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## ORIGINAL ARTICLE

# On non-negative estimation of variance components in mixed linear models



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## ABSTRACT

Alternative estimators have been derived for estimating the variance components according to Iterative Almost Unbiased Estimation (IAUE). As a result two modified IAUEs are introduced. The relative performances of the proposed estimators and other estimators are studied by simulating their bias, Mean Square Error and the probability of getting negative estimates under unbalanced nested-factorial model with two fixed crossed factorial and one nested random factor. Finally the Empirical Quantile Dispersion Graph (EQDG), which provides a comprehensive picture of the quality of estimation, is depicted corresponding to all the studied methods.

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**Introduction**

Quite often, experimental research work requires the empirical identification of the relationship between an observable response variable and a set of associated variables, or factors, believed to have an effect on the response variable. In general, such a relationship, if it exists, is unknown, but is usually assumed to be linear which yields the unknown parameters appear linearly in such a model, then it is called a classical

linear model. It is reasonable to add random effects to the classical linear model which includes fixed effects only. Searle et al. [1] provided a decision tree to assist us to decide whether the parameters are fixed or not. The rule is that if we can reasonably assume the levels of the factor come from a probability distribution, then treat the factor as random; otherwise fixed. If the model contains both fixed and random effects, we can extend classical linear model to mixed linear model which is commonly used.

Variance components estimation has a wide application as it has two major uses as well as many minor ones, the more familiar of the major uses is determining which factors have a significant effect upon the response being studied. The second major use is measuring the relative effect of factors on the dependent factor. Over the years, a plethora of variance components estimation methods has been extensively developed. ANOVA method, Minimum Norm

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Quadratic Unbiased Estimation (MINQUE), IAUE, Maximum Likelihood (ML) method and Restricted Maximum Likelihood (REML) are some of the most important methods available in the literature. A proper and comprehensive review of other methods can be found in Sahai [2], Khuri and Sahai [3] and Khuri [4]. Consequently a number of attempts have been made to study the relative performance and the properties of various estimators in order to determine the best estimator under different criteria such as bias, MSE and computational complexities.

Since most of variance components estimators cannot be explicitly written in various situations, thus conducting the comparisons analytically can be considered as intractable process. Accordingly, the numerical comparisons approach is adopted via many scholars; for instance, Sahai [5] compared between ANOVA, MLE and REML for the three stage nested model when all the factors are random, Swallow and Monahan [6] made a comparison between ANOVA, MLE, REML, and MINQUE methods through running one way model, Rao and Heckler [7] provided some modifications on ANOVA method and presented numerical comparisons among various variance component methods in the case of unbalanced threefold nested random model. Lee and Khuri [8] used the EQDG to make a comparison between the ANOVA and ML estimation methods under two-way random model without interaction terms. Jung et al. [9] compared between ANOVA and MLE under threefold nested random model based upon the EQDG. Subramani [10] introduced a new procedure to estimate the variance components in light of MINQUE approach, further he demonstrated numerically that his proposed estimator has less MSE than both ANOVA and MINQUE method using one way random unbalanced model. Chen and Wei [11] derived parametric empirical Bayes estimators and compared with ANOVA method under one-way random model.

A typical challenge of variance component methods about is that not all of them produce positive estimates, which is not acceptable in the practice. The negative values of the estimates of the variance components might arise for a variety of reasons such as choosing unsuitable set of initial variance components, violating of linearity condition, existing outliers in the data, and closing the actual values of the variance components to zero. However Thompson [12] realized that the natural of the estimator or the used algorithm can be considered as the major reason of the negatively. Further Lamotte [13] proved analytically that the only linear combination of variance components for which satisfies unbiasedness and non-negativity is the single error component estimator in variance components model. Although there is a number of authors replace the negative variance components with zero value, many efforts have been made in order to design non-negative estimator of variance components.

Horn et al. [14] proposed IAUE which avoids the non-negativity of MINQUE. Jennrich and Sampson [15] suggested to replace the negative estimates of the variance components with 5 value as done in some packages, or force the algorithm to take the non-negativity in the consideration by adding non-negative constraints. Kelly and Mathew [16] proposed non-negative quadratic estimator which offers substantial MSE improvement. Khattree [17] suggested a simple modification ensuring the non-negativity of Henderson's ANOVA method. Searle et al.[1] explained EM algorithm for treating the non-negativity problem associated with both ML and REML

methods. Teunissen and Amiri [18] discussed how to modify Least-squares method in order to ensure that the estimated variances are non-negative. Moghtased-Azar et al. [19] suggested a new idea to deal with the negatively related to REML method. The major motivation behind this article is providing a new estimator for estimating variance components through applying simple modifications on IAUE which is so-called (MIAUE).

The rest of the paper is organized as follows: The second section concerns with the REML method introduced by Thompson [12] and Modified REML (MREML) using EM algorithm explained via Searle et al. [1]. The third section reviews the MINQUE method proposed by Rao [20] and Modified MINQUE (MMINQUE) derived by Subramani [10]. The fourth section presents IAUE method suggested by Horn et al. [14]. The fifth section illustrates the proposed estimators Modified IAUE (MIAUE). The followed section summarizes the steps of EQDG approach in depth which are employed in this study. The next section includes the Monte Carlo results using unbalanced nested-factorial model. Finally some conclusions about the work are given in the last section.

### REML and MREML method

Consider the variance components model stated by Subramani [10]

$$Y = X\beta + Z_1\delta_1 + Z_2\delta_2 \cdots + Z_{r-1}\delta_{r-1} + Z_r\delta_r \quad (2.1)$$

where  $Y$  is a  $n \times 1$  vector of observations,  $X$  is a  $n \times m$  matrix with known constants,  $\beta$  is a  $m \times 1$  vector of fixed (unknown) parameters,  $Z_i$  is a  $n \times c_i$  matrix of known constants and  $\delta_i$  is  $c_i \times 1$  random vector has multivariate normal distribution with zero mean and covariance matrix  $\sigma_i^2 I_{c_i}$ . Further it is assumed that  $\delta_i$  and  $\delta_j$ ,  $i \neq j$  are uncorrelated. Model (1) can be expressed in a compact form as:

$$Y = X\beta + Z\delta \quad (2.2)$$

where  $Z = [Z_1 Z_2 \dots Z_r]$  and  $\delta' = [\delta_1' \delta_2' \dots \delta_r']$ . The model (2) is called a mixed linear model. If  $r = 1$ , it becomes a fixed model and if  $m = 1$  it becomes a random model. Thus generally we have  $E(Y) = X\beta$  and  $D(Y) = \sum_{i=1}^r \sigma_i^2 V_i$ , where  $V_i = Z_i Z_i'$ ,  $D$  is called the dispersion matrix and the parameters  $\sigma_1^2 \sigma_2^2 \dots \sigma_r^2$  are the unknown variance components whose values should be estimated.

Since the normality distribution is assumed, thus it is acceptable to operate distribution-based methods. The preferred parametric method for estimating variance components is REML. The original reference to REML is the article by Thompson [12]. One of the interesting features of REML is that it takes account of the implicit degrees of freedom associated with the fixed effects as maximizing the likelihood function of the linear combination of the observations. Moreover, REML estimators are invariant to the fixed effects.

Theoretically, REML can be illustrated as assuming  $K_{n-x,n}$  be a full rank matrix, where  $x$  is the rank of  $X$ , such that  $KX = 0$ , then the likelihood of  $KY$  can be formulated as:

$$L(\sigma/Y) \propto |KDK'|^{-.5} \exp(-.5(Y'K'(KDK')^{-1}KY))$$

the log likelihood of  $KY$  becomes:

$$\ln(L(\sigma/Y)) \propto -.5\ln|KDK'| - .5(Y'K'(KDK')^{-1}KY)$$

in order to obtain REML estimates, it is required to take the partial derivatives of  $\ln(L(\sigma/Y))$  with respect to  $\sigma$  then setting to zero, we obtain

$$\frac{\partial \ln(L(\sigma/Y))}{\partial \sigma_i^2} = -.5(\text{tr}(K'(KDK')^{-1}KV_i) - Y'K'(KDK')^{-1}KV_iK'(KDK')^{-1}KY) = 0$$

using the lemma given in Searle et al.[1] that states:

$$K'(KDK')^{-1}K = P, \text{ where } P = D^{-1} - D^{-1}X(X'D^{-1}X)^{-1}X'D^{-1}.$$

hence we will get

$$\frac{\partial \ln(L(\sigma/Y))}{\partial \sigma_i^2} = (\text{tr}(PV_i) - Y'PV_iPY) = 0 \quad i = 1 \dots r. \quad (2.3)$$

It is obvious that we have  $r$  equations in  $r$  unknowns  $\sigma$ . In some cases these equations can be simplified to yield closed form. Yet, in almost all cases numerical algorithms have to be used to solve the equations. In this study, the algorithm proposed in [19] is devoted. In addition, it should be noted that the system of equations in (3) does not involve the elements of  $K$ , which means no matter what their values, the same result will be reached (see Searle et al. [1]). The main drawback concerning to REML technique is that the solution in (3) can be negative, which is not allowed in the real life problems. This dilemma can be resolved by operating Expectation Maximization (EM) algorithm which is perfectly explained in Searle et al. [1] considered.

EM algorithm is the most well-known technique used in the applied statistics produced firstly by Dempster et al. [21] to obtain ML estimators in the incomplete data. EM algorithm is a mechanism consisting of an expectation followed by maximization stage. Fortunately it is able to apply EM algorithm to estimate the variance components in the mixed linear models. The stages of EM algorithm can be expressed as following:

1. Obtain a starting value of  $\sigma_i^{2(f)}$ .
2. E-step: calculate the  $E(\delta'_i\delta_i|Y)|_{\sigma_i^2=\sigma_i^{2(f)}}$ . Since  $[KY, \delta_i]$  are normally distributed, then  $f(\delta_i|KY)$  is a normal distribution with mean  $\sigma_i^2Z'_iPY$  and variance  $\sigma_i^2I_{c_i} - \sigma_i^4Z'_iPZ_i$ , hence  $E(\delta'_i\delta_i|Y)$  would be:

$$E(\delta'_i\delta_i|Y) = \sigma^4 Y'PV_iPY + \text{tr}(\sigma_i^2I_{c_i} - \sigma_i^4Z'_iPZ_i)$$

3. M-step: determine  $\hat{\sigma}_i^2$  as maximizing the complete data which includes the observed data and the random effects  $\delta$ :

$$\hat{\sigma}_i^{2(f+1)} = \frac{E(\delta'_i\delta_i|Y)}{c_i}$$

4. While  $|\hat{\sigma}_i^{2(f+1)} - \hat{\sigma}_i^{2(f)}|/\hat{\sigma}_i^{2(f)} > .01$  increase  $f$  by one unit and return to step 1, otherwise terminate the calculations and set  $\hat{\sigma}_i^2 = \hat{\sigma}_i^{2(f+1)}$ .

The variance components estimates computed using the EM algorithm is donated hereafter as MREML. Harville [22] stated that the EM algorithm has the property of always yielding positive estimates as long as prior values or initial points are positive, thus using any non-negative variance components estimates may be reliable to be considered as started values for the EM algorithm. Despite EM algorithm can be rather slow

to converge and required heavy iterations, but it is not sensitivity to the initial values (see [23,24]).

### MINQUE and MMINQUE method

Rao [20] decided to estimate the unknown variance components as considering  $Y'AY$  as an estimator to the linear combination of the variance components  $\rho'\sigma$ , where  $\rho$  is known vector and  $\sigma = (\sigma_1^2\sigma_2^2 \dots \sigma_r^2)'$ , then selecting a symmetric matrix  $A$  that satisfies the following criteria:

- Invariance under translation of the  $\beta$  parameter The first criterion should be satisfied by  $A$  is somewhat intuitive as  $A$  should not be sensitivity to location shifting in the fixed parameters. In other words  $A$  should satisfy the following equation:

$$Y'AY = (Y - X\beta_0)'A(Y - X\beta_0)$$

where  $\beta_0$  is a constant vector, which means

$$AX = 0$$

- Unbiasedness The second criterion should be satisfied by  $A$  that is:

$$E(Y'AY) = \rho'\sigma$$

but

$$\begin{aligned} E(Y'AY) &= E((X\beta + Z\delta)'A(X\beta + Z\delta)) \\ &= E(\beta'X'AX\beta + 2\beta'X'AZ\delta + \delta'Z'AZ\delta) \end{aligned}$$

under the Invariant condition, we can get:

$$E(Y'AY) = E(\delta'Z'AZ\delta) = \sum_{i=1}^r E(\delta'_iZ'_iAZ_i\delta_i) = \sum_{i=1}^r \sigma_i^2 \text{trace}(AV_i)$$

hence

$$\sum_{i=1}^r \sigma_i^2 \text{trace}(AV_i) = \sum_{i=1}^r \rho_i \sigma_i^2 \text{ which means :}$$

$$\text{tr}(AV_i) = \rho_i$$

- Minimum Norm The third criterion should be satisfied by  $A$  is that minimize the Euclidean norm of the difference between  $Y'AY$  and the natural unbiased estimator of  $\rho'\sigma$ , which can be formulated as:

$$\begin{aligned} \text{Min} \left\| \delta'Z'AZ\delta - \sum_{i=1}^r \frac{\rho_i}{c_i} \delta'_i\delta_i \right\| &= \text{Min} \|\delta'(Z'AZ - \Delta)\delta\| \\ &\cong \text{Min} \|Z'AZ - \Delta\| \end{aligned}$$

where  $\|\cdot\|$  denotes the Euclidean norm of the matrix,  $\Delta = \text{diag}\left(\frac{\rho_1}{c_1}I_{c_1} \frac{\rho_2}{c_2}I_{c_2} \dots \frac{\rho_r}{c_r}I_{c_r}\right)$ . Thus we can state that  $Y'AY$  is MINQUE of  $\rho'\sigma$  if the symmetric matrix  $A$  is selected such that  $\|Z'AZ - \Delta\|$  is minimum as possible as subject to:

$$AX = 0 \text{ and } \text{tr}(AV_i) = \rho_i$$

For making the optimization more easier, the squared Euclidean norm, the sum of square of all elements in the matrix, will be utilized. Then we get

$$\begin{aligned}\|Z'AZ - \Delta\|^2 &= tr((Z'AZ - \Delta)'(Z'AZ - \Delta)) \\ &= tr(AVAV) + \Delta^*\end{aligned}$$

where  $V = \sum_{i=1}^r V_i = ZZ'$  and  $\Delta^*$  refers to constant quantity does not involve  $A$ . Let  $A$  be a symmetric matrix and  $V$  be a symmetric and invertible matrix. Then the minimum  $tr(AVAV)$  subject to invariant and unbiasedness criteria is attained at, according to Rao [20]:

$$A = \sum_{i=1}^r a_i R V_i R$$

where  $a = S^{-1}\rho$ ,  $S_{ij} = tr(Q'V^{-1}V_iV^{-1}QV_j)$ ,  $i$  and  $j = 1 \dots r$ ,  $Q = I_n - X(X'V^{-1}X)^{-1}X'V^{-1}$  and  $R = Q'V^{-1}$ . Consequently, the MINQUE of  $\rho'\sigma$  is

$$Y'AY = \sum_{i=1}^r a_i Y'RV_iRY = \sum_{i=1}^r a_i b_i^* = d'b^* = \rho'S^{-1}b^*$$

where  $b^* = Y'RV_iRY$ . By equating  $Y'AY$  with  $\rho'\sigma$ , we can get:

$$\sigma_{MINQUE} = S^{-1}b^*$$

In the case of the singularity of the matrix  $S$ , one can resort to calculate the generalized inverse of  $S$ . On another hand, Subramani [10] proposed a new idea to develop the estimation of variance components in light of Rao [20] approach. Instead of dealing with one linear combination, he decided to estimate a set of linear combinations of variance components  $\rho'\sigma$  through a set of quadratic functions  $Y'A_iY$ . In other words, he claimed that estimating variance components obtained by calculating the following normal equations:

$$\begin{bmatrix} \sigma_1^2 \\ \vdots \\ \sigma_r^2 \end{bmatrix} = \begin{bmatrix} \rho_{11} & \cdots & \rho_{1r} \\ \vdots & \ddots & \vdots \\ \rho_{r1} & \cdots & \rho_{rr} \end{bmatrix}^{-1} \begin{bmatrix} Y'A_1Y \\ \vdots \\ Y'A_rY \end{bmatrix} \quad (3.1)$$

Likewise, the symmetric  $A_i$  should be derived based upon certain criteria:

- Invariance under translation of the  $\beta$  parameter It can easily be shown that the invariant condition will be satisfied if:

$$A_iX = 0$$

- Unbiasedness In order to ensure the unbiasedness,  $A_i$  should satisfy:

$$E(Y'A_iY) = \rho_i'\sigma = \sum_{j=1}^r \rho_{ij}\sigma_j$$

under the invariant condition, we can get:

$$\rho_{ij} = tr(A_iV_j) \quad (3.2)$$

- Minimum Norm As already pointed above, in order to minimize the squared Euclidian norm between  $Y'A_iY$  and the natural estimator of  $\rho_i'\sigma$ . According to Subramani [20], the following theorem with our proof guides us the strategy of selecting  $A_i$  that minimizes  $tr(A_iVA_iV)$ . Theorem Let  $V_{n \times n}$

be a given symmetric matrix and  $A_{n \times n}$  be an unknown symmetric matrix such that  $tr(AV) = rank(AV) = p < n$ . Then  $\|AV^2\|$  attains minimum at  $A^*V$ , where  $A^*V$  is any symmetric idempotent matrix.

**Proof.** Since  $rank(AV) = p < n$ , then we have  $p$  non-zero characteristic roots of  $AV$  such that:

$$tr(AV) = \sum_{t=1}^p \lambda_t = p$$

in addition,

$$\|AV\|^2 = tr(AVAV) = \sum_{t=1}^p \lambda_t^2$$

Now, minimizing  $\|AV\|^2$  is as equivalent as minimizing  $\sum_{t=1}^p \lambda_t^2$ . Hence the optimization problem may be reformulated as:

$$\text{Min} \sum_{t=1}^p \lambda_t^2 \text{ subject to } \sum_{t=1}^p \lambda_t = p$$

using the Lagrange multipliers technique, the Lagrangian can be defined as:

$$A(\lambda_1 \dots \lambda_p, \lambda^*) = \sum_{t=1}^p \lambda_t^2 - \lambda^* \left( \sum_{t=1}^p \lambda_t - p \right)$$

where  $\lambda^*$  denotes the constant of the Lagrange multipliers. Lagrange's equations can be obtained:

$$\frac{\partial A(\lambda_1 \dots \lambda_p, \lambda^*)}{\partial \lambda_t} = 2\lambda_t - \lambda^* = 0 \quad t = 1 \dots p$$

and

$$\frac{\partial A(\lambda_1 \dots \lambda_p, \lambda^*)}{\partial \lambda^*} = \sum_{t=1}^p \lambda_t - p = 0.$$

Since  $\lambda_t = \frac{\lambda^*}{2}$ ,  $t = 1 \dots p$ , then  $\sum_{t=1}^p \frac{\lambda^*}{2} - p = 0$ , which yields  $\lambda^* = 2$ , hence  $\lambda_t = 1$ ,  $t = 1 \dots p$ .

Consequently, Subramani [10] deduced the minimum of  $\|AV\|^2$  will be reached if we replace  $A$  with  $A^*$  such that the characteristic roots of  $A^*V$  are only zero's and one's, which refers to the idempotency of the matrix. Thus the steps of MMINQUE can be summarized as: (1) Selecting  $A_i$  such that  $A_iV$  is an idempotent matrix and  $A_iX = 0$ . (2) Substituting (3.2) in (3.1), then calculating the normal equations. The remaining point is the structure of  $A_i$ . Since the solution in the theorem is not unique, Subramani [10] introduced two different formulas of  $A_i$  which can be reliable to obtain MMINQUE.

The first version of  $A_i$  can be derived as assuming:

$$A_{i1} = V^{-1}(I_n - U_i(U_i'V^{-1}U_i)^{-1}U_i'V^{-1}) \quad i = 1 \dots r$$

where  $U_1 = X$ ,  $U_2 = [XZ_1]$ ,  $U_3 = [XZ_2] \dots U_r = [XZ_{r-1}]$ . The second version of  $A_i$  can be derived as assuming:

$$\begin{aligned}A_{i2} &= G_i(G_i'VG_i)^{-1}G_i' \\ &\quad - G_i(G_i'VG_i)^{-1}G_i'X(X'G_i(G_i'VG_i)^{-1}G_i'X)^{-1}X'G_i(G_i'VG_i)^{-1}G_i'\end{aligned}$$

where  $G_i = Z_i$ . In reality, Subramani [20] proposed other shapes of  $U_i$ 's and  $G_i$ 's, yet we confine ourselves to select the preceding shapes as the others lead finally to the same result.<sup>1</sup> The main drawback that may be thrown to MMINQUE is the existence of the condition that  $tr(AV) = rank(AV)$  in the theorem which leads MMINQUE valid only in this class of the matrices. Moreover the negativity is possible which will be resolved in the next section. It should be pointed out that if we replace  $V$  in  $\sigma_{MINQUE}$ ,  $\sigma_{MMINQUE1}$  or  $\sigma_{MMINQUE2}$ ,<sup>2</sup> by  $D$ , then the estimators are called weighted MINQUE, weighed MMINQUE1 and weighed MMINQUE2 respectively.  $\square$

### IAUE method

The concept of IAUE was developed by Horn et al. [14]. IAUE can be considered as an advantageous alternative to MINQUE approach basically when MINQUE produces negative estimates. Lucas [25] stated that though IAUE gives bias estimators, it is far less computation than many variance component methods even though it usually requires more iterations to converge to the same degree of approximation. Analogously to Rao [20], Horn et al. [14] preferred to estimate the variance components  $\sigma_i^2$  with quadratic form  $Y'A_iY$  given  $A_i$  has the following formula:

$$A_i = R^* \tau_i V_i R^*$$

where  $R^* = D^{*-1} - D^{*-1}X(X'D^{*-1}X)^{-1}X'D^{*-1}$ ,  $D^* = \sum_{i=1}^r \tau_i V_i$  and  $\tau_i$  be the prior estimate for  $\sigma_i^2$ . Then the expectation of  $Y'A_iY$  can be obtained as:

$$\begin{aligned} E(Y'A_iY) &= E(Y'R^* \tau_i V_i R^* Y) \\ &= tr(R^* \tau_i V_i R^* D) + \beta' X' R^* \tau_i V_i R^* X \beta \end{aligned}$$

Since  $R^*X = 0$ , hence:

$$E(Y'A_iY) = \sum_{j=1}^r tr(R^* \tau_i V_i R^* \sigma_j^2 V_j) = \sum_{j=1}^r f_j tr(R^* \tau_i V_i R^* \tau_j V_j)$$

where  $f_j = \frac{\sigma_j^2}{\sigma_j}$ . Horn et al. [14] showed that  $R^*D^*R^* = R^*$ , then we can get:

$$\begin{aligned} E(Y'R^* \tau_i V_i R^* Y) &= \sum_{j=1}^r f_j tr(R^* \tau_i V_i R^* \tau_j V_j) + f_i tr(\tau_i V_i R^*) \\ &\quad - f_i tr(\tau_i V_i R^* D^* R^*) = \sum_{j=1}^r f_j tr(R^* \tau_i V_i R^* \tau_j V_j) \\ &\quad + f_i tr(\tau_i V_i R^*) - f_i \sum_{j=1}^r tr(\tau_i V_i R^* \tau_j V_j R^*) \\ &= \sum_{j=1}^r (f_j - f_i) tr(R^* \tau_i V_i R^* \tau_j V_j) + f_i tr(\tau_i V_i R^*) \end{aligned}$$

If all the prior estimates  $\tau_i$  approach to the true values or at least the ratios between  $\tau_i$  and the true values are close, the first term of the previous equation will vanish, and the working equation can be simplified as:

$$E(\tau_i Y'R^* V_i R^* Y) = f_i tr(\tau_i V_i R^*)$$

Which yields:

$$\hat{f}_i = \frac{Y'R^* V_i R^* Y}{tr(V_i R^*)}$$

Consequently, the IAUE can be summarized as: (1) Choose initial value for  $\tau_i$ . (2) Compute  $\hat{f}_i$  based on  $\tau_i$ . (3) Update the values of  $\tau_i$  until all  $\hat{f}_i$ 's approach one by using any iterative procedure. (4) Finally calculate  $\sigma_{iIAUE}^2 = \tau_i \hat{f}_i$ . In other words  $\sigma_{iIAUE}^2$  can be expressed as:

$$\sigma_{iIAUE}^2 = \hat{\tau}_i |_{\hat{f}_i \cong 1}$$

The more significant advantage related to IAUE is its facility computation and non-negativity property as the numerator of  $\hat{f}_i$  in a quadratic form as  $R^* V_i R^*$  is a positive definite matrix and the denominator can be written in a sums of squares as:

$$\begin{aligned} tr(R^* V_i) &= tr(R^* D^* R^* V_i) = \sum_{i=1}^r \tau_i tr(R^* V_i R^* V_j) \\ &= \sum_{j=1}^r \tau_i tr(R^* Z_i Z_i' R^* Z_j Z_j') \\ &= \sum_{j=1}^r \tau_i tr((Z_j' R^* Z_i)(Z_i' R^* Z_j)') \\ &= \sum_{i=1}^r \tau_i tr(Z_j' R^* Z_i)' (Z_j' R^* Z_i) \end{aligned}$$

### MIAUE method

On the other hand, one can easily operate IAUE principle to MMINQUE which generates new non-negative estimators in light of Subramani [10]. Mathematically, the suggested estimators can be expressed as considering the expectation of the quadratic form:

$$E(Y'A_{i1}^* \tau_i V_i A_{i1}^* Y) = tr(A_{i1}^* \tau_i V_i A_{i1}^* D) + \tau_i \beta' X' A_{i1}^* V_i A_{i1}^* X \beta$$

Where  $A_{i1}^* = D^{*-1}(I_n - U_i(U_i' D^{*-1} U_i)^- U_i' D^{*-1})$ . In light of the Invariant condition:

$$E(Y'A_{i1}^* \tau_i V_i A_{i1}^* Y) = tr(A_{i1}^* \tau_i V_i A_{i1}^* D)$$

Since

$$\begin{aligned} A_{i1} D^* A_{i1} &= D^{*-1}(I_n - U_i(U_i' D^{*-1} U_i)^- U_i' D^{*-1})(I_n \\ &\quad - U_i(U_i' D^{*-1} U_i)^- U_i' D^{*-1}). \\ &= D^{*-1}(I_n - 2U_i(U_i' D^{*-1} U_i)^- U_i' D^{*-1} \\ &\quad + U_i(U_i' D^{*-1} U_i)^- U_i' D^{*-1} U_i(U_i' D^{*-1} U_i)^- U_i' D^{*-1}). \\ &= D^{*-1}(I_n - U_i(U_i' D^{*-1} U_i)^- U_i' D^{*-1}) = A_{i1}^* \end{aligned}$$

then we have:

$$\begin{aligned} E(Y'A_{i1}^* \tau_i V_i A_{i1}^* Y) &= \sum_{j=1}^r f_j tr(A_{i1}^* \tau_i V_i A_{i1}^* \tau_j V_j) + f_i tr(\tau_i A_{i1}^* V_i) \\ &\quad - f_i \sum_{j=1}^r tr(\tau_i V_i A_{i1}^* \tau_j V_j A_{i1}^*) \\ &= \sum_{j=1}^r (f_j - f_i) tr(A_{i1}^* \tau_i V_i A_{i1}^* \tau_j V_j) + f_i tr(\tau_i A_{i1}^* V_i) \end{aligned}$$

<sup>1</sup> We concluded this result during recording simulation's results, thus our conclusion is restricted to nested-factorial model with two fixed crossed factorial and one nested random factor.

<sup>2</sup>  $\sigma_{MMINQUE1}$  and  $\sigma_{MMINQUE2}$  are based upon  $A_{i1}$  and  $A_{i2}$  respectively.

thus we can get under neglecting the difference between  $f_i$  and all  $f_j$ :

$$\hat{f}_{i1} = \frac{Y' A_{i1}^* V_i A_{i1}^* Y}{tr(V_i A_{i1}^*)}$$

As previously mentioned during deriving  $\sigma_{iAUE}^2$ ,  $\sigma_{iMIAUE1}^2$  can be computed as:

$$\widehat{\sigma}_{iMIAUE1}^2 = \hat{\tau}_i |_{\hat{f}_i, s \geq 1}$$

Likewise  $\sigma_{iMIAUE2}^2$ , can be calculated as:

$$\widehat{\sigma}_{iMIAUE2}^2 = \hat{\tau}_i |_{\hat{f}_i, s \geq 1},$$

where

$$\hat{f}_{i2} = \frac{Y' A_{i2}^* V_i A_{i2}^* Y}{tr(V_i A_{i2}^*)}$$

and

$$A_{i2}^* = G_i (G_i' D^* G_i)^{-1} G_i' - G_i (G_i' D^* G_i)^{-1} G_i' X (X' G_i (G_i' D^* G_i)^{-1} G_i' X)^{-1} X' G_i (G_i' D^* G_i)^{-1} G_i'$$

It is notable that both  $\sigma_{iMIAUE1}^2$  and  $\sigma_{iMIAUE2}^2$  are not required heavy calculations and not producing negative estimates, which yields that MIAUE1 and MIAUE2 can be considered as a competitor estimators to IAUE.

### Empirical quantile dispersion graphs

Quantile Dispersion Graph (QDG) is a graphical technique used, typically, for comparing and assessing the quality of the variance components estimations. The QDG was suggested by Lee and Khuri [8] as consisting of plots of the maxima and minima, in our view one of them suffices, over some region in the parameter space against the quantiles of a variance component estimator. These plots provide a comprehensive picture of the quality of estimation with a particular variance component method. Since most of variance component methods have not a closed-form expression, so the quantiles can be obtained numerically, in this case QDG is so-called empirical QDG (EQDG). The steps of the EQDG can be outlined, according to Lee and Khuri [8], as follows:

- Select specific variance component method.
- Generate a random sample  $Y$  from the model (2) corresponding to  $\sigma_1^2 \dots \sigma_r^2$ .
- Use the random sample obtained in (b) and the variance component method in (a) to estimate the variance components  $\hat{\sigma}_1^2 \dots \hat{\sigma}_r^2$ .
- Repeat steps (b) and (c) sufficient number of times.
- Compute the quantity  $q_s = \frac{\hat{\sigma}_s^2}{\sigma_1^2}$ , where  $s$  is the index of the times' number in (d).
- Corresponding to certain specific percentiles values  $p_h$ ,<sup>3</sup> obtain the empirical quantiles  $w_{h1}$  of  $q_s$ , where  $h$  is the index of the percentiles' values.
- Repeat steps (e) and (f) to the remaining  $\sigma_i^2$ 's.

- Select another point of  $\beta, \sigma_1^2 \dots \sigma_r^2$  from the determined region space in order to obtain another  $w_{h2}$  for each  $\sigma_i^2$ .
- Repeat step (h) sufficient times until all points in the determined region space are covered.
- Computed the maximum of  $[W_{h1}, W_{h1}, \dots, W_{hh^*}]$ , where  $h^*$  is the number of the points in the determined region space corresponding to each  $p_h$ . This maximum will be so-called here Empirical Quantile Maximum (EQM).
- Turn on another variance component method and obtain EQM associated with each  $\sigma_i^2$ .
- Repeat the step (k)  $k$  times, where  $k$  is the number of the variance component methods under the study.
- For each  $\sigma_i^2$  a line chart is obtained with the percentiles values  $p_h$  on the  $X$ -axis, while the EQM corresponding to each variance component method on the  $Y$ -axis.

As expected whether the specific variance component method is perfect, then the elements of EQM should be identical and close to the one, otherwise it is referred to little quality for estimating the variance components. In other words, the more variability in EQM the less efficiency of the corresponding method. It should be noted that EQM reflects on the variability associated with the estimators not other characteristics e.g. biasedness or getting negative values, etc.

### Simulation study

It may be of interest to make a comparison study among all the preceding variance components estimates. Since it may be impossible to do any theoretical comparisons about the performance of them, thus one has to resort to compare through Monte Carlo simulation. Following to Melo et al. [26], nested factorial design with two crossed factors and one nested factor is adopted in this context in order to identify the behavior of variance components estimators which can be described as:

$$y_{abcd} = \alpha_a + \beta_b + \gamma_{c(a)} + \alpha\beta_{ab} + \beta\gamma_{bc(a)} + \varepsilon_{abcd}$$

$a = 1 \dots I$ ;  $b = 1 \dots J$ ;  $c = 1 \dots K_a$ ;  $d = 1 \dots n_{abc}$  where  $\alpha_a$  is the effect of the  $a$  level of factor A,  $\beta_b$  is the effect of the  $b$  level of factor B,  $\gamma_{c(a)}$  is the effect of the  $c$  level of factor C nested within the  $a$  level of factor A,  $\alpha\beta_{ab}$  is the interaction effect between the factor A and B,  $\beta\gamma_{bc(a)}$  is the interaction effect between the factor B and C instead within the a level of factor A and  $\varepsilon_{abcd}$  is a random term. It is assumed that all the effects in the model are fixed parameters except  $\gamma_{c(a)}$ ,  $\beta\gamma_{bc(a)}$  and  $\varepsilon_{abcd}$  are normally independently distributed such that:

$$\gamma_{c(a)} \sim N(0, \sigma_1^2), \beta\gamma_{bc(a)} \sim N(0, \sigma_2^2) \text{ and } \varepsilon_{abcd} \sim N(0, \sigma_3^2).$$

**Table 1** Variance components configurations used in the simulation.

|    | $\sigma_1^2$ | $\sigma_2^2$ | $\sigma_3^2$ |
|----|--------------|--------------|--------------|
| V1 | .1           | 1            | 1            |
| V2 | .1           | 10           | 1            |
| V3 | 1            | .1           | 1            |
| V4 | 1            | 10           | 1            |
| V5 | 10           | 1            | 1            |
| V6 | 10           | .1           | 1            |

<sup>3</sup> Lee and Khuri [8] selected the values of  $p_h$  as .01, .05, .1, .2, .3, .4, .5, .6, .7, .8, .9, .95 and .99.

**Table 2** The patterns of imbalance rate for each sample size used in the simulation.

|    | $n$ | $I$ | $J$ | $K_i$ | $n_{ijk}$  | $\phi$ |
|----|-----|-----|-----|-------|--|--------|
| P1 | 24  | 2   | 2   | 2,2   | 3, . . . , 3                                     | 1      |
| P2 | 24  | 2   | 2   | 2,2   | 2,5; 3,2; 5,3; 3,1                               | .83    |
| P3 | 24  | 2   | 2   | 2,2   | 1,2; 1,2; 2,8; 7,1                               | .56    |
| P4 | 36  | 3   | 2   | 1,2,3 | 3, . . . , 3                                     | 1      |
| P5 | 36  | 3   | 2   | 1,2,3 | 2,2; 2,5; 2,5; 4,4; 3,3; 2,2                     | .87    |
| P6 | 36  | 3   | 2   | 1,2,3 | 1,1; 2,5; 2,1; 1,10; 3,1; 7,2                    | .53    |
| P7 | 63  | 3   | 3   | 2,3,2 | 3, . . . , 3                                     | 1      |
| P8 | 63  | 3   | 3   | 2,3,2 | 2,2,4; 4,2,4; 2,2,4; 4,3,6; 3,3,4; 3,3,2; 3,2,1  | .87    |
| P9 | 63  | 3   | 3   | 2,3,2 | 2,1,1; 3,2,10; 8,1,2; 1,1,3; 3,3,2; 9,3,3; 3,1,1 | .57    |

Since the fixed effects are out of our interest, thus one can fix all the fixed parameters at one. Oppositely, the comparison process requires to be conducted under a variety of variance components configurations, difference of imbalance degrees and multiple sample sizes. Following to Rao and Heckler [7], Table 1 displays the variance components values used in the simulation. A lot of measures of imbalance have been introduced in the literature, see Khuri et al. [27], which can be selected with the aim of covering different levels of imbalance of  $n_{abc}$  and various sample sizes. According to Qie and Xu [28], the measure which is introduced by Ahrens and Pincus [28] can be reliable for reflecting the imbalance effect of  $n_{abc}$  which can be formulated as:

$$\phi = \frac{1}{m^* \sum_a \sum_b \sum_c \left(\frac{n_{abc}}{n}\right)^2}$$

where  $n$  is the grand sample size and  $m^* = J \times \sum_a K_a$ . Ahrens and Pincus [29] illustrated that the values of  $\phi$  range from  $\frac{1}{m^*}$  up to one, the smaller values refer a greater degree of imbalance, while the larger values are only for balanced case. Table 2 presents the patterns of imbalance according to different sample sizes throughout the simulation.

For each variance components configuration and pattern of imbalance combination, 2000 independent random samples were generated, then all the negative estimates are forced to be zero. The estimated bias, MSE and probability of getting negative estimates<sup>4</sup> are shown in Table 3.

According to Table 3, a number of conclusions are drawn from the results for all the patterns and designs which are summarized in the following points:

- (a) For the completely balanced designs, it does not matter computing MINQUE, MMINQUE1 or MMINQUE2 because they are the same.
- (b) Generally speaking, one can observe that REML has the lowest compound absolute bias among all the estimators in most cases, whereas MREML can be considered as the best estimator in terms of MSE criteria.
- (c) it is reasonable to note that the compound absolute biasedness of MINQUE, MMINQUE1 and MMINQUE2 is lower than IAUE, MIAUE1 and MIAUE2 regardless the sample size or imbalance rate.

<sup>4</sup> The probability of getting negative values is calculated as one minus the number of the samples whose all are non-negative out of 2000.

Oppositely, the compound MSE associated with MINQUE, MMINQUE1 and MMINQUE2 is greater than IAUE, MIAUE1 and MIAUE2 in most cases.

- (d) Among the negative methods, REML estimator has the best behavior in terms of both bias and MSE, while MREML in the case of the non-negative methods.
- (e) It is clear the superiority of MIAUE1 and MIAUE2 over IAUE in terms of biasedness criterion that the latter across ALL cases has bias greater than either MIAUE1 or MIAUE2 or both. However the proposed estimators have MSE less than all MINQUE, MMINQUE1 and MMINQUE2.
- (f) The sample size and the imbalance rate have substantially effect on the behavior of all the estimators, as either increasing the small size or reducing the imbalance rate yield to significant improvement in the two measures of the performance. Furthermore, there is an interaction effect between the sample size and the imbalance rate as the effect of the imbalance rate is downward at high level of the sample size.
- (g) The performance of the estimators depends heavily on the ratio of  $\frac{\sigma_1^2}{\sigma_2^2}$ . It is observed that the compound absolute biasedness of the estimators is acceptable whenever the ratio is greater than one.
- (h) There are negligible differences among MINQUE, MMINQUE1 MMINQUE2 and REML with respect to the frequency of getting negative values, yet in almost cases it is remarkable that the frequency at MMINQUE2 is slightly higher than the remaining and relatively lower at REML. The sample size has strong effect in reducing the probability of getting negative values, while the imbalance effect has weak effect.

In order to enhance the numerical comparison process, EQDG's which provide a powerful graphical tool for the comparisons are exhibited for all the above estimators which are given in Fig. 1. In addition the norm of EQM is computed and obtained as shown in Table 4.

The extracted results from both EQDG and EQM coincide with the above conclusions as MREML can be donated as the best estimator since it has the least MSE, whereas MMINQUE2 has the highest variability among the above estimators. On the other hand all the estimators based on  $A_{I1}$  are better than those based on MINQUE and  $A_{I2}$ . Furthermore, one can notice that the degrees of freedom have substantially negative effect on the norm of all above estimators, thus the norm associated with  $\sigma_2^2$  is lower than the norm

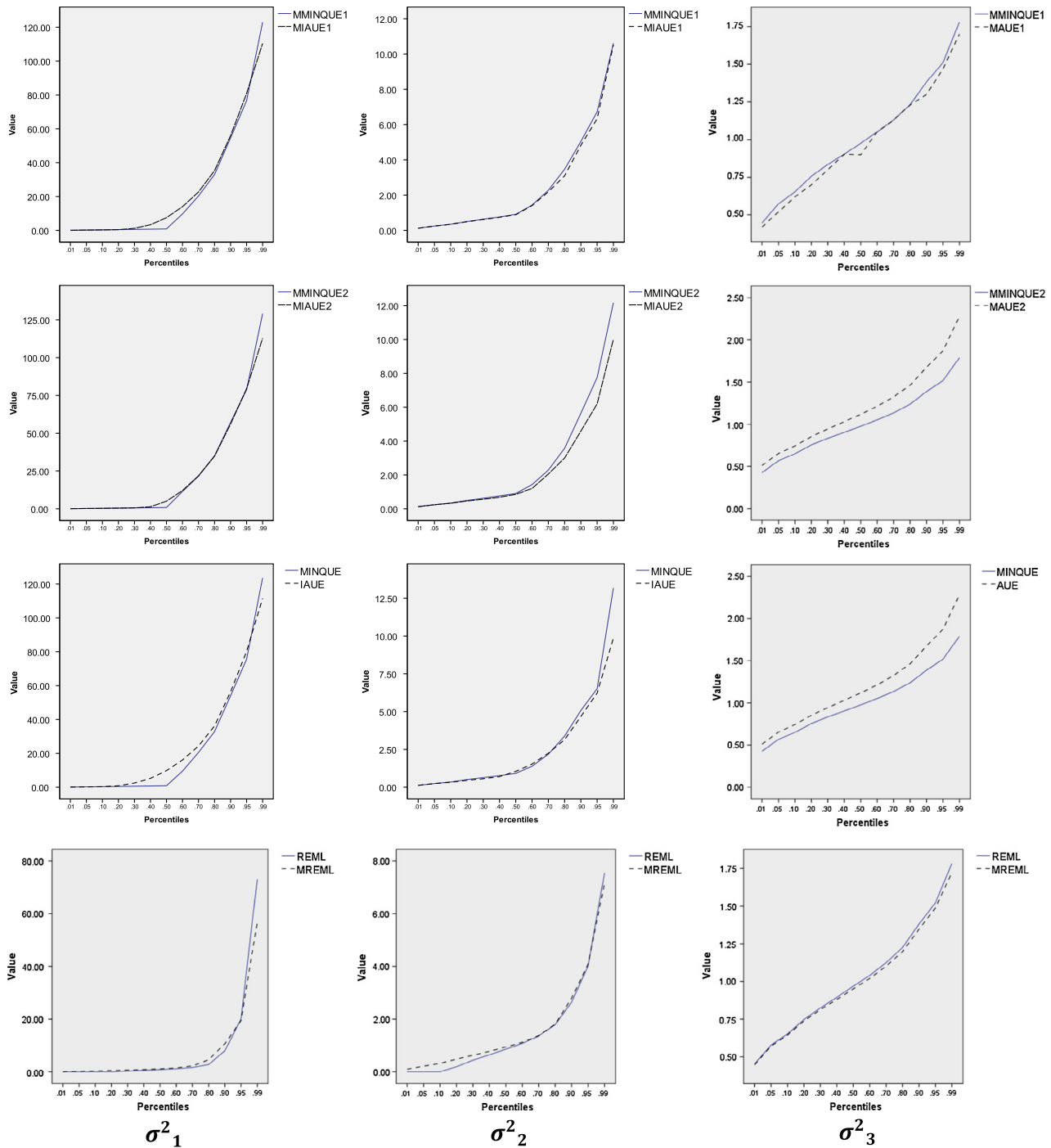
**Table 3** Comparison of MINQUE, MMINQUE, IAUE, MIAUE, REML and MREML estimators based on compound absolute bias, compound MSE and prop. negative values.

|           | MMINQUE1               |              |                       | MMINQUE2               |              |                       | MINQUE                 |              |                       | MIAUE1                 |              | MIAUE2                 |              | IAUE                   |              | REML                   |              |                       | MREML                  |              |
|-----------|------------------------|--------------|-----------------------|------------------------|--------------|-----------------------|------------------------|--------------|-----------------------|------------------------|--------------|------------------------|--------------|------------------------|--------------|------------------------|--------------|-----------------------|------------------------|--------------|
|           | Compound absolute bias | Compound MSE | Prop. negative values | Compound absolute bias | Compound MSE | Prop. negative values | Compound absolute bias | Compound MSE | Prop. negative values | Compound absolute bias | Compound MSE | Compound absolute bias | Compound MSE | Compound absolute bias | Compound MSE | Compound absolute bias | Compound MSE | Prop. negative values | Compound absolute bias | Compound MSE |
| <i>P1</i> |                        |              |                       |                        |              |                       |                        |              |                       |                        |              |                        |              |                        |              |                        |              |                       |                        |              |
| V1        | 0.26                   | 1.31         | 0.56                  | 0.26                   | 1.31         | 0.56                  | 0.26                   | 1.31         | 0.56                  | 0.26                   | 1.14         | 0.46                   | 0.86         | 0.48                   | 0.62         | 0.24                   | 1.30         | 0.56                  | 0.48                   | 0.60         |
| V2        | 2.35                   | 71.58        | 0.52                  | 2.35                   | 71.58        | 0.52                  | 2.35                   | 71.58        | 0.52                  | 2.28                   | 70.70        | 3.87                   | 50.51        | 4.15                   | 50.48        | 2.16                   | 69.84        | 0.52                  | 4.27                   | 37.20        |
| V3        | 0.07                   | 1.10         | 0.51                  | 0.07                   | 1.10         | 0.51                  | 0.07                   | 1.10         | 0.51                  | 0.113                  | 1.180        | 0.115                  | 1.154        | 0.142                  | 1.145        | 0.08                   | 1.15         | 0.52                  | 0.51                   | 0.76         |
| V4        | 2.38                   | 78.14        | 0.46                  | 2.38                   | 78.14        | 0.46                  | 2.38                   | 78.14        | 0.46                  | 1.95                   | 69.88        | 3.49                   | 53.12        | 3.75                   | 52.82        | 2.03                   | 77.22        | 0.46                  | 3.63                   | 40.47        |
| V5        | 0.17                   | 74.94        | 0.12                  | 0.17                   | 74.94        | 0.12                  | 0.17                   | 74.94        | 0.12                  | 0.04                   | 73.54        | 0.04                   | 73.54        | 0.04                   | 73.48        | 0.09                   | 73.98        | 0.11                  | 2.57                   | 51.05        |
| V6        | 0.16                   | 72.47        | 0.45                  | 0.16                   | 72.47        | 0.45                  | 0.16                   | 72.47        | 0.45                  | 1.31                   | 74.07        | 2.01                   | 74.64        | 1.99                   | 74.62        | 0.16                   | 59.58        | 0.45                  | 3.05                   | 43.37        |
| <i>P2</i> |                        |              |                       |                        |              |                       |                        |              |                       |                        |              |                        |              |                        |              |                        |              |                       |                        |              |
| V1        | 0.31                   | 1.60         | 0.59                  | 0.34                   | 1.71         | 0.59                  | 0.30                   | 1.58         | 0.58                  | 0.97                   | 2.14         | 1.06                   | 1.76         | 1.07                   | 1.65         | 0.30                   | 1.49         | 0.58                  | 0.47                   | 0.71         |
| V2        | 2.46                   | 73.83        | 0.51                  | 2.64                   | 76.98        | 0.50                  | 2.48                   | 75.08        | 0.50                  | 2.35                   | 72.49        | 3.83                   | 52.10        | 4.03                   | 47.80        | 2.30                   | 71.62        | 0.51                  | 4.62                   | 39.94        |
| V3        | 0.10                   | 1.27         | 0.57                  | 0.11                   | 1.29         | 0.57                  | 0.09                   | 1.26         | 0.55                  | 0.11                   | 1.23         | 0.02                   | 1.20         | 0.05                   | 1.18         | 0.10                   | 1.21         | 0.53                  | 0.41                   | 0.79         |
| V4        | 1.88                   | 77.52        | 0.45                  | 2.16                   | 85.20        | 0.46                  | 1.86                   | 78.37        | 0.45                  | 2.10                   | 72.07        | 3.51                   | 59.58        | 3.81                   | 55.81        | 2.06                   | 71.80        | 0.46                  | 3.92                   | 39.09        |
| V5        | 0.13                   | 73.67        | 0.13                  | 0.16                   | 74.56        | 0.20                  | 0.14                   | 73.65        | 0.13                  | 0.16                   | 81.97        | 0.11                   | 82.74        | 0.13                   | 81.89        | 0.26                   | 70.26        | 0.13                  | 2.73                   | 52.06        |
| V6        | 0.13                   | 71.18        | 0.46                  | 0.18                   | 71.46        | 0.49                  | 0.13                   | 71.68        | 0.45                  | 1.17                   | 70.60        | 1.17                   | 70.64        | 1.18                   | 70.60        | 0.20                   | 62.32        | 0.45                  | 2.82                   | 44.62        |
| <i>P3</i> |                        |              |                       |                        |              |                       |                        |              |                       |                        |              |                        |              |                        |              |                        |              |                       |                        |              |
| V1        | 0.34                   | 1.75         | 0.61                  | 0.43                   | 2.07         | 0.62                  | 0.32                   | 1.67         | 0.60                  | 0.35                   | 1.70         | 0.52                   | 1.31         | 0.51                   | 1.17         | 0.30                   | 1.62         | 0.60                  | 0.51                   | 0.79         |
| V2        | 2.13                   | 71.98        | 0.53                  | 2.73                   | 84.52        | 0.53                  | 2.11                   | 74.92        | 0.52                  | 2.37                   | 75.17        | 4.07                   | 57.85        | 4.22                   | 51.77        | 2.16                   | 72.28        | 0.52                  | 4.58                   | 39.35        |
| V3        | 0.16                   | 1.33         | 0.58                  | 0.20                   | 1.40         | 0.62                  | 0.14                   | 1.31         | 0.58                  | 0.19                   | 1.42         | 0.10                   | 1.36         | 0.12                   | 1.35         | 0.18                   | 1.37         | 0.56                  | 0.55                   | 0.88         |
| V4        | 1.90                   | 83.40        | 0.49                  | 2.44                   | 98.91        | 0.51                  | 1.87                   | 85.07        | 0.49                  | 2.22                   | 79.91        | 3.58                   | 63.87        | 3.72                   | 57.87        | 2.10                   | 76.30        | 0.47                  | 4.04                   | 42.02        |
| V5        | 0.34                   | 78.32        | 0.21                  | 0.38                   | 81.91        | 0.30                  | 0.35                   | 78.71        | 0.22                  | 0.10                   | 81.02        | 0.07                   | 80.37        | 0.07                   | 80.91        | 0.28                   | 70.95        | 0.18                  | 2.81                   | 51.72        |
| V6        | 0.31                   | 72.42        | 0.49                  | 0.43                   | 72.52        | 0.53                  | 0.32                   | 73.06        | 0.50                  | 0.53                   | 69.95        | 0.45                   | 70.24        | 0.49                   | 69.96        | 0.37                   | 65.51        | 0.47                  | 2.97                   | 48.21        |
| <i>P4</i> |                        |              |                       |                        |              |                       |                        |              |                       |                        |              |                        |              |                        |              |                        |              |                       |                        |              |
| V1        | 0.19                   | 0.87         | 0.49                  | 0.19                   | 0.87         | 0.49                  | 0.19                   | 0.87         | 0.49                  | 0.22                   | 0.93         | 0.37                   | 0.71         | 0.40                   | 0.69         | 0.21                   | 0.82         | 0.50                  | 0.42                   | 0.50         |
| V2        | 1.74                   | 47.46        | 0.52                  | 1.74                   | 47.46        | 0.52                  | 1.74                   | 47.46        | 0.52                  | 1.97                   | 50.14        | 3.23                   | 33.67        | 3.57                   | 33.94        | 1.60                   | 44.45        | 0.51                  | 4.01                   | 27.32        |
| V3        | 0.07                   | 0.72         | 0.42                  | 0.07                   | 0.72         | 0.42                  | 0.07                   | 0.72         | 0.42                  | 0.11                   | 0.72         | 0.05                   | 0.71         | 0.07                   | 0.70         | 0.05                   | 0.72         | 0.41                  | 0.29                   | 0.56         |
| V4        | 1.55                   | 49.22        | 0.41                  | 1.55                   | 49.22        | 0.41                  | 1.55                   | 49.22        | 0.41                  | 1.82                   | 50.21        | 2.44                   | 35.56        | 2.80                   | 35.21        | 1.52                   | 50.01        | 0.42                  | 3.42                   | 29.36        |
| V5        | 0.07                   | 44.26        | 0.05                  | 0.07                   | 44.26        | 0.05                  | 0.07                   | 44.26        | 0.05                  | 0.07                   | 46.18        | 0.06                   | 46.19        | 0.06                   | 46.17        | 0.02                   | 45.96        | 0.04                  | 1.78                   | 37.21        |
| V6        | 0.08                   | 41.14        | 0.40                  | 0.08                   | 41.14        | 0.40                  | 0.08                   | 41.14        | 0.40                  | 0.12                   | 40.03        | 0.06                   | 40.02        | 0.08                   | 40.03        | 0.12                   | 38.72        | 0.40                  | 2.15                   | 31.08        |
| <i>P5</i> |                        |              |                       |                        |              |                       |                        |              |                       |                        |              |                        |              |                        |              |                        |              |                       |                        |              |
| V1        | 0.19                   | 0.89         | 0.51                  | 0.20                   | 0.92         | 0.50                  | 0.19                   | 0.88         | 0.50                  | 0.23                   | 0.93         | 0.38                   | 0.70         | 0.41                   | 0.67         | 0.18                   | 0.82         | 0.51                  | 0.44                   | 0.52         |
| V2        | 1.81                   | 48.00        | 0.50                  | 1.91                   | 49.65        | 0.49                  | 1.81                   | 48.16        | 0.50                  | 1.93                   | 46.84        | 3.40                   | 33.10        | 3.74                   | 32.83        | 1.60                   | 46.59        | 0.50                  | 4.02                   | 28.13        |
| V3        | 0.07                   | 0.74         | 0.43                  | 0.08                   | 0.74         | 0.44                  | 0.07                   | 0.73         | 0.42                  | 0.14                   | 0.65         | 0.08                   | 0.64         | 0.10                   | 0.63         | 0.07                   | 0.74         | 0.45                  | 0.31                   | 0.56         |
| V4        | 1.52                   | 49.82        | 0.43                  | 1.60                   | 51.85        | 0.43                  | 1.54                   | 50.18        | 0.43                  | 1.79                   | 48.26        | 3.00                   | 36.57        | 3.37                   | 35.86        | 1.55                   | 52.40        | 0.44                  | 3.43                   | 30.87        |
| V5        | 0.05                   | 47.28        | 0.07                  | 0.08                   | 47.46        | 0.09                  | 0.05                   | 47.34        | 0.07                  | 0.21                   | 45.51        | 0.21                   | 45.79        | 0.20                   | 45.50        | 0.05                   | 45.44        | 0.06                  | 1.84                   | 34.95        |
| V6        | 0.15                   | 39.08        | 0.41                  | 0.15                   | 39.28        | 0.44                  | 0.16                   | 39.13        | 0.40                  | 0.26                   | 42.02        | 0.20                   | 42.07        | 0.22                   | 42.02        | 0.16                   | 34.61        | 0.40                  | 2.03                   | 34.78        |
| <i>P6</i> |                        |              |                       |                        |              |                       |                        |              |                       |                        |              |                        |              |                        |              |                        |              |                       |                        |              |
| V1        | 0.27                   | 1.22         | 0.56                  | 0.34                   | 1.42         | 0.58                  | 0.26                   | 1.17         | 0.54                  | 0.24                   | 0.91         | 0.41                   | 0.71         | 0.43                   | 0.68         | 0.27                   | 1.14         | 0.51                  | 0.47                   | 0.58         |
| V2        | 2.01                   | 52.2         | 0.51                  | 2.4                    | 60.76        | 0.5                   | 1.97                   | 53.47        | 0.51                  | 2.07                   | 45.43        | 3.53                   | 33.59        | 3.83                   | 32.92        | 1.81                   | 48.7         | 0.5                   | 4.05                   | 29.28        |
| V3        | 0.11                   | 0.86         | 0.5                   | 0.15                   | 0.93         | 0.55                  | 0.1                    | 0.84         | 0.5                   | 0.12                   | 0.78         | 0.05                   | 0.77         | 0.08                   | 0.76         | 0.12                   | 0.86         | 0.48                  | 0.43                   | 0.64         |
| V4        | 1.85                   | 50.84        | 0.42                  | 2.17                   | 58.89        | 0.44                  | 1.92                   | 52.13        | 0.41                  | 1.74                   | 28.59        | 1.74                   | 50.89        | 4.99                   | 47.00        | 1.48                   | 68.84        | 0.43                  | 5.48                   | 43.34        |
| V5        | 0.04                   | 48.46        | 0.11                  | 0.09                   | 50.79        | 0.21                  | 0.04                   | 48.76        | 0.12                  | 0.08                   | 48.76        | 0.07                   | 48.92        | 0.08                   | 48.75        | 0.08                   | 46.03        | 0.1                   | 1.8                    | 35.17        |
| V6        | 0.38                   | 43.31        | 0.46                  | 0.48                   | 43.41        | 0.48                  | 0.39                   | 43.58        | 0.48                  | 0.19                   | 43.45        | 0.16                   | 43.46        | 0.16                   | 43.46        | 0.32                   | 41.36        | 0.44                  | 2.17                   | 33.22        |
| <i>P7</i> |                        |              |                       |                        |              |                       |                        |              |                       |                        |              |                        |              |                        |              |                        |              |                       |                        |              |
| V1        | 0.11                   | 0.38         | 0.44                  | 0.11                   | 0.38         | 0.44                  | 0.11                   | 0.38         | 0.44                  | 0.10                   | 0.36         | 0.17                   | 0.31         | 0.19                   | 0.31         | 0.12                   | 0.40         | 0.42                  | 0.21                   | 0.28         |
| V2        | 0.96                   | 18.95        | 0.53                  | 0.96                   | 18.95        | 0.53                  | 0.96                   | 18.95        | 0.53                  | 1.20                   | 18.84        | 1.94                   | 15.42        | 2.17                   | 15.58        | 0.96                   | 19.54        | 0.52                  | 2.43                   | 14.22        |
| V3        | 0.02                   | 0.49         | 0.33                  | 0.02                   | 0.49         | 0.33                  | 0.02                   | 0.49         | 0.33                  | 0.08                   | 0.51         | 0.03                   | 0.50         | 0.05                   | 0.50         | 0.05                   | 0.51         | 0.31                  | 0.26                   | 0.42         |
| V4        | 0.79                   | 22.14        | 0.41                  | 0.79                   | 22.14        | 0.41                  | 0.79                   | 22.14        | 0.41                  | 0.81                   | 20.43        | 1.85                   | 17.60        | 2.09                   | 17.44        | 0.77                   | 21.33        | 0.39                  | 1.6                    | 14.65        |
| V5        | 0.15                   | 40.31        | 0.02                  | 0.15                   | 40.31        | 0.02                  | 0.15                   | 40.31        | 0.02                  | 0.07                   | 38.37        | 0.13                   | 38.37        | 0.13                   | 38.37        | 0.01                   | 37.67        | 0.01                  | 1.5                    | 29.98        |
| V6        | 0.06                   | 33.64        | 0.31                  | 0.06                   | 33.64        | 0.31                  | 0.06                   | 33.64        | 0.31                  | 0.26                   | 33.32        | 0.25                   | 33.31        | 0.28                   | 33.31        | 0.05                   | 34.64        | 0.30                  | 1.43                   | 27.34        |
| <i>P8</i> |                        |              |                       |                        |              |                       |                        |              |                       |                        |              |                        |              |                        |              |                        |              |                       |                        |              |
| V1        | 0.10                   | 0.42         | 0.44                  | 0.11                   | 0.43         | 0.44                  | 0.10                   | 0.41         | 0.45                  | 0.12                   | 0.39         | 0.21                   | 0.34         | 0.23                   | 0.33         | 0.11                   | 0.42         | 0.42                  | 0.25                   | 0.29         |
| V2        | 0.87                   | 18.82        | 0.53                  | 0.92                   | 19.26        | 0.52                  | 0.87                   | 19.15        | 0.52                  | 1.08                   | 18.47        | 2.39                   | 15.52        | 2.59                   | 15.33        | 0.93                   | 19.12        | 0.51                  | 2.37                   | 13.74        |
| V3        | 0.05                   | 0.52         | 0.33                  | 0.05                   | 0.51         | 0.34                  | 0.05                   | 0.52         | 0.32                  | 0.09                   | 0.50         | 0.01                   | 0.49         | 0.04                   | 0.49         | 0.04                   | 0.54         | 0.30                  | 0.26                   | 0.45         |
| V4        | 0.68                   | 20.07        | 0.4                   | 0.74                   | 20.7         | 0.41                  | 0.68                   | 20.44        | 0.4                   | 0.91                   | 21.43        | 1.76                   | 18.74        | 2.01                   | 18.06        | 0.5                    | 21.23        | 0.40                  | 1.71                   | 14.52        |
| V5        | 0.02                   | 37.66        | 0.01                  | 0.02                   | 37.74        | 0.01                  | 0.02                   | 37.7         | 0.01                  | 0.14                   | 35.13        | 0.19                   | 35.10        | 0.20                   | 35.12        | 0.12                   | 35.91        | 0.01                  | 1.61                   | 29.0         |
| V6        | 0.1                    | 35.23        | 0.32                  | 0.11                   | 35.19        | 0.35                  | 0.10                   | 35.3         | 0.31                  | 0.05                   | 35.41        | 0.06                   | 35.40        | 0.07                   | 35.41        | 0.04                   | 33.19        | 0.29                  | 1.55                   | 26.66        |
| <i>P9</i> |                        |              |                       |                        |              |                       |                        |              |                       |                        |              |                        |              |                        |              |                        |              |                       |                        |              |
| V1        | 0.11                   | 0.48         | 0.45                  | 0.14                   | 0.53         | 0.46                  | 0.11                   | 0.46         | 0.45                  | 0.14                   | 0.46         | 0.25                   | 0.41         | 0.26                   | 0.37         | 0.11                   | 0.45         | 0.42                  | 0.25                   | 0.31         |
| V2        | 1.02                   | 20.63        | 0.51                  | 1.25                   | 22.64        | 0.5                   | 1.05                   | 22.22        | 0.51                  | 1.23                   | 18.97        | 2.62                   | 17.27        | 2.76                   | 16.18        | 0.9                    | 20.14        | 0.51                  | 2.41                   | 14.77        |
| V3        | 0.08                   | 0.61         | 0.38                  | 0.09                   | 0.6          | 0.4                   | 0.07                   | 0.60         | 0.36                  | 0.11                   | 0.55         | 0.03                   | 0.53         | 0.07                   | 0.54         | 0.05                   | 0.57         | 0.32                  | 0.28                   | 0.46         |
| V4        | 0.75                   | 22.45        | 0.41                  | 0.98                   | 25.13        | 0.42                  | 0.74                   | 23.91        | 0.41                  | 0.91                   | 23.35        | 1.99                   | 21.65        | 2.14                   | 19.51        | 0.74                   | 21.8         | 0.37                  | 1.73                   | 15.0         |
| V5        | 0.11                   | 35.74        | 0.01                  | 0.10                   | 36.54        | 0.03                  | 0.10                   | 35.95        | 0.02                  | 0.03                   | 36.21        | 0.10                   | 36.98        | 0.11                   | 36.20        | 0.02                   | 38.19        | 0.01                  | 1.54                   | 30.42        |
| V6        | 0.22                   | 34.73        | 0.37                  | 0.25                   | 34.86        | 0.41                  | 0.21                   | 34.86        | 0.36                  | 0.17                   | 33.34        | 0.17                   | 33.26        | 0.19                   | 33.33        | 0.14                   | 35.54        | 0.32                  | 1.49                   | 28.09        |



**Table 4** The norm of EQM corresponding to MINQUE, MMINQUE, IAUE, MIAUE, REML and MREML estimators at each variance component.

|              | MMINQUE1 | MMINQUE2 | MINQUE | MIAUE1 | MIAUE2 | IAUE   | REML   | MREML  |
|--------------|----------|----------|--------|--------|--------|--------|--------|--------|
| $\sigma_1^2$ | 159.91   | 167.24   | 160.69 | 154.60 | 155.09 | 155.97 | 153.56 | 122.53 |
| $\sigma_2^2$ | 14.31    | 16.20    | 16.19  | 13.18  | 13.28  | 13.32  | 13.83  | 12.76  |
| $\sigma_3^2$ | 3.98     | 3.98     | 3.99   | 4.18   | 4.13   | 4.11   | 3.97   | 3.93   |



**Fig. 1** EQDG's corresponding to MINQUE, MMINQUE, AUE, MAUE, REML and MREML estimators for each variance component.

associated with  $\sigma_2^2$  which the latter is lower than the norm associated with  $\sigma_1^2$ .

## Conclusions

In this article, two new estimators based on IAUE principle are introduced for estimating the variance components in the mixed linear model. The aim of this article was to evaluate the performance of the proposed estimators relative to various estimators via simulation studies. The model we used is nested-factorial model with two fixed crossed factorial and one nested random factor under regularity assumptions. Several criteria such as bias, MSE, probability of getting negative values and the norm of EQM are used to show the performance of the estimators under the study. From the numerical analysis, we have found that the estimators based on restricted likelihood function have desirable properties as long as the data have normal distribution. Further, the proposed estimators may be appropriate estimators since they have less bias and less MSE than the estimator based on almost unbiased approach it may be important to study some details in the proposed algorithms in the literature which used for computing the variance components estimates and its effect to the statistical characteristics e.g. [19,23].

## Conflict of interest

*The authors have declared no conflict of interests.*

## Compliance with Ethics Requirements

*This article does not contain any studies with human or animal subjects.*

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