Available online at www.sciencedirect.com

ScienceDirect

Applied Mathematics Letters 21 (2008) 510–515

Applied Mathematics Letters

www.elsevier.com/locate/aml

Sobolev exponents of Butterworth refinable functions☆

Hong Oh Kim[a], Rae Young Kim[b,*]

[a] Department of Mathematical Sciences, KAIST, 373-1, Guseong-dong, Yuseong-gu, Daejeon, 305-701, Republic of Korea
[b] Department of Mathematics, Yeungnam University, 214-1, Dae-dong, Gyeongsan-si, Gyeongsangbuk-do, 712-749, Republic of Korea

Received 14 August 2006; received in revised form 11 April 2007; accepted 30 May 2007

Abstract

The precise Sobolev exponent $s_\infty(\varphi_n)$ of the Butterworth refinable function φ_n associated with the Butterworth filter of order n, $b_n(\xi) := \frac{\cos^{2n}(\xi/2)}{\cos^{2n}(\xi/2)+\sin^{2n}(\xi/2)}$, is shown to be $s_\infty(\varphi_n) = n\log_2 3 + \log_2(1+3^{-n})$. This recovers the previously given asymptotic estimate of $s_\infty(\varphi_n)$ of Fan and Sun, and gives more accurate regularity of Butterworth refinable function φ_n.

Keywords: Sobolev exponent; Butterworth filter; Butterworth refinable function; Orthonormal cardinal function; Blaschke product

The Sobolev exponent $s_\infty(f)$ of a function f is defined in terms of its Fourier transform

$$s_\infty(f) = \sup\{s \mid \sup_\xi |\hat{f}(\xi)|(1+|\xi|)^s < \infty\}.$$

This gives the regularity of f as $f \in C^s$ for any $s < s_\infty(f) - 1$.

The Butterworth filter of order n is defined by

$$b_n(\xi) := \cos^{2n}(\xi/2)\mathcal{L}_n(\xi),$$

where

$$\mathcal{L}_n(\xi) := \frac{1}{\cos^{2n}(\xi/2) + \sin^{2n}(\xi/2)}.$$

Then the corresponding refinable function φ_n, called the Butterworth refinable function, is given by

$$\hat{\varphi}_n(\xi) := \prod_{j=1}^{\infty} b_n(2^{-j}\xi) = \prod_{j=1}^{\infty} \cos^{2n}(2^{-j-1}\xi) \prod_{j=1}^{\infty} \mathcal{L}_n(2^{-j}\xi)$$

☆ This work was supported by the Korea Science and Engineering Foundation (KOSEF) Grant funded by the Korea Government (MOST; R01-2006-000-10424-0: the first author) and by the Korea Research Foundation Grant funded by the Korean Government (MOEHRD, Basic Research Promotion Fund; KRF-2006-331-C00014: the second author).

* Corresponding author.
E-mail addresses: hkim@amath.kaist.ac.kr (H.O. Kim), rykim@ynu.ac.kr (R.Y. Kim).

0893-9659/$ - see front matter
doi:10.1016/j.aml.2007.05.016

$$= \left(\frac{\sin(\xi/2)}{\xi/2}\right)^{2n} \prod_{j=1}^{\infty} \mathcal{L}_n(2^{-j}\xi).$$

Fan and Sun [1] obtained the estimate

$$n \log_2 3 \le s_\infty(\varphi_n) \le n \log_2 3 + \log_2(1 + 3^{-n}).$$

We prove here that the precise Sobolev exponent is their upper bound of $s_\infty(\varphi_n)$:

$$s_\infty(\varphi_n) = n \log_2 3 + \log_2(1 + 3^{-n}).$$

As an application, we also give the precise Sobolev exponents of the special class of refinable orthonormal cardinal functions from Blaschke products in [2].

We recall a method for estimating the decay of $\hat{\varphi}$ of a refinable function φ adapted for our particular purpose in the following proposition. See [3, Lemma 7.1.5, Lemma 7.1.6].

Proposition 1. *For $\mathcal{L} \in C^1(\mathbb{T})$, let b be the filter of the refinable function φ of the form*

$$|b(\xi)| = \cos^{2n}(\xi/2)|\mathcal{L}(\xi)|, \quad \xi \in [-\pi, \pi].$$

Suppose that $[-\pi, \pi] = D_1 \cup D_2 \cup D_3$ and that

$$|\mathcal{L}(\xi)| \le \left|\mathcal{L}\left(\frac{2\pi}{3}\right)\right|, \quad \xi \in D_1;$$

$$|\mathcal{L}(\xi)\mathcal{L}(2\xi)| \le \left|\mathcal{L}\left(\frac{2\pi}{3}\right)\right|^2, \quad \xi \in D_2;$$

$$|\mathcal{L}(\xi)\mathcal{L}(2\xi)\mathcal{L}(4\xi)| \le \left|\mathcal{L}\left(\frac{2\pi}{3}\right)\right|^3, \quad \xi \in D_3.$$

Then

$$|\hat{\varphi}(\xi)| \le C(1 + |\xi|)^{-2n+\kappa},$$

where $\kappa = \log_2(|\mathcal{L}(2\pi/3)|)$, and this decay is optimal; i.e., $s_\infty(\varphi) = 2n - \kappa$. Consequently, $\varphi \in C^s$ for any $s < 2n - \kappa - 1$.

The idea is to divide the interval $[-1/2, 1/2]$ as a union of three sets to have the relevant estimates on each set as in the following lemma.

Lemma 2. *Let $Q_n(x) := (1/2 - x)^n + (1/2 + x)^n$. Then:*

(a) $Q_n(x) \ge Q_n(\frac{1}{4}),\ x \in [-\frac{1}{2}, -\frac{1}{4}] \cup [\frac{1}{4}, \frac{1}{2}]$;

(b) $Q_n(x)Q_n(1/2 - 4x^2) \ge (Q_n(\frac{1}{4}))^2,\ x \in [-\frac{1}{4}, -\frac{1}{10}] \cup [\frac{1}{10}, \frac{1}{4}]$;

(c) $Q_n(x)Q_n(1/2 - 4x^2)Q_n(-64x^4 + 16x^2 - 1/2) \ge (Q_n(\frac{1}{4}))^3,\ x \in [-\frac{1}{10}, \frac{1}{10}]$.

Proof. Note that $Q_n(x)$, $Q_n(1/2 - 4x^2)$ and $Q_n(-64x^4 + 16x^2 - 1/2)$ are symmetric about the origin. Thus we only assume that $x \in [0, \frac{1}{2}]$.

The condition (a) follows from the fact that Q_n is increasing on $[0, \frac{1}{2}]$, since

$$Q_n'(x) = n(-(1/2 - x)^{n-1} + (1/2 + x)^{n-1}) \ge 0, \quad x \in \left[0, \frac{1}{2}\right].$$

We now prove the condition (b). For $n = 1$, we have

$$Q_1(x)Q_1\left(\frac{1}{2} - 4x^2\right) = 1 = \left(Q_1\left(\frac{1}{4}\right)\right)^2.$$

For $n = 2$, let $f(x) := Q_2(x)Q_2(\frac{1}{2} - 4x^2)$. Then a direct calculation shows that for $0 < x < \frac{1}{4}$,

$$f'(x) = 4x(-1 + 96x^4) < 0.$$

Thus we have

$$Q_2(x)Q_2(1/2-4x^2) \ge f(1/4) = (Q_2(1/4))^2, \quad \text{for } x \in \left[\frac{1}{10}, \frac{1}{4}\right].$$

Assume that $n \ge 3$. Since $(\frac{1}{2}-x)^n \ge (\frac{1}{3}(\frac{1}{2}+x))^n$ for $x \in [\frac{1}{10}, \frac{1}{4}]$, we have

$$\begin{aligned} Q_n(x)Q_n\left(\frac{1}{2}-4x^2\right) &= \left(\left(\frac{1}{2}-x\right)^n + \left(\frac{1}{2}+x\right)^n\right)((1-4x^2)^n+(4x^2)^n) \\ &\ge \left(\left(\frac{1}{3}\left(\frac{1}{2}+x\right)\right)^n + \left(\frac{1}{2}+x\right)^n\right)((1-4x^2)^n+(4x^2)^n) \\ &= \left(\left(\frac{1}{3}\right)^n+1\right)\left(\frac{1}{2}+x\right)^n((1-4x^2)^n+(4x^2)^n). \end{aligned} \tag{1}$$

Let $g_n(x) := (\frac{1}{2}+x)^n\left((1-4x^2)^n+(4x^2)^n\right)$. We claim that

$$g_n(x) \ge \left(\frac{3}{4}\right)^n\left(\left(\frac{3}{4}\right)^n+\left(\frac{1}{4}\right)^n\right), \quad \text{for } x \in \left[\frac{1}{10}, \frac{1}{4}\right]. \tag{2}$$

Indeed, we divide into two cases. Suppose that $x \in [\frac{1}{10}, \frac{1}{5}]$. Then

$$\begin{aligned} (g_n(x))^{1/n} &\ge \left(\frac{1}{2}+x\right)(1-4x^2) \ge \left(\frac{1}{2}+\frac{1}{10}\right)\left(1-4\left(\frac{1}{10}\right)^2\right) = 0.576 \\ &\ge \left(\frac{3}{4}\right)\left(\left(\frac{3}{4}\right)^3+\left(\frac{1}{4}\right)^3\right)^{1/3} \approx 0.569. \end{aligned}$$

Noticing that $((\frac{3}{4})^n+(\frac{1}{4})^n)^{1/n}$ is decreasing on n, we obtain Condition (2). Suppose, on the other hand, that $x \in [\frac{1}{5}, \frac{1}{4}]$. We first derive g_n' as follows:

$$\begin{aligned} g_n'(x) &= n\left(x+\frac{1}{2}\right)^{n-1}\left\{(1-4x^2)^n+(4x^2)^n\right\} + n\left(x+\frac{1}{2}\right)^n\left\{-8x(1-4x^2)^{n-1}+8x(4x^2)^{n-1}\right\} \\ &= n\left(x+\frac{1}{2}\right)^{n-1}\left\{(1-4x^2)^n+(4x^2)^n+\left(x+\frac{1}{2}\right)(-8x(1-4x^2)^{n-1}+8x(4x^2)^{n-1})\right\} \\ &= n\left(x+\frac{1}{2}\right)^{n-1}\left\{(1-4x^2)^{n-2}\left((1-4x^2)^2-8x(1-4x^2)\left(x+\frac{1}{2}\right)\right)\right. \\ &\quad \left. +(4x^2)^{n-2}\left((4x^2)^2+8x(4x^2)\left(x+\frac{1}{2}\right)\right)\right\}. \end{aligned}$$

Since $4x^2 \le 1-4x^2$ for $x \in [\frac{1}{5}, \frac{1}{4}]$, we obtain

$$\begin{aligned} g_n'(x) &\le n\left(x+\frac{1}{2}\right)^{n-1}\left\{(1-4x^2)^{n-2}\left((1-4x^2)^2-8x(1-4x^2)\left(x+\frac{1}{2}\right)\right)\right. \\ &\quad \left. +(1-4x^2)^{n-2}\left((4x^2)^2+8x(4x^2)\left(x+\frac{1}{2}\right)\right)\right\} \\ &= n\left(x+\frac{1}{2}\right)^{n-1}(1-4x^2)^{n-2}\left\{(1-4x^2)^2-8x(1-4x^2)\left(x+\frac{1}{2}\right)+(4x^2)^2+8x(4x^2)\left(x+\frac{1}{2}\right)\right\} \\ &= n\left(x+\frac{1}{2}\right)^{n-1}(1-4x^2)^{n-2}(96x^4+32x^3-16x^2-4x+1). \end{aligned} \tag{3}$$

Let $h(x) := 96x^4 + 32x^3 - 16x^2 - 4x + 1$. Then

$$h'(x) = 384\left(x - \frac{1}{4}\right)\left(x - \frac{-3+\sqrt{3}}{12}\right)\left(x - \frac{-3-\sqrt{3}}{12}\right).$$

Thus $h'(x) \le 0$ for $x \in [\frac{1}{5}, \frac{1}{4}]$. Since $h(\frac{1}{5}) = -\frac{19}{625} < 0$, $h(x) < 0$ for $x \in [\frac{1}{5}, \frac{1}{4}]$. This together with (3) implies that $g_n'(x) < 0$ for $x \in [\frac{1}{5}, \frac{1}{4}]$. Hence

$$g_n(x) \ge g_n\left(\frac{1}{4}\right) = \left(\frac{3}{4}\right)^n\left(\left(\frac{3}{4}\right)^n + \left(\frac{1}{4}\right)^n\right) \quad \text{for } x \in \left[\frac{1}{5}, \frac{1}{4}\right].$$

This concludes the claim. Putting this back to Condition (1), we obtain that for $x \in [\frac{1}{10}, \frac{1}{4}]$,

$$\begin{aligned} Q_n(x)Q_n\left(\frac{1}{2} - 4x^2\right) &\ge \left(\left(\frac{1}{3}\right)^n + 1\right)\left(\frac{3}{4}\right)^n\left(\left(\frac{3}{4}\right)^n + \left(\frac{1}{4}\right)^n\right) \\ &= \left(\left(\frac{3}{4}\right)^n + \left(\frac{1}{4}\right)^n\right)^2 \\ &= \left(Q_n\left(\frac{1}{4}\right)\right)^2. \end{aligned}$$

Finally, we check the condition (c). Note that since $Q_1(y) \equiv 1$, the condition (3) is obviously true for $n = 1$. Suppose that $n \ge 2$. It is obvious by elementary calculation that for $x \in [0, \frac{1}{10}]$,

$$\begin{aligned} &Q_n(x) = \left(\frac{1}{2} - x\right)^n + \left(\frac{1}{2} + x\right)^n \ge \left(\frac{1}{3}\right)^n\left(\frac{1}{2} + x\right)^n + \left(\frac{1}{2} + x\right)^n; \\ &Q_n\left(\frac{1}{2} - 4x^2\right) = (1 - 4x^2)^n + (4x^2)^n \ge (1 - 4x^2)^n \ge \left(1 - 4\left(\frac{1}{10}\right)^2\right)^n = \left(\frac{96}{100}\right)^n; \\ &Q_n\left(-64x^4 + 16x^2 - \frac{1}{2}\right) = (1 - 8x^2)^{2n} + (16x^2(1 - 4x^2))^n \ge (1 - 8x^2)^{2n}. \end{aligned}$$

These imply that

$$\left(Q_n(x)Q_n\left(\frac{1}{2} - 4x^2\right)Q_n\left(-64x^4 + 16x^2 - \frac{1}{2}\right)\right)^{1/n} \ge \left(\left(\frac{1}{3}\right)^n + 1\right)^{1/n}\left(\frac{1}{2} + x\right)\frac{96}{100}(1 - 8x^2)^2.$$

Since $(\frac{1}{2} + x)(1 - 8x^2)^2 \ge \frac{1}{2}$ for $x \in [0, \frac{1}{10}]$, we have

$$\left(\frac{1}{2} + x\right)\frac{96}{100}(1 - 8x^2)^2 \ge \frac{48}{100} > \frac{15}{32} = \frac{3}{4}\left(\left(\frac{3}{4}\right)^2 + \left(\frac{1}{4}\right)^2\right)^{2/2}.$$

Noticing that $((\frac{3}{4})^n + (\frac{1}{4})^n)^{2/n}$ is decreasing on n, we have

$$\begin{aligned} \left(Q_n(x)Q_n\left(\frac{1}{2} - 4x^2\right)Q_n\left(-64x^4 + 16x^2 - \frac{1}{2}\right)\right)^{1/n} &> \left(\left(\frac{1}{3}\right)^n + 1\right)^{1/n}\frac{3}{4}\left(\left(\frac{3}{4}\right)^n + \left(\frac{1}{4}\right)^n\right)^{2/n} \\ &= \left(\left(\frac{3}{4}\right)^n + \left(\frac{1}{4}\right)^n\right)^{3/n} \\ &= (Q_n(1/4))^{3/n}. \end{aligned}$$

This completes the proof. □

Theorem 3. *Let φ_n be the Butterworth refinable function of order n. Then*

$$|\hat{\varphi}_n(\xi)| \le C(1 + |\xi|)^{-2n+\kappa_n}, \tag{4}$$

where $\kappa_n = \log_2(Q_n(\frac{1}{4})) = 2n - n\log_2 3 - \log_2(1+3^{-n})$ *and this decay is optimal; i.e.,* $s_\infty(\varphi_n) = 2n - \kappa_n = n\log_2 3 + \log_2(1+3^{-n})$. *In particular,* $|\hat{\varphi}_n(\xi)| \le C(1+|\xi|)^{-n\log_2 3}$ *and* $\varphi_n \in C^s$ *for any* $s < n\log_2 3 - 1$.

Proof. Recall that

$$b_n(\xi) = \frac{\cos^{2n}(\xi/2)}{\cos^{2n}(\xi/2) + \sin^{2n}(\xi/2)}.$$

Since

$$Q_n(\sin^2(\xi/2) - 1/2) = \cos^{2n}(\xi/2) + \sin^{2n}(\xi/2),$$

$|\mathcal{L}(w)|$ in Proposition 1 is exactly $(Q_n(\sin^2(\xi/2) - 1/2))^{-1}$ here. Let $x := \sin^2(\xi/2) - 1/2$. Then we have

$$|\mathcal{L}(2\xi)| = \left(Q_n\left(\sin^2(\xi) - \frac{1}{2}\right)\right)^{-1}$$
$$= \left(Q_n\left(4\sin^2\left(\frac{\xi}{2}\right)\left(1 - \sin^2\left(\frac{\xi}{2}\right)\right) - \frac{1}{2}\right)\right)^{-1} = \left(Q_n\left(\frac{1}{2} - 4x^2\right)\right)^{-1}.$$

Similarly,

$$|\mathcal{L}(4\xi)| = \left(Q_n(-64x^4 + 16x^2 - 1/2)\right)^{-1}.$$

We take

$$D_1 := [-\pi, -2\pi/3] \cup [2\pi/3, \pi];$$
$$D_2 := [-2\pi/3, -2\sin^{-1}(\sqrt{3/5})] \cup [2\sin^{-1}(\sqrt{3/5}), 2\pi/3];$$
$$D_3 := [-2\sin^{-1}(\sqrt{3/5}), 2\sin^{-1}(\sqrt{3/5})].$$

Then it is easy to see that

$$\xi \in D_1 \Leftrightarrow x \in [-1/2, -1/4] \cup [1/4, 1/2];$$
$$\xi \in D_2 \Leftrightarrow x \in [-1/4, -1/10] \cup [1/10, 1/4];$$
$$\xi \in D_3 \Leftrightarrow x \in [-1/10, 1/10].$$

Hence, by Proposition 1 and Lemma 2, φ satisfies

$$|\hat{\varphi}(\xi)| \le C(1+|\xi|)^{-2n+\kappa},$$

where $\kappa = \log_2(|\mathcal{L}(2\pi/3)|)$ and this decay is optimal. This leads to $\varphi \in C^s$ for any $s < 2n - \kappa - 1$. □

We can also give the precise Sobolev exponent of a special class of refinable orthonormal cardinal functions from Blaschke products in [2].

Example 4. Consider the rational filter a_n defined by

$$a_n(\xi) = \frac{(1+e^{-i\xi})^{2n+1}}{(1+e^{-i\xi})^{2n+1} - (1-e^{-i\xi})^{2n+1}},$$

which yields the refinable orthonormal cardinal function φ_n. See [2]. We have

$$|a_n(\xi)| = \left(\frac{\cos^{2(2n+1)}(\xi/2)}{\cos^{2(2n+1)}(\xi/2) + \sin^{2(2n+1)}(\xi/2)}\right)^{1/2}.$$

Hence, by Theorem 3, we obtain

$$|\hat{\varphi}_n(\xi)| \le C(1+|\xi|)^{-\frac{1}{2}\{(2n+1)\log_2 3 + \log_2(1+3^{-2n-1})\}}$$

and this decay is optimal; i.e.,

$$s_\infty(\varphi_n) = \frac{1}{2}\{(2n+1)\log_2 3 + \log_2(1+3^{-2n-1})\}.$$

In particular,

$$|\hat{\varphi}_n(\xi)| \leq C(1+|\xi|)^{-\frac{1}{2}(2n+1)\log_2 3}$$

and

$$\varphi_n \in C^s \quad \text{for any } s < \frac{1}{2}(2n+1)\log_2 3 - 1. \quad \square$$

Acknowledgment

The authors thank the anonymous referee for careful and valuable comments.

References

[1] A. Fan, Q. Sun, Regularity of Butterworth refinable function, Asian J. Math. 5 (3) (2001) 433–440.
[2] M. Cotronei, M.L. LoCascio, H.O. Kim, C.A. Micchelli, T. Sauer, Refinable functions from Blaschke products, Rend. Mat. Appl. (7) 26 (2006) 267–290.
[3] I. Daubechies, Ten Lectures on Wavelets, in: CBMS-NSF Series in Applied Mathematics, SIAM, Philadelphia, 1992.