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Variational-Type Inequalities for (η, θ, δ)-Pseudomonotone-Type Set-Valued Mappings in Nonreflexive Banach Spaces

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Abstract—In this paper, we introduce (η, θ, δ)-pseudomonotone-type set-valued mappings and consider the existence of solutions to variational-type inequality problems for (η, θ, δ)-pseudomonotone-type set-valued mappings in nonreflexive Banach spaces.

Keywords—Variational-type inequalities, (η, θ, δ)-pseudomonotone-type, Nonreflexive Banach spaces, Hemicontinuity, Finite-dimensional upper semicontinuity.

1. INTRODUCTION

In 1995, Chang *et al.* [1] first considered variational inequalities for monotone single-valued mappings in nonreflexive Banach spaces. Later on, Verma [2] and Watson [3] studied variational inequalities for strong pseudomonotonicity and pseudomonotonicity for single-valued mappings in nonreflexive Banach spaces, respectively. Recently, Lee *et al.* [4] introduced (η, θ)-pseudomonotonicity, which generalizes and extends monotonicities mentioned above. And then, they showed the existence theorem of solutions to generalized variational-like inequalities for single-valued mappings in nonreflexive Banach spaces, which generalizes and extends some results in [2] and [1,3], respectively.

In this paper, we first introduce (η, θ, δ)-pseudomonotone-type, which generalizes and extends (η, θ)-pseudomonotonicity to set-valued case. And then, we show that the existence theorem of solutions to variational-type inequalities for (η, θ, δ)-pseudomonotone-type set-valued mappings in nonreflexive Banach spaces.

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Let X be a real Banach space, $T : X \to 2^{X^*}$, where X^* is the dual space of X, be a set-valued mapping, $K \subset X$ be a set, $\eta : K \times K \to X^{**}$, the second dual of X, be an operator, and $\delta : K \times K \to \mathbb{R}$ be a function. The so-called scalar variational-type inequality problem for set-valued mappings is to find an $x_0 \in K$ such that for all $x \in K$ there exists a $v_0 \in T(x_0)$ such that

$$\langle v_0, \eta(x, x_0)\rangle + \delta(x, x_0) \geq 0,$$

the single-valued case of which was first considered by Behera and Panda [5], and extended to vector case for single-valued mappings using generalized Minty's lemma [6] and for set-valued mappings [7] using Fan's geometrical lemma [8].

DEFINITION 1.1. *Let X be a real nonreflexive Banach space with the dual X^* and X^{**} be the dual of X^*. Let $T : K \to 2^{X^*} \setminus \{\emptyset\}$ be a set-valued mapping, $\eta, \theta : K \times K \to X^{**}$ operators and $\delta : K \times K \to \mathbb{R}$ a function, where K is a subset of X^{**}. T is said to be (η, θ, δ)-pseudomonotone-type, if there exists a constant r (called (η, θ, δ)-pseudomonotone-type constant of T) such that for all $x, y \in K$ and $v \in T(y)$, there exists $u \in T(x)$ such that*

$$\langle v, \eta(x, y)\rangle + \delta(x, y) \geq 0 \text{ implies } \langle u, \eta(x, y)\rangle + \delta(x, y) \geq r\|\theta(x, y)\|^2,$$

where $\|\cdot\|$ denotes the norm.

EXAMPLE. Let X be a nonreflexive Banach space c_0, then $l_1 = {c_0}^*$ and $l_\infty = {c_0}^{**}$. Put $K = \{x \in l_\infty : \|x\| < 1\}$. Define $\eta : K \times K \to l_\infty$ by $\eta(x, y) = ({x_i}^2 + {y_i}^2)_{i=1}^\infty$ for $x = (x_i), y = (y_i) \in K \setminus \{0\}$ and $\eta(0, 0) = (1)_{i=1}^\infty$, $\delta : K \times K \to \mathbb{R}$ and $\theta : K \times K \to l_\infty$ by $\theta(x, y) \neq 0$ for $x, y \in K$. Define $T : K \subset l_\infty \to 2^{l_1} \setminus \{\emptyset\}$ by for $x = (x_i)_{i=1}^\infty \in K \setminus \{0\}$,

$$T(x) = \{n \cdot \hat{x}_i : \hat{x}_i = (0, \ldots, 0, x_i, 0, \ldots), n \in \mathbb{N} \cup (-\mathbb{N}), i \in \mathbb{N}\} \quad \text{and} \quad T(0) = \left(\frac{1}{i^2}\right)_{i=1}^\infty,$$

then T is (η, θ, δ)-pseudomonotone-type mapping. In fact, for all $x = (x_i)_{i=1}^\infty, y = (y_i)_{i=1}^\infty \in K$ and $v = n \cdot \hat{y}_i \in T(y)$. Choose $u = k \cdot \hat{x}_i \in T(x)$ such that $k \cdot x_i > n \cdot y_i$ for some $k \in \mathbb{N} \cup (-\mathbb{N})$. Then

$$\begin{aligned} \frac{\langle u - v, \eta(x, y)\rangle}{\|\theta(x, y)\|^2} &= \frac{\langle k \cdot \hat{x}_i - n \cdot \hat{y}_i, \eta(x, y)\rangle}{\|\theta(x, y)\|^2} \\ &= \frac{(k \cdot x_i - n \cdot y_i)\left({x_i}^2 + {y_i}^2\right)}{\|\theta(x, y)\|^2} \quad \text{or} \quad \frac{k \cdot x_i - n \cdot y_i}{\|\theta(x, y)\|^2} \\ &\geq r, \qquad \text{for some } r > 0. \end{aligned}$$

Definition 1.1 generalizes the (η, θ)-pseudomonotonicity for single-valued mappings in [4].

DEFINITION 1.2. *Let $\eta : K \times K \to X^{**}$ be a function. A set-valued mapping $T : K \subset X^{**} \to 2^{X^*}$ is said to be hemicontinuous, if for any $x, y \in K$ with $x + t(y - x) \in K$ for any $t \in [0, 1]$, the multifunction*

$$t \in [0, 1] \longmapsto T(x + t(y - x)) \cdot \eta(y, x)$$

is upper semicontinuous (shortly, u.s.c.) at 0^+, where

$$T(x + t(y - x)) \cdot \eta(y, x) = \{\langle s, \eta(y, x)\rangle : s \in T(x + t(y - x))\}.$$

The set-valued mapping T is said to be finite-dimensional u.s.c. if for any finite-dimensional subspace F of X^{**} with $K_F = K \cap F \neq \emptyset$, $T : K_F \to 2^{X^*}$ is u.s.c. in the norm topology.

LEMMA 1.1. *(See [9].) Let X, Y be topological vector spaces and $T : X \to 2^Y$ be a set-valued mapping.*

(1) *If K is a compact subset of X, and T is u.s.c. and compact-valued, then $T(K)$ is compact.*
(2) *If T is u.s.c. and compact-valued, then T is closed.*

Let K be a subset of a topological vector space X. Then a mapping $T : K \to 2^X$ is called a Knaster-Kuratowski-Mazurkiewicz (in short, KKM) mapping [10] if for each nonempty finite subset N of K, $\text{co}N \subset T(N)$, where co denotes the convex hull and $T(N) = \cup\{T(x) : x \in N\}$.

THEOREM 1.1. KKM THEOREM. *(See [8].) Let K be an arbitrary nonempty subset of a Hausdorff topological vector space X. Let a set-valued mapping $T : K \to 2^X$ be a KKM mapping such that $T(x)$ is closed for all $x \in K$ and compact for at least one $x \in K$. Then*

$$\bigcap_{x \in K} T(x) \neq \emptyset.$$

2. MAIN RESULTS

In this section, we first consider some result with (η, θ, δ)-pseudomonotone-type hemicontinuous set-valued mappings in nonreflexive Banach spaces. And then, we show that the existence theorem of solutions to variational-type inequality problems for (η, θ, δ)-pseudomonotone-type finite-dimensional u.s.c. compact set-valued mappings in nonreflexive Banach spaces.

THEOREM 2.1. *Let X be a real nonreflexive Banach space and K a nonempty convex subset of X^{**}. Let $T : K \to 2^{X^*}$ be an (η, θ, δ)-pseudomonotone-type hemicontinuous set-valued mapping, and η, $\theta : K \times K \to X^{**}$ operators and $\delta : K \times K \to \mathbb{R}$ a function such that*

(i) *$\eta(x, x) = \bar{0}$, $\theta(x, x) = \bar{0}$, and $\delta(x, x) = 0$, for all $x \in K$ and*
(ii) *for each $y \in K$,*

$$x \mapsto \eta(x, y) \qquad \text{and} \qquad x \mapsto \theta(x, y) \text{ are affine, and } x \mapsto \delta(x, y) \text{ is convex.}$$

Then the following variational-type inequality problems are equivalent.
Find $x_0 \in K$ such that for all $x \in K$ there exists $v_0 \in T(x_0)$ such that

$$\langle v_0, \eta(x, x_0) \rangle + \delta(x, x_0) \geq 0. \tag{2.1}$$

Find $x_0 \in K$ such that for all $x \in K$ there exists $v \in T(x)$ such that

$$\langle v, \eta(x, x_0) \rangle + \delta(x, x_0) \geq r \|\theta(x, x_0)\|^2, \tag{2.2}$$

where r is the (η, θ, δ)-pseudomonotone-type constant of T.

PROOF. Equation (2.2) follows from (2.1) by the definition of (η, θ, δ)-pseudomonotone-type of T.

Conversely, suppose that (2.2) holds and set $x_t = x_0 + t(x - x_0)$, where $x \in K$ and $t \in [0, 1]$. Then $x_t \in K$, and we have $v_t \in T(x_t)$ such that

$$\langle v_t, \eta(x_t, x_0) \rangle + \delta(x_t, x_0) \geq r \|\theta(x_t, x_0)\|^2.$$

Hence, by (ii), we have

$$t\{\langle v_t, \eta(x, x_0) \rangle + \delta(x, x_0)\} + (1 - t)\{\langle v_t, \eta(x_0, x_0) \rangle + \delta(x_0, x_0)\} \geq r \|t\theta(x, x_0) + (1 - t)\theta(x_0, x_0)\|^2.$$

Consequently, we have

$$\langle v_t, \eta(x, x_0) \rangle + \delta(x, x_0) \geq rt \|\theta(x, x_0)\|^2. \tag{2.3}$$

Suppose that (2.1) does not hold. Then there exists an $x \in K$ such that for any $v_0 \in T(x_0)$

$$\langle v_0, \eta(x, x_0) \rangle + \delta(x, x_0) < 0.$$

By hemicontinuity of T, there exists $t_0 > 0$ such that for any $t \in (0, t_0)$ and $s \in T(x_t)$,

$$\langle s, \eta(x, x_0) \rangle + \delta(x, x_0) < 0.$$

Hence, we have that for any $t \in (0, t_0)$ and $s \in T(x_t)$

$$\langle s, \eta(x, x_0) \rangle + \delta(x, x_0) < rt \|\theta(x, x_0)\|^2,$$

which contradicts (2.3). So (2.1) holds.

COROLLARY 2.2. *Considering $T : K \to X^*$ instead of $T : K \to 2^{X^*}$ and a zero function δ in Theorem 2.1, we can obtain Theorem 2.1 in [4], which generalizes Theorem 2.1 in [2] and Lemma 1 in [3].*

Now, we consider the existence theorem of solutions to variational-type inequality problems for (η, θ, δ)-pseudomonotone-type finite-dimensional u.s.c. compact set-valued mappings in nonreflexive Banach spaces.

THEOREM 2.3. *Let X be a real nonreflexive Banach space and K a nonempty bounded closed convex subset of X^{**}. Let $T : K \to 2^{X^*}$ be an (η, θ, δ)-pseudomonotone type finite-dimensional u.s.c. compact set-valued mapping, and η, $\theta : K \times K \to X^{**}$ operators and $\delta : K \times K \to \mathbb{R}$ a function such that*

(i) $\eta(x, x) = \bar{0}$, $\theta(x, x) = \bar{0}$, *and* $\delta(x, x) = 0$, *for all* $x \in K$,
(ii) *for each* $y \in K$,

$$x \mapsto \eta(x, y) \qquad \text{and} \qquad x \mapsto \theta(x, y) \text{ are affine, and } x \mapsto \delta(x, y) \text{ is convex.}$$

(iii) *for each* $x \in K$,

$$y \mapsto \eta(x, y), \quad y \mapsto \theta(x, y), \quad \text{and} \quad y \mapsto \delta(x, y) \text{ are continuous.}$$

Then there exists $x_0 \in K$ such that for all $x \in K$ there exists $v_0 \in T(x_0)$ such that

$$\langle v_0, \eta(x, x_0) \rangle + \delta(x, x_0) \geq 0.$$

PROOF. For each finite-dimensional subspace F of X^{**} with $K_F = K \cap F \neq \emptyset$, we first consider the following variational-type inequality.

Find $x_0 \in K_F$ such that for all $x \in K_F$ there exists $v_0 \in T(x_0)$ such that

$$\langle v_0, \eta(x, x_0) \rangle + \delta(x, x_0) \geq 0. \tag{2.4}$$

Since K_F is a nonempty bounded closed convex set in a finite-dimensional space F and $T : K_F \to 2^{X^*}$ is u.s.c. compact-valued, (2.4) has a solution $x_0 \in K_F$. In fact, define a set-valued map $G : K_F \to 2^F$ for each $y \in K_F$ by

$$G(y) := \{x \in K_F : \text{there exists } v \in T(x) \text{ such that } \langle v, \eta(y, x) \rangle + \delta(y, x) \geq 0\}.$$

First, we prove that G is a KKM map. Suppose to the contrary that G is not a KKM map. Then there exists a finite set $\{x_1, x_2, \dots, x_n\}$ in K_F, $\alpha_i \geq 0, i = 1, 2, \dots, n$ and $x \in \mathrm{co}\{x_1, x_2, \dots, x_n\}$ such that

$$\sum_{i=1}^{n} \alpha_i = 1 \qquad \text{and} \qquad x = \sum_{i=1}^{n} \alpha_i x_i \notin \bigcup_{i=1}^{n} G(x_i).$$

So, by Condition (ii) and definition of a set-valued map G, we have for all $v \in T(x)$,

$$\begin{aligned} \langle v, \eta(x, x) \rangle + \delta(x, x) &= \left\langle v, \eta\left(\sum_{i=1}^{n} \alpha_i x_i, x\right)\right\rangle + \delta\left(\sum_{i=1}^{n} \alpha_i x_i, x\right) \\ &\leq \left\langle v, \sum_{i=1}^{n} \alpha_i \eta(x_i, x)\right\rangle + \sum_{i=1}^{n} \alpha_i \delta(x_i, x) \\ &= \sum_{i=1}^{n} \left(\langle v, \eta(x_i, x)\rangle + \delta(x_i, x)\right) \\ &< 0, \qquad \text{for all } i = 1, 2, \dots, n. \end{aligned}$$

By Condition (i), this is impossible. Therefore, G is a KKM map.

Next, we show that $G(y)$ is closed in F, for all $y \in K_F$. Let $\{x_n\}$ be a sequence in $G(y)$ converging to $x_0 \in F$. Then we have there exists $v_n \in T(x_n)$ for each n such that

$$\langle v_n, \eta(y, x_n)\rangle + \delta(y, x_n) \geq 0.$$

By (1) of Lemma 1.1, $T(K_F)$ is compact, there exists $v_0 \in T(K_F)$ such that $v_n \to v_0$. Since T is closed by (2) of Lemma 1.1, $v_0 \in T(x_0)$. And by Condition (iii) and the fact that $\{\|\eta(y, x_n)\|\}$ is bounded, we have

$$\begin{aligned}
&|\langle v_n, \eta(y, x_n)\rangle + \delta(y, x_n) - \{\langle v_0, \eta(y, x_0)\rangle + \delta(y, x_0)\}| \\
&\quad \leq |\langle v_n, \eta(y, x_n)\rangle - \langle v_0, \eta(y, x_0)\rangle| + |\delta(y, x_n) - \delta(y, x_0)| \\
&\quad \leq |\langle v_n - v_0, \eta(y, x_n)\rangle| + |\langle v_0, \eta(y, x_n) - \eta(y, x_0)\rangle| + |\delta(y, x_n) - \delta(y, x_0)| \\
&\quad \leq \|v_n - v_0\| \cdot \|\eta(y, x_n)\| + \|v_0\| \cdot \|\eta(y, x_n) - \eta(y, x_0)\| + |\delta(y, x_n) - \delta(y, x_0)| \\
&\quad \longrightarrow 0, \qquad \text{as } n \to \infty.
\end{aligned}$$

Consequently, there exists $v_0 \in T(x_0)$ such that $\langle v_0, \eta(y, x_0)\rangle + \delta(y, x_0) \geq 0$. Hence, $x_0 \in G(y)$, thus, $G(y)$ is closed in F. Moreover, $G(y)$ is compact from the compactness of K_F. By KKM theorem, $\bigcap_{x \in K_F} G(x)$ is nonempty.

Letting $x_0 \in \bigcap_{x \in K_F} G(x)$, for all $x \in K_F$, we have some $v_0 \in T(x_0)$ with

$$\langle v_0, \eta(x, x_0)\rangle + \delta(x, x_0) \geq 0.$$

Now we will prove that $T : K_F \to 2^{X^*}$ is hemicontinuous.

Since F is a finite-dimensional subspace of X^{**}, $x_\lambda \to x_0$ strongly. Therefore, by Condition (iii), there exists $v \in T(x)$ such that

$$\langle v, \eta(x, x_0)\rangle + \delta(x, x_0) \geq r\|\theta(x, x_0)\|^2, \qquad \text{for all } x \in K.$$

Since $T : K_F \to 2^{X^*}$ is u.s.c., for any $x, y \in K_F$, if $\eta(y, x) = 0$, then it is clear that the multifunction

$$t \in [0, 1] \longmapsto T(x + t(y - x)) \cdot \eta(y, x)$$

is u.s.c. at 0^+. Suppose that $\eta(y, x) \neq 0$ and $\varepsilon > 0$. Then by the upper semicontinuity of T, there exists $t_0 \in (0, 1)$ such that for $t \in (0, t_0)$

$$T(x + t(y - x)) \subset T(x) + \frac{\varepsilon}{\|\eta(y, x)\|} B_{X^*},$$

where $B_{X^*} = \{x^* \in X^* : \|x^*\| < 1\}$.

So for $t \in (0, t_0)$ and $z \in T(x + t(y - x))$, there exists $\bar{z} \in T(x)$ and $x^* \in B_{X^*}$ such that

$$z = \bar{z} + \frac{\varepsilon}{\|\eta(y, x)\|} x^*.$$

Note that

$$\begin{aligned}
|\langle z, \eta(y, x)\rangle - \langle \bar{z}, \eta(y, x)\rangle| &= \frac{\varepsilon}{\|\eta(y, x)\|} |\langle x^*, \eta(y, x)\rangle| \\
&\leq \frac{\varepsilon}{\|\eta(y, x)\|} \|x^*\| \cdot \|\eta(y, x)\| \\
&\leq \varepsilon \|x^*\| \\
&< \varepsilon.
\end{aligned}$$

So we have that for $t \in (0, t_0)$,

$$T(x + t(y - x)) \cdot \eta(y, x) \subset T(x) \cdot \eta(y, x) + \varepsilon B_{\mathbb{R}}.$$

Thus, $T : K_F \to 2^{X^*}$ is hemicontinuous.

On the other hand, since T is (η, θ, δ)-pseudomonotone-type, by Theorem 2.1, there exists $v \in T(x)$ such that

$$\langle v, \eta(x, x_0)\rangle + \delta(x, x_0) \geq r\|\theta(x, x_0)\|^2, \qquad \text{for all } x \in K_F. \tag{2.5}$$

Let $\Im = \{F \subset X^{**} : \dim(F) < +\infty \text{ and } K_F \neq \emptyset\}$, and associate each $F \in \Im$ with a set

$$\begin{aligned} W_F := \big\{x_0 \in K_F &: \text{there exists } v \in T(x) \text{ such that} \\ \langle v, \eta(x, x_0)\rangle + \delta(x, x_0) &\geq r\|\theta(x, x_0)\|^2, \text{ for all } x \in K_F\big\}. \end{aligned} \tag{2.6}$$

By (2.5), we know that W_F is nonempty. Since $W_F \subset K$ and K is weak*-closed, the weak*-closure $\overline{W}^*_F$ of W_F is contained in K. Next, for any n elements $F_1, F_2, \dots, F_n$ in $\Im$, let F be the subspace spanned by the union $\bigcup_{i=1}^n F_i$, then it is obvious that $\dim(F)$ is finite and K_F is nonempty, hence, F lies in $\Im$. Moreover, W_F is nonempty and contained in W_{F_i}, for each $i = 1, 2, \dots, n$. Therefore, we have

$$\bigcap_{i=1}^n \overline{W}^*_{F_i} \neq \emptyset.$$

This implies that $\{\overline{W}^*_F : F \in \Im\}$ has the finite intersection property. Since K is bounded, by Banach-Alaoglu theorem, K is weak*-compact, hence, we know that $\{\overline{W}^*_F : F \in \Im\}$ has the nonempty intersection; i.e.,

$$\bigcap_{F \in \Im} \overline{W}^*_F \neq \emptyset.$$

Take $x_0 \in \bigcap_{F \in \Im} \overline{W}^*_F$. Then for each $F \in \Im$, $x_0 \in K$.

Next, we prove that there exists $v \in T(x)$ such that

$$\langle v, \eta(x, x_0)\rangle + \delta(x, x_0) \geq r\|\theta(x, x_0)\|^2, \qquad \text{for all } x \in K.$$

For a given $x \in K$, take $F \in \Im$ such that $x \in F$. Since $x_0 \in \overline{W}^*_F$, there exists a net x_λ in W_F such that $x_\lambda \to x_0$ in the weak*-topology. Hence, by (2.6), there exists $v \in T(x)$ such that

$$\langle v, \eta(x, x_\lambda)\rangle + \delta(x, x_\lambda) \geq r\|\theta(x, x_\lambda)\|^2, \qquad \text{for all } x \in K.$$

Hence, by Theorem 2.1, we see that there exists $v_0 \in T(x_0)$ such that

$$\langle v_0, \eta(x, x_0)\rangle + \delta(x, x_0) \geq 0, \qquad \text{for all } x \in K.$$

COROLLARY 2.4. *Considering $T : K \to X^*$ instead of $T : K \to 2^{X^*}$ and a zero function δ in Theorem 2.3, we can obtain Theorem 2.3 in [4], which generalizes Theorem 2.2 in [1] and Theorem 2.2 in [2].*

REMARK. We can obtain the same results as Theorem 2.1 and Theorem 2.3 for (η, θ, δ)-pseudomonotone set-valued mappings, which also generalize some results in [2–4] and [1,2,4], respectively.

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