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Harmonic mappings onto stars

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Abstract

A general version of the Radó–Kneser–Choquet theorem implies that a piecewise constant sense-preserving mapping of the unit circle onto the vertices of a convex polygon extends to a univalent harmonic mapping of the unit disk onto the polygonal domain. This paper discusses similarly generated harmonic mappings of the disk onto nonconvex polygonal regions in the shape of regular stars. Calculation of the Blaschke product dilatation allows a determination of the exact range of parameters that produce univalent mappings.

Keywords: Harmonic mappings; Dilatation

1. Introduction

A *harmonic mapping* of the unit disk $\mathbb{D}$ onto a region in the plane is a complex-valued harmonic function. Every harmonic function $f(z)$ in $\mathbb{D}$ has a unique representation $f = h + \bar{g}$, where h and g are analytic in $\mathbb{D}$ and $g(0) = 0$. By a theorem of Lewy [6], the Jacobian of a locally univalent harmonic mapping never vanishes. If we take f to be *sense-*

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preserving, so that its Jacobian $|h'|^2 - |g'|^2$ is positive everywhere in $\mathbb{D}$, then its *dilatation* $\omega = g'/h'$ is an analytic function with $|\omega(z)| < 1$ in $\mathbb{D}$.

Our point of departure is the classical theorem of Radó [7], Kneser [5], and Choquet [1] (see also [2]). Suppose Ω is a convex domain bounded by a Jordan curve Γ, and let $w = f(e^{it})$ be a sense-preserving homeomorphism of the unit circle $\mathbb{T}$ onto Γ. Then the Poisson extension

$$f(z) = \frac{1}{2\pi}\int_0^{2\pi} \frac{1-|z|^2}{|e^{it}-z|^2}\, f\big(e^{it}\big)\, dt$$

is a *univalent* harmonic mapping of $\mathbb{D}$ onto Ω. The proof generalizes to show that for any piecewise constant sense-preserving mapping f of $\mathbb{T}$ onto the vertices of a convex polygonal region Ω, the Poisson extension represents a univalent harmonic mapping of $\mathbb{D}$ onto Ω. Suppose in particular that Ω is a regular polygon with m vertices at the mth roots of unity $\alpha, \alpha^2, \ldots, \alpha^m$, where $\alpha = e^{2\pi i/m}$. Recall the formula

$$u(z) = \frac{1}{\pi}\left(\arg\left\{\frac{z-e^{i\tau}}{z-e^{i\sigma}}\right\} - \frac{\tau-\sigma}{2}\right)$$

for the harmonic measure in the disk of the boundary arc $(e^{i\sigma}, e^{i\tau})$ extending counterclockwise from $e^{i\sigma}$ to $e^{i\tau}$, where $\sigma < \tau < \sigma + 2\pi$. Let $\beta = \sqrt{\alpha} = e^{i\pi/m}$. Then if the boundary correspondence is prescribed by

$$f\big(e^{it}\big) = \alpha^k \quad \text{for } e^{it} \in \big(\alpha^k\bar{\beta}, \alpha^k\beta\big),\ k = 1, 2, \ldots, m,$$

the harmonic extension takes the form (cf. [2])

$$f(z) = \frac{1}{\pi}\sum_{k=1}^{m} \alpha^k \arg\left\{\frac{z-\beta^{2k+1}}{z-\beta^{2k-1}}\right\}, \tag{1}$$

since $\alpha + \alpha^2 + \cdots + \alpha^m = 0$. It can also be shown that the dilatation of the function f defined by (1) is $\omega(z) = z^{m-2}$. More generally, Sheil-Small [8] has shown that for any harmonic extension of a piecewise constant boundary function with m values, the dilatation is a Blaschke product with $m-2$ factors, some of which may, however, have their zeros on or outside the unit circle. For mappings onto convex polygons, all zeros of this Blaschke product lie in the unit disk.

Hengartner and Schober [4] obtained a result in the converse direction, to the effect that a univalent harmonic mapping "onto" a convex domain must in fact be a mapping onto an inscribed polygon if its dilatation is a finite Blaschke product (see also [2, Section 7.4]).

We begin with the observation that the harmonic extension of a piecewise constant boundary function is univalent in $\mathbb{D}$ if and only if all zeros of its dilatation lie in $\mathbb{D}$. We then focus on regions in the shape of regular stars and determine the precise range of parameters for which the harmonic extension is univalent.

2. Criterion for univalence

Suppose now that Ω is a general polygon with vertices $c_1, c_2, \dots, c_m$, taken in counterclockwise order on the boundary $\Gamma = \partial\Omega$. For

$$0 \leqslant t_0 < t_1 < \cdots < t_m = t_0 + 2\pi,$$

let the points

$$b_k = e^{it_k}, \quad k = 0, 1, \dots, m,$$

determine an arbitrary partition of the unit circle into m subarcs. Note that $b_m = b_0$. Given the boundary correspondence

$$f\left(e^{it}\right) = c_k \quad \text{for } e^{it} \in (b_{k-1}, b_k),\ k = 1, 2, \dots, m,$$

construct the harmonic extension

$$f(z) = \frac{1}{\pi}\sum_{k=1}^{m} c_k \arg\left\{\frac{z - b_k}{z - b_{k-1}}\right\} - \hat{c}, \quad z \in \mathbb{D}, \tag{2}$$

where

$$\hat{c} = \frac{1}{2\pi}\sum_{k=1}^{m} c_k \arg\{b_k/b_{k-1}\}.$$

Observe that $f(0) = \hat{c}$, and that $\hat{c}$ belongs to the convex hull of the region Ω, although it need not lie in Ω.

According to a result of Sheil-Small [8], the dilatation of any function f of the form (2) is a Blaschke product with at most $m - 2$ factors of the form

$$\varphi_\zeta(z) = \frac{\zeta - z}{1 - \bar{\zeta} z}, \qquad |\zeta| \neq 1.$$

Some zeros ζ of the dilatation may be situated outside $\mathbb{D}$. However, the following theorem gives a criterion for the univalence of f.

Theorem 1. *Let f be a harmonic function of the form* (2), *constructed as above from a piecewise constant boundary function with values on the m vertices of a polygonal region Ω, so that the dilatation ω of f is a Blaschke product with at most $m - 2$ factors. Then f is univalent in $\mathbb{D}$ if and only if all zeros of ω lie in $\mathbb{D}$. In this case, f is a harmonic mapping of $\mathbb{D}$ onto Ω.*

Proof. Suppose first that f is univalent in $\mathbb{D}$. Observe that the finite Blaschke product ω can have no zeros on the unit circle, since $\varphi_\zeta(z) \equiv \zeta$ when $|\zeta| = 1$. If ω has a zero at some point ζ outside $\bar{\mathbb{D}}$, then it has a pole at $1/\bar{\zeta} \in \mathbb{D}$. If it also has zeros in $\mathbb{D}$, then there are points in $\mathbb{D}$ where $|\omega(z)| < 1$ and other points where $|\omega(z)| > 1$. The Jacobian of f then changes sign in $\mathbb{D}$, which contradicts to Lewy's theorem. Thus if f is univalent, there are only two possibilities. Either all zeros of ω lie in $\mathbb{D}$, or all lie outside $\bar{\mathbb{D}}$. But if all zeros of ω lie outside $\bar{\mathbb{D}}$, then $|\omega(z)| > 1$ in $\mathbb{D}$ and f has negative Jacobian, contradicting its construction from a sense-preserving boundary function. Therefore, all zeros of ω must lie in $\mathbb{D}$.

Conversely, if all zeros of ω are in $\mathbb{D}$, then $|\omega(z)| < 1$ in $\mathbb{D}$ and an application of the argument principle for harmonic functions [3] shows that f is univalent in $\mathbb{D}$ and it maps $\mathbb{D}$ onto Ω. To be more specific, choose an arbitrary point $w_0 \in \Omega$ and let C_ε be the path in $\bar{\mathbb{D}}$ consisting of arcs of the unit circle alternating with small circular arcs of radius ε about the points b_k. If ε is sufficiently small, the image curve $f(C_\varepsilon)$ will have winding number $+1$ about the point w_0. Since $|\omega(z)| < 1$ inside C_ε, it follows from the argument principle that $f(z) - w_0$ has one simple zero inside C_ε. Thus $\Omega \subset f(\mathbb{D})$. If $w_0 \notin \Omega$, a similar construction shows that $w_0 \notin f(\mathbb{D})$. Thus f maps $\mathbb{D}$ univalently onto Ω. □

3. Star mappings

We now specialize the construction to harmonic mappings onto regions in the shape of regular stars. For $n \geqslant 2$, our target region Ω will be an n-pointed star with its "inner vertices" at the points $r\alpha^{2k}$ for $k = 1, 2, \ldots, n$, where $\alpha = e^{i\pi/n}$ and $0 < r \leqslant 1$, and its "outer vertices" at the points α^{2k+1}. Thus the inner vertices lie in the directions of the nth roots of unity, at distance r from the origin, whereas the outer vertices lie on the unit circle. Simple geometric considerations show that Ω is convex if and only if $\cos(\pi/n) \leqslant r \leqslant 1$. We will explore the behavior of the canonical harmonic mapping as r varies and the star changes shape. With $\beta = \sqrt{\alpha} = e^{i\pi/2n}$, we first prescribe the symmetric boundary correspondence

$$f\left(e^{it}\right) = \begin{cases} r\alpha^{2k}, & e^{it} \in (\alpha^{2k}\bar{\beta}, \alpha^{2k}\beta), \\ \alpha^{2k+1}, & e^{it} \in (\alpha^{2k+1}\bar{\beta}, \alpha^{2k+1}\beta), \end{cases} \tag{3}$$

where $k = 1, 2, \ldots, n$. Then the harmonic extension to $\mathbb{D}$ is

$$f(z) = \frac{r}{\pi} \sum_{k=1}^{n} \alpha^{2k} \arg \frac{z - \alpha^{2k}\beta}{z - \alpha^{2k}\bar{\beta}} + \frac{1}{\pi} \sum_{k=1}^{n} \alpha^{2k+1} \arg \frac{z - \alpha^{2k+1}\beta}{z - \alpha^{2k+1}\bar{\beta}}. \tag{4}$$

For $\cos(\pi/n) \leqslant r \leqslant 1$ the target region Ω is convex and the Radó–Kneser–Choquet theorem ensures that f maps $\mathbb{D}$ univalently onto Ω. We shall see, however, that the univalence persists for a larger interval including values of r that generate nonconvex configurations of Ω. In fact, we will determine the exact range of values of the parameter r for which f is univalent. The transition from univalence to nonunivalence will be explained in terms of the dilatation of f, which we now calculate.

First note that f has the canonical decomposition $f = h + \bar{g}$ with

$$h(z) = \frac{r}{2\pi i} \sum_{k=1}^{n} \alpha^{2k} \log \frac{z - \alpha^{2k}\beta}{z - \alpha^{2k}\bar{\beta}} + \frac{1}{2\pi i} \sum_{k=1}^{n} \alpha^{2k+1} \log \frac{z - \alpha^{2k+1}\beta}{z - \alpha^{2k+1}\bar{\beta}},$$

$$g(z) = \frac{r}{2\pi i} \sum_{k=1}^{n} \bar{\alpha}^{2k} \log \frac{z - \alpha^{2k}\beta}{z - \alpha^{2k}\bar{\beta}} + \frac{1}{2\pi i} \sum_{k=1}^{n} \bar{\alpha}^{2k+1} \log \frac{z - \alpha^{2k+1}\beta}{z - \alpha^{2k+1}\bar{\beta}}.$$

Now calculate the derivatives

$$h'(z) = \frac{r-\alpha}{2\pi i}\sum_{k=1}^{n}\frac{\alpha^{2k}}{z-\alpha^{2k}\beta} - \frac{r-\bar{\alpha}}{2\pi i}\sum_{k=1}^{n}\frac{\alpha^{2k}}{z-\alpha^{2k}\bar{\beta}},$$
$$g'(z) = \frac{r-\bar{\alpha}}{2\pi i}\sum_{k=1}^{n}\frac{\bar{\alpha}^{2k}}{z-\alpha^{2k}\beta} - \frac{r-\alpha}{2\pi i}\sum_{k=1}^{n}\frac{\bar{\alpha}^{2k}}{z-\alpha^{2k}\bar{\beta}}, \tag{5}$$

using the identities $\alpha^{2k}\beta = \alpha^{2k+1}\bar{\beta}$ and $\alpha^{2(k+1)}\bar{\beta} = \alpha^{2k+1}\beta$. In view of the partial fraction expansions

$$\frac{1}{z^n - i} = \frac{\beta}{in}\sum_{k=1}^{n}\frac{\alpha^{2k}}{z-\alpha^{2k}\beta}, \qquad \frac{1}{z^n + i} = -\frac{\bar{\beta}}{in}\sum_{k=1}^{n}\frac{\alpha^{2k}}{z-\alpha^{2k}\bar{\beta}},$$
$$\frac{z^{n-2}}{z^n - i} = \frac{\bar{\beta}}{n}\sum_{k=1}^{n}\frac{\bar{\alpha}^{2k}}{z-\alpha^{2k}\beta}, \qquad \frac{z^{n-2}}{z^n + i} = \frac{\beta}{n}\sum_{k=1}^{n}\frac{\bar{\alpha}^{2k}}{z-\alpha^{2k}\bar{\beta}}, \tag{6}$$

these formulas reduce to

$$h'(z) = \frac{n}{\pi}\,\frac{(1+r)\operatorname{Im}\{\beta\} - (1-r)\operatorname{Re}\{\beta\}z^n}{z^{2n}+1},$$
$$g'(z) = \frac{nz^{n-2}}{\pi}\,\frac{(1+r)\operatorname{Im}\{\beta\}z^n - (1-r)\operatorname{Re}\{\beta\}}{z^{2n}+1}.$$

Thus f has dilatation

$$\omega(z) = \frac{g'(z)}{h'(z)} = z^{n-2}\frac{(1+r)\operatorname{Im}\{\beta\}z^n - (1-r)\operatorname{Re}\{\beta\}}{(1+r)\operatorname{Im}\{\beta\} - (1-r)\operatorname{Re}\{\beta\}z^n}$$
$$= z^{n-2}\frac{z^n - c}{1 - cz^n}, \tag{7}$$

where

$$c = \frac{(1-r)\operatorname{Re}\{\beta\}}{(1+r)\operatorname{Im}\{\beta\}} = \frac{1-r}{1+r}\cot\frac{\pi}{2n}. \tag{8}$$

Note that $c \geqslant 0$ for $0 < r \leqslant 1$.

4. Univalence of star mappings

On the basis of our dilatation formula (7) we can now determine the exact range of parameters n and r for which the function f provides a univalent harmonic mapping of $\mathbb{D}$ onto the corresponding star-shaped region Ω. According to Theorem 1, this will be the case if and only if all zeros of the function ω lie in $\mathbb{D}$. Thus we have arrived at the following result.

Theorem 2. *Let f be the harmonic function* (4) *in the unit disk $\mathbb{D}$ with boundary values* (3), *where $n \geqslant 2$ and $0 < r \leqslant 1$. Then f is univalent in $\mathbb{D}$ if and only if $r_1 \leqslant r \leqslant 1$, where*

$$r_1 = \frac{1 - \sin\frac{\pi}{n}}{\cos\frac{\pi}{n}}.$$

In this case f is a univalent harmonic mapping of $\mathbb{D}$ onto the star-shaped domain Ω with vertices $r\alpha^{2k}$ and α^{2k+1}, where $k = 1, 2, \ldots, n$.

Proof. In view of Theorem 1, it suffices to examine the zeros of the dilatation of f, as given by the formula (7). Clearly, all zeros of ω lie in $\mathbb{D}$ if and only if $c \leqslant 1$, where c is given by (8). If $r = 1$, then $c = 0$ and $\omega(z) = z^{2n-2}$, as predicted by the general considerations of Section 1. The formula (8) shows that $c > 0$ for $r < 1$, and that $c = 1$ when

$$r = \frac{1 - \tan\frac{\pi}{2n}}{1 + \tan\frac{\pi}{2n}} = \frac{1 - \sin\frac{\pi}{n}}{\cos\frac{\pi}{n}} = r_1.$$

If $c = 1$, then $\omega(z) = z^{n-2}$, with all of its zeros at the origin. If $r_1 < r < 1$, then $c < 1$ and the formula (7) shows that ω has all of its zeros in $\mathbb{D}$. Thus if $r_1 \leqslant r \leqslant 1$, Theorem 1 says that f maps $\mathbb{D}$ univalently onto Ω. If $r < r_1$, then $c > 1$ and ω has n zeros outside $\bar{\mathbb{D}}$ in addition to a zero of order $n - 2$ at the origin, so by Theorem 1 the function f is not univalent in $\mathbb{D}$. □

It should be observed that $r_1 < \cos(\pi/n)$ for $n \geqslant 3$, since

$$1 - \sin\theta < \cos^2\theta, \quad 0 < \theta < \pi/2.$$

For $r_1 < r < \cos(\pi/n)$ the harmonic extension f maps $\mathbb{D}$ univalently onto Ω for some values of r where Ω is not convex.

As an illustration of Theorem 2, let $n = 6$, so that $r_1 = 1/\sqrt{3} = 0.577\ldots$ and $\cos(\pi/n) = \sqrt{3}/2 = 0.866\ldots$. Figure 1 shows the images under f of concentric circles

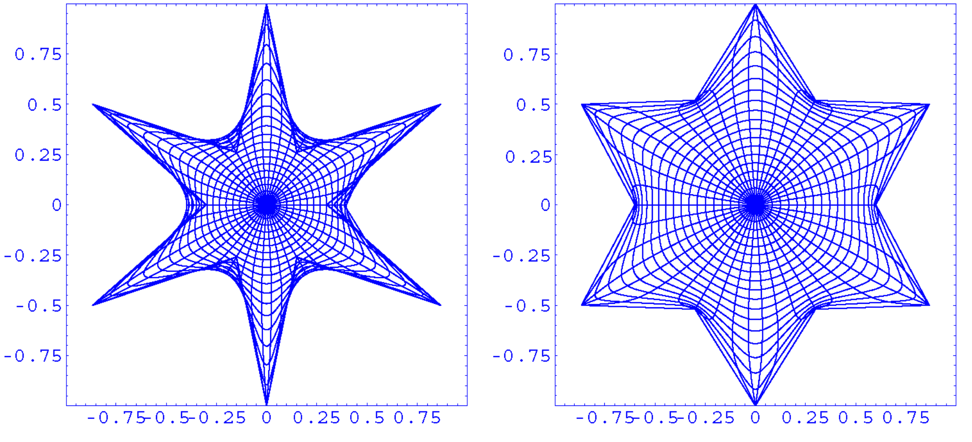

Fig. 1. Images of mapping f for $n = 6$ and $r = 0.3, 0.6$.

and radial segments, as produced by *Mathematica* graphics. For $r = 0.3 < r_1$, the mapping f is not univalent and folding occurs near the inner vertices of Ω. For $r = 0.6 > r_1$, the figure confirms that f is univalent although Ω is not convex.

5. Unequal arcs

We now consider harmonic mappings whose boundary correspondence is a step function defined on a system of arcs of two different sizes. This construction will enable us to produce a univalent harmonic mapping onto an arbitrarily prescribed star, with piecewise constant boundary correspondence.

Again let $\alpha = e^{i\pi/n}$, where $n \geqslant 2$. Let $\beta = e^{ip\pi/n}$ and $\gamma = e^{i(1-p)\pi/n}$ for $0 < p < 1$, so that $\beta\gamma = \alpha$. Given $r \in (0, 1]$, let f be the harmonic mapping of $\mathbb{D}$ with boundary correspondence

$$f\left(e^{it}\right) = \begin{cases} r\alpha^{2k}, & e^{it} \in (\alpha^{2k}\bar{\beta}, \alpha^{2k}\beta), \\ \alpha^{2k+1}, & e^{it} \in (\alpha^{2k+1}\bar{\gamma}, \alpha^{2k+1}\gamma), \end{cases} \tag{9}$$

where $k = 1, 2, \ldots, n$. Then

$$f(z) = \frac{r}{\pi}\sum_{k=1}^{n} \alpha^{2k} \arg \frac{z - \alpha^{2k}\beta}{z - \alpha^{2k}\bar{\beta}} + \frac{1}{\pi}\sum_{k=1}^{n} \alpha^{2k+1} \arg \frac{z - \alpha^{2k+1}\gamma}{z - \alpha^{2k+1}\bar{\gamma}}. \tag{10}$$

Note that $\beta = \gamma$ when $p = \frac{1}{2}$, and then f reduces to the earlier form (4).

The dilatation of $f = h + \bar{g}$ is calculated as before, with minor adjustments. The formulas (5) for $h'(z)$ and $g'(z)$ remain valid, but now with $\beta = e^{ip\pi/n}$. Suitable modifications of the partial fraction expansions (6) then lead to the expressions

$$h'(z) = \frac{n}{2\pi i}\left\{\frac{(r-\alpha)\bar{\beta}e^{ip\pi}}{z^n - e^{ip\pi}} - \frac{(r-\bar{\alpha})\beta e^{-ip\pi}}{z^n - e^{-ip\pi}}\right\},$$
$$g'(z) = \frac{nz^{n-2}}{2\pi i}\left\{\frac{(r-\bar{\alpha})\beta}{z^n - e^{ip\pi}} - \frac{(r-\alpha)\bar{\beta}}{z^n - e^{-ip\pi}}\right\},$$

which produce the dilatation formula

$$\omega(z) = \frac{g'(z)}{h'(z)} = z^{n-2}\frac{z^n - c}{1 - cz^n}, \tag{11}$$

where c is now defined by

$$c = c(p) = \frac{\operatorname{Im}\{(r-\alpha)\bar{\beta}e^{ip\pi}\}}{\operatorname{Im}\{(r-\alpha)\bar{\beta}\}} = \frac{\operatorname{Im}\{\gamma^{n-1}\} - r\operatorname{Im}\{\beta^{n-1}\}}{\operatorname{Im}\{\gamma\} + r\operatorname{Im}\{\beta\}}$$
$$= \frac{\sin((1-\frac{1}{n})(1-p)\pi) - r\sin((1-\frac{1}{n})p\pi)}{\sin(\frac{1}{n}(1-p)\pi) + r\sin(\frac{1}{n}p\pi)}. \tag{12}$$

Observe that for $p = \frac{1}{2}$ the expression (12) reduces to (8).

With the dilatation formula in hand, the proof of Theorem 2 can be adapted to yield the following generalization.

Theorem 3. *Let f be the harmonic function* (10) *with boundary values* (9)*, where $n \geqslant 2$, $0 < r \leqslant 1$, and $0 < p < 1$. Let c be defined by* (12)*. Then f is univalent in $\mathbb{D}$ if and only if $-1 \leqslant c \leqslant 1$. In this case, f is a univalent harmonic mapping of $\mathbb{D}$ onto the star-shaped domain with vertices $r\alpha^{2k}$ and α^{2k+1} for $k = 1, 2, \ldots, n$.*

Consequently, the question of univalence comes down to the behavior of the function $c(p)$ as defined by (12). It is easily seen that $c(0) = 1$ and $c(1) = -1$. Straightforward calculations lead to the simple formulas

$$c'(0) = \pi \sin\frac{\pi}{n}\left(\cos\frac{\pi}{n} - r\right), \qquad c'(1) = \pi \sin\frac{\pi}{n}\left(\cos\frac{\pi}{n} - \frac{1}{r}\right)$$

for the derivatives at the two endpoints of the interval $0 \leqslant p \leqslant 1$. If $r \geqslant \cos(\pi/n)$, then the target region Ω is convex and f is univalent for $0 \leqslant p \leqslant 1$, by the Radó–Kneser–Choquet theorem. However, if $r < \cos(\pi/n)$, we see that $c'(0) > 0$ and therefore $c(p) > c(0) = 1$ for all $p > 0$ sufficiently small, so that the corresponding mapping f is not univalent. On the other hand, for each $r > 0$ we see that $c'(1) < 0$ and so $-1 = c(1) < c(p) < 1$ for all $p < 1$ sufficiently large. Thus for an arbitrary target region Ω it is possible to adjust the boundary correspondence (9) to produce a harmonic function f that maps the disk univalently onto Ω. These conclusions are summarized in the following theorem.

Theorem 4. *Let Ω be an arbitrarily prescribed star-shaped domain with $n \geqslant 2$ and $0 < r \leqslant 1$. Then for all $p < 1$ sufficiently large, the function f defined by* (10) *is a univalent harmonic mapping of $\mathbb{D}$ onto Ω. If $0 < r < \cos(\pi/n)$, so that Ω is not convex, then for each $p > 0$ sufficiently small the function f is not univalent in $\mathbb{D}$.*

Intuitively, Theorem 4 is true because for p near 1, relatively large boundary arcs are mapped to the inner vertices; whereas for $r < \cos(\pi/n)$ and p near 0, relatively large arcs are mapped to the outer vertices. Figure 2 illustrates Theorem 4 by displaying the same regions Ω as in Fig. 1, but now for $r = 0.3$ and $p = 0.81$ the mapping f is univalent,

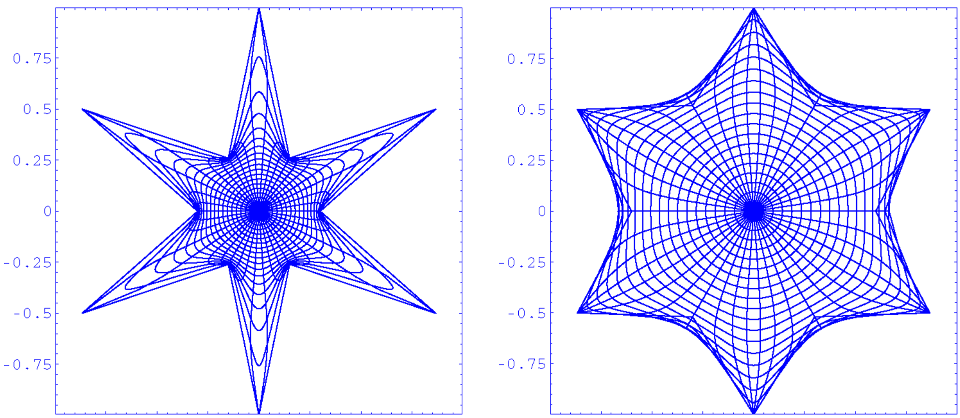

Fig. 2. Images for $n = 6$ and $r = 0.3$, $p = 0.81$; $r = 0.6$, $p = 0.2$.

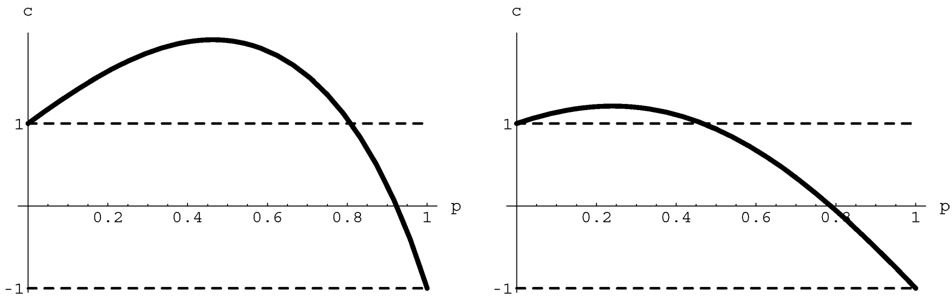

Fig. 3. Graphs of $c = c(p)$ for $n = 6$ and $r = 0.3, 0.6$.

whereas for $r = 0.6$ and $p = 0.2$ it is not univalent. Graphs of the corresponding functions $c(p)$ are shown in Fig. 3.

It is an interesting open question whether every simply connected domain with polygonal boundary is the univalent image of some harmonic function with piecewise constant boundary function.

Curiously, the analysis shows that $c = 0$ and thus f has dilatation $\omega(z) = z^{2n-2}$ in certain cases of irregular boundary distribution; that is, for $r < 1$ and $p \neq \frac{1}{2}$. According to the formula (12), this happens precisely when

$$\sin\left(\left(1 - \frac{1}{n}\right)(1-p)\pi\right) = r \sin\left(\left(1 - \frac{1}{n}\right)p\pi\right) \quad \text{or}$$

$$r = \sin\frac{\pi}{n}\cot\left(\left(1 - \frac{1}{n}\right)p\pi\right) + \cos\frac{\pi}{n}. \tag{13}$$

For $n \geqslant 2$ and $0 < r < 1$, the relation (13) holds for some uniquely determined p in the interval $\frac{1}{2} < p < 1$. For $n \geqslant 2$ and $\frac{1}{2} < p < 1$, the formula (13) produces a radius r in the interval $0 < r < 1$.

References

[1] G. Choquet, Sur un type de transformation analytique généralisant la répresentation conforme et définie au moyen de fonctions harmoniques, Bull. Sci. Math. 69 (1945) 156–165.
[2] P. Duren, Harmonic Mappings in the Plane, Cambridge Univ. Press, Cambridge, UK, 2004.
[3] P. Duren, W. Hengartner, R.S. Laugesen, The argument principle for harmonic functions, Amer. Math. Monthly 103 (1997) 411–415.
[4] W. Hengartner, G. Schober, On the boundary behavior of orientation-preserving harmonic mappings, Complex Variables Theory Appl. 5 (1986) 197–208.
[5] H. Kneser, Lösung der Aufgabe 41, Jahresber. Deutsch. Math.-Verein. 35 (1926) 123–124.
[6] H. Lewy, On the non-vanishing of the Jacobian in certain one-to-one mappings, Bull. Amer. Math. Soc. 42 (1936) 689–692.
[7] T. Radó, Aufgabe 41, Jahresber. Deutsch. Math.-Verein. 35 (1926) 49.
[8] T. Sheil-Small, On the Fourier series of a step function, Michigan Math. J. 36 (1989) 459–475.