j}{j!}. \]

By Lemma 0.6 we can choose functions $\eta_j(t) \in C^\infty(\mathbb{R})$ such that for all $i$, $1 \leq i \leq m$ the series

\[ S_i(\bar{x}, t) = \sum_{j=0}^{\infty} \frac{\partial^j A_i}{\partial t^j}(\bar{x}, 0) \eta_j(t) \frac{t^j}{j!} \]

converge in the $C^\infty$ topology, and so, $s_i(\bar{x}, t) \in C^\infty(\mathbb{R}^{k+1})$. Moreover, for all $\alpha, j$,

\[ \frac{\partial^{\alpha + i} S_i}{\partial t^i \partial \bar{x}^\alpha}(\bar{x}, 0) = \frac{\partial^{\alpha + i} A_i}{\partial t^i \partial \bar{x}^\alpha}(\bar{x}, 0). \]

(In fact, Lemma 0.6 is stated for a single series $S$. But it is easily seen from the proof of Lemma 0.6 that if we have a finite number of series $S_i (1 \leq i \leq m)$, the functions $\eta_j$ can be chosen to be the same for all the $S_i$'s). Now, notice the following:

(i) \[ \sum_{i=0}^{m} s_i(\bar{x}, t) \varphi_i(\bar{x}) = 0, \]

(ii) \[ A_i(\bar{x}, t) - s_i(\bar{x}, t) \text{ is flat at } \{t=0\} \subseteq \mathbb{R}^{k+1}. \]

Point (ii) is immediate. To see that (i) is true notice first that from (a) above, it follows that

\[ 0 = \frac{\partial^j}{\partial t^j} (q_1 - q_2)(\bar{x}, 0) = \sum_{i=1}^{m} \frac{\partial^j}{\partial t^j} A_i(\bar{x}, 0) \varphi_i(\bar{x}) \]

Then

\[ \sum_{i=1}^{m} \frac{\partial^j}{\partial t^j} A_i(\bar{x}, 0) \frac{t^j}{j!} \eta_j(t) \varphi_i(\bar{x}) = 0 \]

and then

\[ 0 = \sum_{j=0}^{\infty} \left( \sum_{i=1}^{m} \frac{\partial^j}{\partial t^j} A_i(\bar{x}, 0) \frac{t^j}{j!} \eta_j(t) \varphi_i(\bar{x}) \right) \]

\[ = \sum_{i=1}^{m} \left( \sum_{j=0}^{\infty} \frac{\partial^j A_i}{\partial t^j}(\bar{x}, 0) \frac{t^j}{j!} \eta_j(t) \right) \varphi_i(\bar{x}) \]

\[ = \sum_{i=1}^{m} \varphi_i(\bar{x}) s_i(\bar{x}, t). \]
From (i) and (ii) the result follows: from (a) above and (i) it follows that \( q_1 - q_2 = \sum_{i=1}^{m} (A_i(\tilde{x}, t) - S_i(\tilde{x}, t)) \phi_i(\tilde{x}) \). Now consider the Heaviside function

\[
H(t) = \begin{cases} 
0 & \text{if } t \geq 0, \\
1 & \text{if } t < 0.
\end{cases}
\]

\[
q_1(t, \tilde{x}) = H(t)(q_1(t, \tilde{x}) - q_2(t, \tilde{x}))
\]

\[
= \sum_{i=1}^{m} H(t)(A_i(\tilde{x}, t) - S_i(\tilde{x}, t)) \phi_i(\tilde{x}).
\]

To see that \( q_1 \in I(\tilde{x}, t) \) it suffices now to show that \( H(t) \cdot (A_i(\tilde{x}, t) - S_i(\tilde{x}, t)) \in C^\infty(\mathbb{R}^{k+1}) \). This is immediate from (ii). This finishes the proof in the case \( C = \mathcal{F} \).

The case \( C = \mathcal{G} \) follows easily from the previous one since taking local nature closure preserves intersection. The proof in the case \( C = \mathcal{F} \) is an easy consequence of Theorem 0.1. \( \Box \)

**Section 3**

In this section we prove that the arrows \( t^n : \mathbb{R} \to \mathbb{R} \) (odd) and \( t^n : \mathbb{R} \to \mathbb{R}_{\geq 0} \) (even, \( n > 2 \)) are not stable effective epics in either of the categories \( \mathcal{L}, \mathcal{F}, \mathcal{G} \) (as always \( R_{\geq 0} = C^\infty(\mathbb{R})/m_{\mathbb{R}_{\geq 0}} \)).

As was said in the introduction, this question was affirmatively answered in the case \( n = 2 \) (see [S]). We do not prove the following two lemmas since their proofs are identical to the easy first part of the proof of [8, Theorem 3].

**3.1. Lemma.** Let \( n \) be an odd number. Then \( t^n : \mathbb{R} \to \mathbb{R} \) (see Example 0.2) would be a stable effective epic in \( C \) (\( C = \mathcal{L}, \mathcal{F}, \mathcal{G} \)) iff for every \( k \in \mathbb{N} \), for every \( C \)-closed ideal \( I \) and for every \( g \in C^\infty(\mathbb{R}^{k+1}) \) we had that:

(i) For every \( f \in C^\infty(\mathbb{R}^{k+1}) \), \( f(\tilde{x}, t) - f(\tilde{x}, s) \in Cl_C(I(\tilde{x}, t, s), g(\tilde{x}) - t^n, t^n - s^n) \) implies that there exists \( h \in C^\infty(\mathbb{R}^{k+1}) \) such that \( f(\tilde{x}, t) - h(\tilde{x}) \in Cl_C(I(\tilde{x}, t) + (g(\tilde{x}) - t^n)). \)

(ii) \( h(\tilde{x}) \in Cl_C(I(\tilde{x}, t) + (g(\tilde{x}) - t^n)) \) implies \( h(\tilde{x}) \in I. \) \( \Box \)

**3.2. Lemma.** Let \( n \) be an even number. Then \( t^n : \mathbb{R} \to \mathbb{R}_{\geq 0} \) would be a stable effective epic in \( C \) (\( C = \mathcal{L}, \mathcal{F}, \mathcal{G} \)) iff for every \( k \in \mathbb{N} \), for every \( C \)-closed ideal \( I \subseteq C^\infty(\mathbb{R}^{k}) \) and for every \( g \in C^\infty(\mathbb{R}^{k}) \) such that for every \( q \in m_{\mathbb{R}_{\geq 0}}, g(q(\tilde{x})) \in I \) we had that (i) and (ii) in Lemma 3.1 hold. \( \Box \)

**Remark.** Assume the function \( g(\tilde{x}) \) in Lemmas 3.1 and 3.2 to be such that \( (g(\tilde{x}))^{1/n} \) is defined and smooth. Then conditions (i) and (ii) are immediately verified.

**Remark.** Notice that points (i) in Lemmas 3.1 and 3.2 are related to certain existence properties while points (ii) are related to certain uniqueness properties. In each of the categories \( \mathcal{G}, \mathcal{F}, \mathcal{L} \) one of them holds while the other fails to hold. The following theorem resumes the situation:
3.3. **Theorem.** Existence, i.e. (i) in Lemma 3.1 (n odd) and (i) in Lemma 3.2 (n even), hold in the categories $C = \mathbb{L}$ and $C = \mathbb{G}$ but they do not hold in the category $C = \mathbb{F}$ if $n > 2$.

Uniqueness, i.e. (ii) in Lemma 3.1 (n odd) and (ii) in Lemma 3.2 (n even), hold in the category $C = \mathbb{F}$ but they do not hold in the categories $C = \mathbb{L}$ and $C = \mathbb{G}$ if $n > 2$.

We split the proof of Theorem 3.3 in five parts: 3.3.1 through 3.3.5.

3.3.1. **Proof that existence holds for $C = \mathbb{L}$ if $n$ is either odd or even**

Let $I$ be any ideal and $g \in C^\infty(\mathbb{R}^k)$ (if $n$ is even our hypothesis allows us to take $g \in C^\infty(\mathbb{R}^k)$ such that for every $p \in \mathbb{N}_{>0}$, $g(p(x)) \in I$, but this is not necessary in this proof). Take $f \in C^\infty(\mathbb{R}^{k+1})$ such that $f(\bar{x}, t) - f(\bar{x}, s) \in (I(\bar{x}, t, s) + (g(\bar{x}) - t^n) + (t^n - s^n))$. This means that there exist $r \in \mathbb{R}$ and functions $a_j(\bar{x}, t, s) \in C^\infty(\mathbb{R}^{k+2})$, $\phi_{j} \in I$ (1 ≤ $j$ ≤ $r$), $b(\bar{x}, t, s)$, $c(\bar{x}, t, s) \in C^\infty(\mathbb{R}^{k+2})$ such that $f(\bar{x}, t) - f(\bar{x}, s) = \sum_{j=1}^{r} a_j(\bar{x}, t, s) \times \phi_j(t) + b(\bar{x}, t, s)(g(\bar{x}) - t^n) + c(\bar{x}, t, s)(t^n - s^n)$. Call $T_s$ the ‘Taylor expansion in the variable $s$ at $s = 0$’ ring homomorphism.

Analogously, $T_{t,s}$ means ‘expansion at $t = 0$, $s = 0$’ and $T_t$ ‘expansion at $t = 0$’.

Applying $T_{t,s}$ to the equality above, we deduce that, in the ring $C^\infty(\mathbb{R}^k)[[t, s]]$ we have $(T_{t,s} f)(\bar{x}, t) - (T_{t,s} f)(\bar{x}, s) = \sum_{j=1}^{r} (T_{t,s} a_j)(\bar{x}, t, s) \times \phi_j(t) + (T_{t,s} b)(\bar{x}, t, s)(g(\bar{x}) - t^n) + (T_{t,s} c)(\bar{x}, t, s)(t^n - s^n)$. Let $\omega$ be an $n$th root of unity in $\mathbb{C}$. We can now evaluate both sides of this equality at $s = \omega t$. We obtain the equality

$$
(T_{t,s} f)(\bar{x}, t) - (T_{t,s} f)(\bar{x}, \omega t) = \sum_{j=1}^{r} (T_{t,s} a_j)(\bar{x}, t, \omega t) \times \phi_j(t) + (T_{t,s} b)(\bar{x}, t, \omega t) \cdot (g(\bar{x}) - t^n).
$$

We now need the following remark:

3.4. **Remark.** Let $\mathbb{G}_n$ be the group of $n$th roots of unity in $\mathbb{C}$. Let $j \in \mathbb{N}$. We have that

$$
\sum_{\omega \in \mathbb{G}_n} \omega^j = \begin{cases} n & \text{if } n \mid j, \\
0 & \text{if } n \nmid j.
\end{cases}
$$

It follows that

(a) The series $(1/n) \sum_{\omega \in \mathbb{G}_n} (T_{t,s} f)(\bar{x}, \omega t)$ is equal to $S(\bar{x}, t^n) \in C^\infty(\mathbb{R}^k)[[t]]$ for certain $S \in C^\infty(\mathbb{R}^k)[[t]]$ ($S$ has real-valued functions as coefficients).

(b) The series

$$
K_j(\bar{x}, t) = \frac{1}{n} \sum_{\omega \in \mathbb{G}_n} (T_{t,s} a_j)(\bar{x}, t, \omega t)
$$

and

$$
L(\bar{x}, t) = \frac{1}{n} \sum_{\omega \in \mathbb{G}_n} (T_{t,s} b)(\bar{x}, t, \omega t)
$$

have real valued functions as coefficients, i.e., they belong to $C^\infty(\mathbb{R}^k)[[t]]$. 

So, adding both sides of (3) over every \( \omega \in \mathcal{G}_n \) (taking into account (a) and (b)) and dividing by \( n \) yields

\[
(T_l f)(\bar{x}, t) - S(\bar{x}, t^n)
= \sum_{j=1}^r K_j(\bar{x}, t) \varphi_j(\bar{x}) + L(\bar{x}, t)(g(\bar{x}) - t^n).
\] (4)

By Lemma 0.6, we can choose functions \( s, k_j, l \in C^\infty(\mathbb{R}^{k+1}) \) such that

\[
(T_l s)(\bar{x}, t) = S(\bar{x}, t), \quad (T_l k_j) = K_j(\bar{x}, t), \quad (T_l l)(\bar{x}, t) = L(\bar{x}, t).
\] (5)

In view of (4) and (5), the function

\[
p(\bar{x}, t) = f(\bar{x}, t) - s(\bar{x}, t^n) - \sum_{j=1}^r k_j(\bar{x}, t) \varphi_j(\bar{x}) - l(\bar{x}, t)(g(\bar{x}) - t^n)
\] (6)

is flat at \( t = 0 \).

We now consider two cases depending on the parity of \( n \).

**n odd.** \( n \) being odd and \( p \) flat at \( t = 0 \), the function \( q(\bar{x}, t) = p(\bar{x}, t^{1/n}) \) is smooth. Then calling \( d(\bar{x}, t) = s(\bar{x}, t) + q(\bar{x}, t) \), we have from (6)

\[
f(\bar{x}, t) - d(\bar{x}, t^n) \in (I(\bar{x}, t) + (g(\bar{x}) - t^n)).
\]

From Lemma 0.7 it now follows that the function \( h(\bar{x}) = d(\bar{x}, g(\bar{x})) \) is such that

\[
f(\bar{x}, t) - h(\bar{x}) \in (I(\bar{x}, t) + (g(\bar{x}) - t^n)).
\]

**n even.** From (6) we obtain

\[
\frac{p(\bar{x}, t) + p(\bar{x}, -t)}{2} = \frac{f(\bar{x}, t) + f(\bar{x}, -t)}{2} - \sum_{j=1}^r \frac{k_j(\bar{x}, t) + k_j(\bar{x}, -t)}{2} \varphi_j(\bar{x})
\]

\[
- \left( \frac{l(\bar{x}, t) + l(\bar{x}, -t)}{2} \right)(g(\bar{x}) - t^n) - s(\bar{x}, t^n).
\] (7)

Call \( p_0(\bar{x}, t) = \frac{1}{2}(p(\bar{x}, t) + p(\bar{x}, -t)) \). The function \( q(\bar{x}, t) = p_0(|t|^{1/n}) \) is smooth since \( p_0 \) is flat at \( t = 0 \); and since \( p_0 \) is an even function, \( q(\bar{x}, t^n) = p_0(\bar{x}, t) \). Calling \( d(\bar{x}, t) = s(\bar{x}, t) + q(\bar{x}, t) \), it follows from (7) that

\[
f(\bar{x}, t) - d(\bar{x}, t^n) \in (I(\bar{x}, t) + (g(\bar{x}) - t^n)).
\]

Let \( h(\bar{x}) = d(\bar{x}, g(\bar{x})) \). In the same way as in the case \( n \) odd, it follows now that

\[
f(\bar{x}, t) - d(\bar{x}, t^n) \leq h(\bar{x}) \in (I(\bar{x}, t) + (g(\bar{x}) - t^n)).
\]

But now, being \( n \) even it follows from our hypothesis (setting \( s = -t \)) that

\[
f(\bar{x}, t) - f(\bar{x}, -t) \leq \frac{1}{2} \in (I(\bar{x}, t) + (g(\bar{x}) - t^n)).
\]

This finishes the proof of 3.3.1.
3.3.2. Proof that existence holds for $C = \mathbb{G}$

Let $I$ be any ideal of local character and $g \in C^\infty(\mathbb{R}^k)$. Take $f \in C^\infty(\mathbb{R}^{k+1})$ such that $f(\tilde{x}, t) - f(\tilde{x}, s) \in \text{Cl}_{\mathbb{C}}(I(\tilde{x}, t, s) + (g(\tilde{x}) - t^n + (t^n - s^n))$).

**Assertion 1.** Let $(\tilde{x}_0, t_0) \in Z(I(\tilde{x}, t) + (g(\tilde{x}) - t^n))$. Then there exists a neighborhood $W_{\tilde{x}_0, t_0}$ of $(\tilde{x}_0, t_0)$ in $\mathbb{R}^{k+1}$ and $l_{\tilde{x}_0, t_0}(\tilde{x}) \in C^\infty(\mathbb{R}^k)$ such that there exists a function $\psi_{\tilde{x}_0, t_0}(\tilde{x}, t) \in (I(\tilde{x}, t) + (g(\tilde{x}) - t^n) \text{ which equals } f(\tilde{x}, t) - l_{\tilde{x}_0, t_0}(\tilde{x}) \text{ for } (\tilde{x}, t) \in W_{\tilde{x}_0, t_0}$.

**Proof.** Case 1. $g(\tilde{x}_0) \neq 0$. In this case we have that the function $g(\tilde{x})^{1/n}$ is defined and smooth in a neighborhood of $\tilde{x} = \tilde{x}_0$ (Notice that if $n$ is even the conditions $g(\tilde{x}_0) \neq 0$ and $(\tilde{x}_0, t_0) \in Z(I(\tilde{x}, t) + (g(\tilde{x}) - t^n))$ imply $g(\tilde{x}_0) > 0$.) The results hold easily.

Case 2. $g(\tilde{x}_0) = 0$. The proof of assertion 1 in this case is exactly the same as the proof of 3.3.1, starting this time with the equality $f(\tilde{x}, t) - f(\tilde{x}, s) = \sum_{j=1}^{r} a_j(\tilde{x}, t, s) \phi_j(\tilde{x}) + b(\tilde{x}, t, s)(g(\tilde{x}) - t^n) + c(\tilde{x}, t, s)(t^n - s^n)$ in a neighborhood of $(\tilde{x}_0, 0, 0)$ instead of all of $\mathbb{R}^{k+2}$.

**Assertion 2.** If $n$ is odd, for each $\tilde{x}_0 \in Z(I)$ there exists exactly one $t_0 \in \mathbb{R}$ such that $(\tilde{x}_0, t_0) \in Z(I(\tilde{x}, t) + (g(\tilde{x}) - t^n))$, namely $t_0 = \frac{1}{n} g(\tilde{x})$. If $n$ is even and $\tilde{x}_0 \in Z(I)$ is such that $g(\tilde{x}_0) > 0$, there are two $t_0$'s: $t_0 = \pm \frac{1}{n} g(\tilde{x}_0)$. In this case, $W_{\tilde{x}_0, t_0}$, $W_{\tilde{x}_0, t_0}$, $l_{\tilde{x}_0, t_0}$ and $l_{\tilde{x}_0, t_0}$ can be chosen in such a way that $l_{\tilde{x}_0, t_0} = l_{\tilde{x}_0, t_0}$, say $l_{\tilde{x}_0, t_0} = h_{\tilde{x}_0}$, $W_{\tilde{x}_0, t_0} \subset \mathbb{R}^k \times [0, \infty)$, $W_{\tilde{x}_0, t_0} \subset \mathbb{R}^k \times [0, \infty)$ and $f(\tilde{x}, t) - h_{\tilde{x}_0}(\tilde{x})$ equal an element of $(I(\tilde{x}, t) + (g(\tilde{x}) - t^n))$ in $W_{\tilde{x}_0, t_0} \cup W_{\tilde{x}_0, t_0}$.

**Proof.** The only thing we have to prove is that in the case $n$ even, $l_{\tilde{x}_0, t_0}$ and $l_{\tilde{x}_0, t_0}$ can be chosen to be the same, the rest being easy. Take $n$ even, the two couples $(\tilde{x}_0, t_0)$, $(\tilde{x}_0, t_0) \in Z(I(\tilde{x}, t) + (g(\tilde{x}) - t^n))$ $(t_0 > 0)$, and consider the function $l_{\tilde{x}_0, t_0} \in C^\infty(\mathbb{R}^k)$ given by Assertion 1, defined in $W_{\tilde{x}_0, t_0}$. Setting $s = -t$ in our hypothesis it follows that, in a neighborhood of $(\tilde{x}_0, t_0)$, $\frac{1}{2}(f(\tilde{x}, t) - f(\tilde{x}, -t))$ equals an element $\eta \in (I(\tilde{x}, t) + (g(\tilde{x}) - t^n))$. Taking a smaller $W_{\tilde{x}_0, t_0}$ we can assume both equalities below to hold in $W_{\tilde{x}_0, t_0}$:

$$\frac{f(\tilde{x}, t) - f(\tilde{x}, -t)}{2} = \eta(\tilde{x}, t), \quad f(\tilde{x}, t) - l_{\tilde{x}_0, t_0}(\tilde{x}) = \psi_{\tilde{x}_0, t_0}(\tilde{x}, t)$$

(where $\psi_{\tilde{x}_0, t_0}$ is the function given by Assertion 1. Both $\psi_{\tilde{x}_0, t_0}$ and $\eta$ are in $(I(\tilde{x}, t) + g(\tilde{x}) - t^n)$). Then, the equality $\frac{1}{2}(f(\tilde{x}, t) + f(\tilde{x}, -t)) - l_{\tilde{x}_0, t_0}(\tilde{x}) = \psi_{\tilde{x}_0, t_0}(\tilde{x}, t) - \eta(\tilde{x}, t)$ holds in $W_{\tilde{x}_0, t_0}$. Let us say that

$$\psi_{\tilde{x}_0, t_0}(\tilde{x}, t) - \eta(\tilde{x}, t) = \sum_{j=1}^{r} a_j(\tilde{x}, t) \phi_j(\tilde{x}) + b(\tilde{x}, t)(g(\tilde{x}) - t^n)$$
for certain \( \varphi_j \in I, \ a_j, \ b \in C^\infty(\mathbb{R}^{k+1}) \). That is to say, for \((\bar{x}, t)\) in \( W_{x_0, t_0} \) we have

\[
\frac{f(\bar{x}, t) + f(\bar{x}, -t)}{2} - l_{x_0, t_0}(\bar{x}) = \sum_{j=1}^{r} a_j(\bar{x}, t)\varphi_j(\bar{x}) + b(\bar{x}, t)(g(\bar{x}) - t^n).
\]

Then, for \((x, t) \in W_{x_0, -t_0} = \{(\bar{x}, t) \in \mathbb{R}^{k+1}: (\bar{x}, -t) \in W_{x_0, t_0}\}\) we have

\[
\frac{f(\bar{x}, -t) + f(\bar{x}, t)}{2} - l_{x_0, t_0}(\bar{x}) = \sum_{j=1}^{r} a_j(\bar{x}, -t)\varphi_j(\bar{x}) + b(\bar{x}, -t)(g(\bar{x}) - t^n).
\]

Again from our hypothesis, we can assume that in \( W_{x_0, t_0}, \frac{1}{2}(f(\bar{x}, t) - f(\bar{x}, -t)) \) equals an element of \( (I(\bar{x}, t) + (g(\bar{x}) - t^n)) \) (taking a smaller \( W_{x_0, t_0} \) if necessary). Since \( f(\bar{x}, t) = \frac{1}{2}(f(\bar{x}, -t) + f(\bar{x}, t)) + \frac{1}{2}(f(x, t) - f(x, -t)) \) it follows that in \( W_{x_0, -t_0}, f(\bar{x}, t) - l_{x_0, t_0}(\bar{x}) \) equals an element of \( (I(\bar{x}, t) + (g(\bar{x}) - t^n)) \). This is what we wanted to show. \( \square \)

**Assertion 3.** If \((x_0, t_0) \in Z(I(\bar{x}, t) + (g(\bar{x}) - t^n))\), there exists a neighborhood \( U_{x_0} \) of \( x_0 \), \( h_{x_0} \in C^\infty(\mathbb{R}^k) \) and \( \psi_{x_0} \in (I(\bar{x}, t) + (g(\bar{x}) - t^n)) \) such that the following equality holds for \((x, t) \in U_{x_0} \times \mathbb{R}^k\):

\[
f(\bar{x}, t) - h_{x_0}(\bar{x}) = \psi_{x_0}(\bar{x}, t).
\]

**Proof.** From Assertions 1 and 2 we know that for \((x_0, t_0) \in Z(I(\bar{x}, t) + (g(\bar{x}) - t^n))\) there exists a neighborhood \( W_{x_0, t_0} \) of \((x_0, t_0)\) (if \( n \) is even and \( t_0 > 0 \), two neighborhoods of \((x_0, t_0)\) and \((x_0, -t_0)\), respectively \( W_{x_0, t_0} \) and \( W_{x_0, -t_0} \)) and a function \( h_{x_0}(x) \in C^\infty(\mathbb{R}^k) \) such that for \((x, t) \in W_{x_0, t_0}\) (for \((x, t) \in W_{x_0, t_0} \cup W_{x_0, -t_0}\) if \( n \) is even and \( t_0 > 0 \)) \( f(\bar{x}, t) - h_{x_0}(x) \) equals an element \( \eta \in (I(\bar{x}, t) + (g(\bar{x}) - t^n)) \). Then the function \( g(\bar{x}, t) = f(\bar{x}, t) - h_{x_0}(x) - \eta(\bar{x}, t) \) vanishes in a neighborhood of \((x_0, t_0)\) (in a neighborhood of \((x_0, t_0)\) and \((x_0, -t_0)\) if \( n \) is even).

Having in mind that \( t_0 = \pm \sqrt{g(x_0)} \), it is readily verified that there exists a neighborhood \( U_{x_0} \) of \( x_0 \) in \( \mathbb{R}^k \) such that in \( U_{x_0} \times \mathbb{R}^k \), \( g(x, t) = l(\bar{x}, t)(g(\bar{x}) - t^n) \) for certain \( l(\bar{x}, t) \in C^\infty(\mathbb{R}^{k+1}) \). Then, in \( U_{x_0} \times \mathbb{R}^k \), \( f(\bar{x}, t) - h_{x_0}(\bar{x}) - \eta(\bar{x}, t) + l(\bar{x}, t)(g(\bar{x}) - t^n) \in (I(\bar{x}, t) + (g(\bar{x}) - t^n)) \). This finishes the proof of Assertion 3. \( \square \)

Let us return to 3.3.2: From Assertion 2 we know that for every \( x_0 \in Z(I) \) if \( n \) is odd and for every \( x_0 \in Z(I) \) such that \( g(x_0) \geq 0 \) if \( n \) is even, there exists a neighborhood \( U_{x_0} \) of \( x_0 \) and \( h_{x_0} \in C^\infty(\mathbb{R}^k) \) such that \( f(\bar{x}, t) - h_{x_0}(\bar{x}) \) equals an element of \( (I(\bar{x}, t) + (g(\bar{x}) - t^n)) \) in \( U_{x_0} \times \mathbb{R}^k \). The proof is finished by gluing the \( h_{x_0} \)'s with a partition of unity.
3.3.3. Proof that existence does not hold for $C = \mathbb{F}$ if $n > 2$.

Let $k = 1$, $n > 2$ and $p(x) \in C^\infty(\mathbb{R}) = C^\infty(\mathbb{R}^2)$ be any function flat at zero and positive for $x \neq 0$. To fix ideas we can choose $p(x) = e^{-1/|x|^2}$. By Lemma 3.5 below (with $i = 2$) there exists $g \in C^\infty(\mathbb{R})$ positive except for $x = 0$ such that $g(x)^{2/n} \cdot p(x)$ is not smooth (this is the point where we need $n > 2$). Let $f(x, t) = t^2 \cdot p(x) \in C^\infty(\mathbb{R}^2)$ (the reason why we need $t^2 \cdot p(x)$ and, therefore, $g^{2/n}(x) \cdot p(x)$ and not $t \cdot p(x)$ is that we need $f$ to be even in the variable $t$). We will see that

(a) $f(x, t) - f(x, s) \in C_{(t, s)}(g(x) - t^n, t^n - s^n)$, but

(b) there is not any $h \in C^\infty(\mathbb{R})$ such that $f(x, t) - h(x) \in C_{(t, s)}(g(x) - t^n)$.

To see (a), consider the ideal $K$ generated in $C^\infty(\mathbb{R})$ by $\{q(g(x)) \mid q(t) \text{ is flat at } t = 0\}$. The only zero of $K$ is $x = 0$. By Theorem 0.1 there exists a sequence of elements of $K$, $C_{-\infty} \cdot CO$ converging to $p(x)$, say $p(x) = \lim_{r \to \infty} p_r(x)$, $p_r \in K$, where $p_r$ is a linear combination of the form $p_r(x) = \sum_{i=1}^{\infty} \lambda_{i,r}(x)q_{i,r}(g(x))$ for some $\lambda_{i,r} \in C^\infty(\mathbb{R})$ and flat functions $q_{i,r}$.

Notice that, for every $l \in \mathbb{N}$, $p_r(x)/g^{1/n}(x) \in C^\infty(\mathbb{R})$.

By Lemma 0.6, there exist functions $\varphi_l \in C^\infty(\mathbb{R})$ which are even, of compact support and for each $l$, $\varphi_l = 1$ in a certain neighborhood $U_l$ of the origin such that

$$b_l(x, t) = \sum_{i=0}^{\infty} \varphi_i(t) \cdot \frac{p_i(x)}{g(x)^{1+i} \cdot t^{i-n}}$$

is $C^\infty$-CO convergent and so, $b_r \in C^\infty(\mathbb{R}^2)$. Notice that if $n$ is even, $b_r$ is even in the variable $t$. It is easily seen that the functions $b_r(x, t)(g(x) - t^n)$ and $p_r(x)$ have the same derivatives at $t = 0$. Thus, the difference $d_r(x, t) = t^2 p_r(x) - t^2 b_r(x, t)(g(x) - t^n)$ is a flat function at $t = 0$ (and even in the variable $t$ if $n$ is even). Call $f_r(x, t) = t^2 p_r(x)$.

Notice that $f_r(x, t)$ $C^\infty$-CO converges to $f(x, t)$. We have $f_r(x, t) = t^2 b_r(x, t) \times (g(x) - t^n) + d_r(x, t)$. Since $d_r$ is flat at $t = 0$ (and even in the variable $t$ if $n$ is even), calling

$$q_r(x, t) \begin{cases} 
  d_r(x, t^{1/n}) & \text{if } n \text{ is odd}, \\
  d_r(x, |t|^{1/n}) & \text{if } n \text{ is even},
\end{cases}$$

we have $q_r(x, t^n) = d_r(x, t)$, and $q_r \in C^\infty(\mathbb{R}^2)$. Thus, $f_r(x, t) - f_r(x, s) = t^2 b_r(x, t) \times (g(x) - t^n) - s^2 b_r(x, s) \cdot (g(x) - s^n) + q_r(x, t^n) - q_r(x, s^n)$.

Since $q_r(x, t^n) - q_r(x, s^n) \in ((t^n - s^n))$ (as may be seen using Lemma 0.7) it immediately follows that $f_r(x, t) - f_r(x, s) \in ((g(x) - t^n) + (t^n - s^n))$, and since $f_r(x, t) - f(x, t)$ we see that $f(x, t) - f(x, s) \in C_{(t^n, t^n - s^n)}(g(x) - t^n, t^n - s^n)$, i.e., we have shown (a) to hold.

Now we deal with (b). Suppose there exists $h \in C^\infty(\mathbb{R})$ such that $f(x, t) - h(x) \in C_{(t^n, t^n - s^n)}(g(x) - t^n)$. In this case (recall that $g$ was taken positive for $x \neq 0$) we have $f(x, g(x)) - h(x) = 0$, or, by definition of $f$, $g^{2/n}(x) \cdot p(x) = h(x) \in C^\infty(\mathbb{R})$, a contradiction. This completes the proof of 3.3.3.
3.3.4. Proof that uniqueness holds for $C = \mathbb{F}$

Take a closed ideal $I \subset C^\infty(\mathbb{R}^k)$ and $h(\vec{x}) \in C^\infty(\mathbb{R}^k)$. Assume $h(\vec{x}) \in \text{Cl}_I(I(\vec{x}, t) + (g(\vec{x}) - t^n))$. We have to prove that $h(\vec{x}) \in I$. To do this ($I$ is closed) we use Theorem 0.1. It suffices to check that $T_{x_0}(h) \in T_{x_0}(I)$ for $x_0 \in Z(I)$. In the case $n$ even, we have from our hypothesis that $q(g(\vec{x})) \in I$ for every $q \in \mathcal{M}_{\mathbb{R}^k}$ which implies that $g(x_0) \geq 0$ for $x_0 \in Z(I)$. We have two cases: (i) $g(x_0) \neq 0$ ($g(x_0) > 0$ if $n$ is even) and (ii) $g(x_0) = 0$.

(i) In this case, $\sqrt{n}g(\vec{x})$ is a smooth function in a neighborhood of $x_0$. It is thus easily seen that $T_{x_0}(h) \in T_{x_0}(I)$.

(ii) Notice that, since $g(x_0) = 0$, it follows that $(x_0, 0)$ is a zero of $(I(\vec{x}, t) + (g(\vec{x}) - t^n))$.

Again from Theorem 0.1, we know that $T_{x_0}(h) = T_{x_0,0}(h)$ is the Taylor expansion at $(x_0,0)$ of certain $f \in (I(\vec{x}, t) + (g(\vec{x}) - t^n))$, i.e., $T_{x_0}(h) = T_{x_0,0}(f)$. Let us say that $f(\vec{x}, t) = \sum_{i=1}^r a_i(\vec{x}, t) \phi_i(\vec{x}) + b(\vec{x}, t)(g(\vec{x}) - t^n)$ for certain $a_i, b \in C^\infty(\mathbb{R}^{k+1})$, and $\phi_i \in I$. Taking Taylor series expansion at $(x_0, 0)$ we have $T_{x_0}(h) = \sum_{i=1}^r T_{x_0,0}(a_i)T_{x_0}(\phi_i) + T_{x_0,0}(b)(T_{x_0}(g) - t^n)$.

Replacing $t$ by $\omega t$ ($\omega \in \mathbb{G}_n$) and adding over $\omega \in \mathbb{G}_n$ yields (see Remark 3.4)

$$T_{x_0}(h) = \sum_{i=1}^r S_i(\vec{x}, t^n)T_{x_0}(\phi_i)(\vec{x}) + U(\vec{x}, t^n)(T_{x_0}(g)(\vec{x}) - t^n)$$

for certain series $S_i$, $U \in \mathbb{R}[[[x - x_0], t]]$.

By Lemma 0.6, there exist functions $s_i$, $u \in C^\infty(\mathbb{R}^{k+1})$ such that $T_{x_0,0}s_i = S_i$, $T_{x_0,0}(u) = U$. So, the function $q(\vec{x}, t) = h(\vec{x}) - \sum_{i=1}^r s_i(\vec{x}, t^n)\phi_i(\vec{x}) - u(\vec{x}, t^n)(g(\vec{x}) - t^n)$ is flat at $(x_0, 0)$, and it is clear from its very definition that $q(\vec{x}, t) = p(\vec{x}, t^n)$ for some smooth $p$ which is seen to be flat at $(x_0, 0)$ since $q$ is. Therefore, we have

\[h(\vec{x}) = \sum_{i=1}^r S_i(\vec{x}, t^n)\phi_i(\vec{x}) + u(\vec{x}, t^n)(g(\vec{x}) - t^n) + p(\vec{x}, t^n).\]

We now have two cases:

(a) $n$ odd. In this case, we replace $t - \sqrt{n}g(\vec{x})$ in (*) and obtain $h(\vec{x}) = \sum_{i=1}^r S_i(\vec{x}, g(\vec{x}))(\phi_i(\vec{x}) + p(\vec{x}, g(\vec{x})))$. Since $p$ is flat at $(x_0, 0)$ and $g(x_0) = 0$, it follows that $T_{x_0}(h) \in T_{x_0}(I)$.

(b) $n$ even. It follows from (*) that the function $h(\vec{x}) - \sum_{i=1}^r S_i(\vec{x}, t^n)\phi_i(\vec{x}) - u(\vec{x}, t^n)(g(\vec{x}) - t^n) - p(\vec{x}, t^n)$ vanishes for $t \geq 0$ and is, therefore, flat at $t = 0$ and so, in particular, it is flat at $(x_0, 0)$. Since $g(x_0) = 0$, replacing $t = g(\vec{x})$ shows that $h(\vec{x}) - \sum S_i(\vec{x}, g(\vec{x}))(\phi_i(\vec{x})$ is flat at $x_0$ (since $p(\vec{x}, g(\vec{x}))$ is also flat at $x_0$), and so, $T_{x_0}(h) \in T_{x_0}(I)$. This finishes the proof of 3.3.4.

3.3.5. Proof that uniqueness does not hold for $C = \mathbb{L}$ and $C = \mathcal{G}$ if $n > 2$

We have two cases:

(a) $n$ odd. Let $n$ be odd, $n > 2$, $I = (e^{-1/x^2}) \subset C^\infty(\mathbb{R})$ and $h(x) = \sqrt{n}e^{-1/x^2} \in C^\infty(\mathbb{R})$. We are to show that $h(x) \in (I(x, t) + (x - t^n))$ — while clearly $h(x) \notin I$. It suffices to
show that there exists $b(x,t), C^\infty(\mathbb{R}^2)$ such that \( \sqrt{x} e^{-1/x^2} = t e^{-1/x^2} + b(x,t)(x - t^n) \). And, to do this it suffices to show that the function

\[
b(x,t) = \frac{e^{-1/x^2} (\sqrt{x} - t)}{x - t^n}
\]

declared for $x \neq t^n$ extends smoothly to $\mathbb{R}^2$. Now

\[
x - t^n = (\sqrt{x} - t) \sum_{i=1}^{n} (\sqrt{x})^{n-i} t^{i-1} = (\sqrt{x} - t) \cdot (\sqrt{x})^{n-1} \cdot \eta(x,t)
\]

where $\eta(x,t) = \sum_{i=1}^{n} (t/\sqrt{x})^{i-1}$. So, for $0 \neq x \neq t^n$,

\[
b(x,t) = \frac{e^{-1/x^2} (\sqrt{x} - t)}{(\sqrt{x})^{n-1} \cdot \eta(x,t)} = \frac{e^{-1/2x^2}}{(\sqrt{x})^{n-1}} \cdot \frac{e^{-1/2x^2}}{\eta(x,t)}.
\]

Of course, the first factor is in (extends to an element of) $C^\infty(\mathbb{R})$. And so does the second factor, as is readily verified. (Hint: the function $\eta(x,t)$ has a positive lower bound:

(i) it does not vanish at any point since the function \( (1 - (t/\sqrt{x})) = (1 - t/\sqrt{x}) \cdot \eta(x,t) \) considered as a function of $t/\sqrt{x}$ has the only (simple) zero $t/\sqrt{x} = 1$.

(ii) $\eta(x,t)$ tends to $+\infty$ as $t/\sqrt{x}$ tends to $\infty$. Now, for $x \neq 0$, the second factor is $C^\infty$. So, it has to be shown that its derivatives tend to zero as $(x,t)$ tends to $(0,t_0)$ ($t_0 \in \mathbb{R}$).

This finishes with the case $n$ odd.

(b) $n$ even. Let $n$ be even, $n > 2$, $I = (e^{-1/x^2}) \subseteq C^\infty(\mathbb{R})$ and $h(x) = \sqrt{x} \cdot e^{-1/x^2}$. We are to show that $h(x) \in I(x,t) + (x^2 - t^n) -$ while $h(x) \notin I$ since $n > 2$. (In this case $(n$ even) we have to fulfill the condition $\varrho(g(x)) \in I$ for every $\varrho \in \mathfrak{m}_{\mathbb{R}^n}$. But we have taken $g(x) = x^2$ and so, $\varrho(g(x)) = 0 \in I$ for $\varrho \in \mathfrak{m}_{\mathbb{R}^n}$.) We will show that there exists $b(x,t) \in C^\infty(\mathbb{R}^2)$ such that \( \sqrt{x} \cdot e^{-1/x^2} = t^2 \cdot e^{-1/x^2} + b(x,t)(x^2 - t^n) \). It suffices to show that the function

\[
b(x,t) = e^{-1/x^2} (\sqrt{x}^4 - t^2) \frac{x^2 - t^n}{x^2 - t^n}
\]

declared for $x^2 \neq t^n$ extends to $b \in C^\infty(\mathbb{R}^2)$. Now,

\[
x^2 - t^n = ((|x|^{4/n})^{n/2} - (t^2)^{n/2})
\]
\[
= (|x|^{4/n} - t^2) \cdot \sum_{i=1}^{n/2} (|x|^{4/n})^{(n/2 - i)} \cdot t^{2(i-1)}
\]

and so, for $0 \neq x^2 \neq t^n$,

\[
b(x,t) = \frac{e^{-1/x^2}}{|x|^{4/n} \cdot (n/2 - 1)} \cdot \sum_{i=1}^{n/2} \frac{(t^2/|x|^{4/n})^{(n/2 - i)}}.\]
The proof now follows in the same way as in the case \( n \) odd \( \eta(x,t) = \sum_{i=1}^{n/2} (i^2/|x|^{i/n})^{i-1} > 1 \).

Since finitely generated ideals are of local character, this finishes the proof of 3.3.5, and also finishes the proof of Theorem 3.3. \( \square \)

We now prove the lemma we used in 3.3.3. This lemma was inspired in an example given in [9] of a function \( f \), flat at \( x=0 \), and \( f > 0 \) in \( \mathbb{R}_{\neq 0} \) such that \( \sqrt[2]{f(x)} \) is not smooth.

3.5. Lemma. Let \( p \in C^\infty(\mathbb{R}) \) be any function such that \( p(x) > 0 \) for \( x \neq 0 \). Let \( n, i \in \mathbb{N}, i < n \). Then there exists a function \( g(x) \in C^\infty(\mathbb{R}) \), \( g(x) > 0 \) except for \( x = 0 \) and flat at \( x=0 \) such that \( g(x)^{i/n} \cdot p(x) \) is not smooth.

Proof. Let \( \phi \in C^\infty(\mathbb{R}) \) be a function such that
- \( \phi = 1 \) in a neighborhood of \( 0 \in \mathbb{R} \),
- \( \phi \) is positive in \((-1, 1)\),
- \( \phi \) is null outside \((-1, 1)\).

For \( r \in \mathbb{N} \) we define
\[
g_r(x) = \phi(r(r+1)(x-1/r)) \cdot \left( e^{-r^2/(r+1)} \left( e^{-r^2/(r+1)} + \frac{r}{p(1/r)} (x-1/r)^2 \right) \right) \in C^\infty(\mathbb{R}).
\]

(i) The function \( g_r \) is positive in \( I_r = (1/(r+1), (r+2)/(r(r+1))) \) and null outside \( I_r \).

(ii) The interval \( I_r \) does not contain any element of the sequence \( \{1/l\} \) other than \( 1/r \).

(iii) \( \bigcup_{i=1}^{\infty} I_i = (0, \frac{1}{2}) \).

Let \( g_0 \) be a function \( C^\infty \) in \( \mathbb{R}_{>0} \) such that \( g_0((0, 1]) = 0 \) and \( g_0 \) is positive for \( x > 1 \).

It is easily verified that the series \( g = \sum_{r=0}^{\infty} g_r \) represents a smooth positive function defined in the set of positive reals. It is also easy to see that all the derivatives of \( g \) tend to zero as \( x \) tends to zero so that the formulae \( g(0) = 0 \) and \( g(-x) = g(x) \) \( (x > 0) \) define a smooth function throughout \( \mathbb{R} \).

It is clear that the function \( g_r \) as well as all its derivatives vanish at \( x = 1/r \) for \( l \neq r \). Then, at \( x=1/r \), \( g(x) \) and its derivatives are equal to \( g_r \) and its respective derivatives. We will show that the second derivative of the function \( a(x) = [g(x)]^{i/n} \cdot p(x) \) at the point \( x=1/r \) tends to infinite as \( x \) tends to zero. This will finish the proof. Since \( g^r(1/r) - 0 \), we have
\[
a'' \left( \frac{1}{r} \right) = \frac{i}{n} g \left( \frac{1}{r} \right)^{i-n/n} \cdot g'' \left( \frac{1}{r} \right) \cdot p \left( \frac{1}{r} \right) + g \left( \frac{1}{r} \right)^{i/n} \cdot p'' \left( \frac{1}{r} \right).
\]

The second term tends to zero. Let us analyse the behavior of the first term, which equals
\[
\frac{i}{n} g \left( \frac{1}{r} \right)^{i-n/n} \cdot \frac{2r}{p(1/r)} \cdot p \left( \frac{1}{r} \right) = \frac{2ir}{n} g \left( \frac{1}{r} \right)^{i-n/n}.
\]
Since $i < n$, this clearly tends to infinite as $r$ tends to infinite. \( \square \)

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References